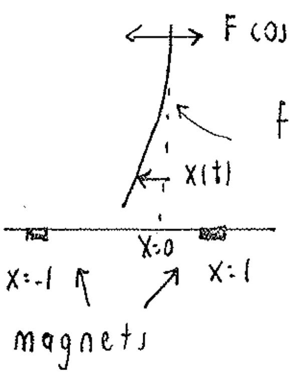
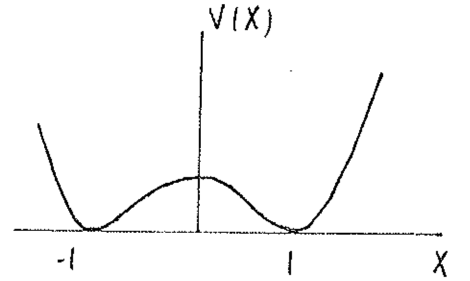
FORCED DUFFING EQUATION





double-well potential

$$V = \frac{1}{4}(x^2-1)^2.$$

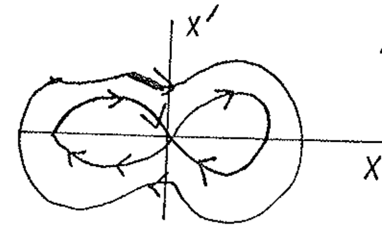
EQUATION OF MOTION

$$x'' + \delta x' + dV/dx = F\cos t.$$

THUS $\quad x'' + \delta x' - x + x^3 = F\cos t.$

SECOND-ORDER NONLINEAR NON-AUTONOMOUS

IF $F=0$ AND $\delta=0$ THEN PHASE PLANE IS



$X=\pm 1$ CENTER

PLOT 1 $\quad \delta>0,\ F=0$ SIMILAR TO DAMPED PENDULUM

PLOT 2a $\quad \delta=.25,\ F=.40$ PLOT y vs X. TRAJECTORIES FOR $0\le t\le 100$ STARTING AT (0,0). TRAJECTORIES "INTERSECT" IN PHASE PLANE.

- PLOT 2b X vs t $\qquad$ PLOT 2c y vs t.

PLOT 3a $\quad \delta=.25,\ F=.25$ TRANSIENT CHAOS. PLOT IN y vs X PLANE WITH $X(0)=.20$, $y(0)=0.10$. EVENTUALLY OSCILLATIONS AROUND ONLY ONE WELL.

- PLOT 3b $\quad X(0)=.20,\ y(0)=0.10$ OSCILLATES IN RIGHT WELL
- PLOT 3c $\quad X(0)=.195,\ y(0)=.10$ " " " LEFT WELL

PLOT 4 $\quad$ POINCARE MAP $\delta=.25,\ F=0.40$

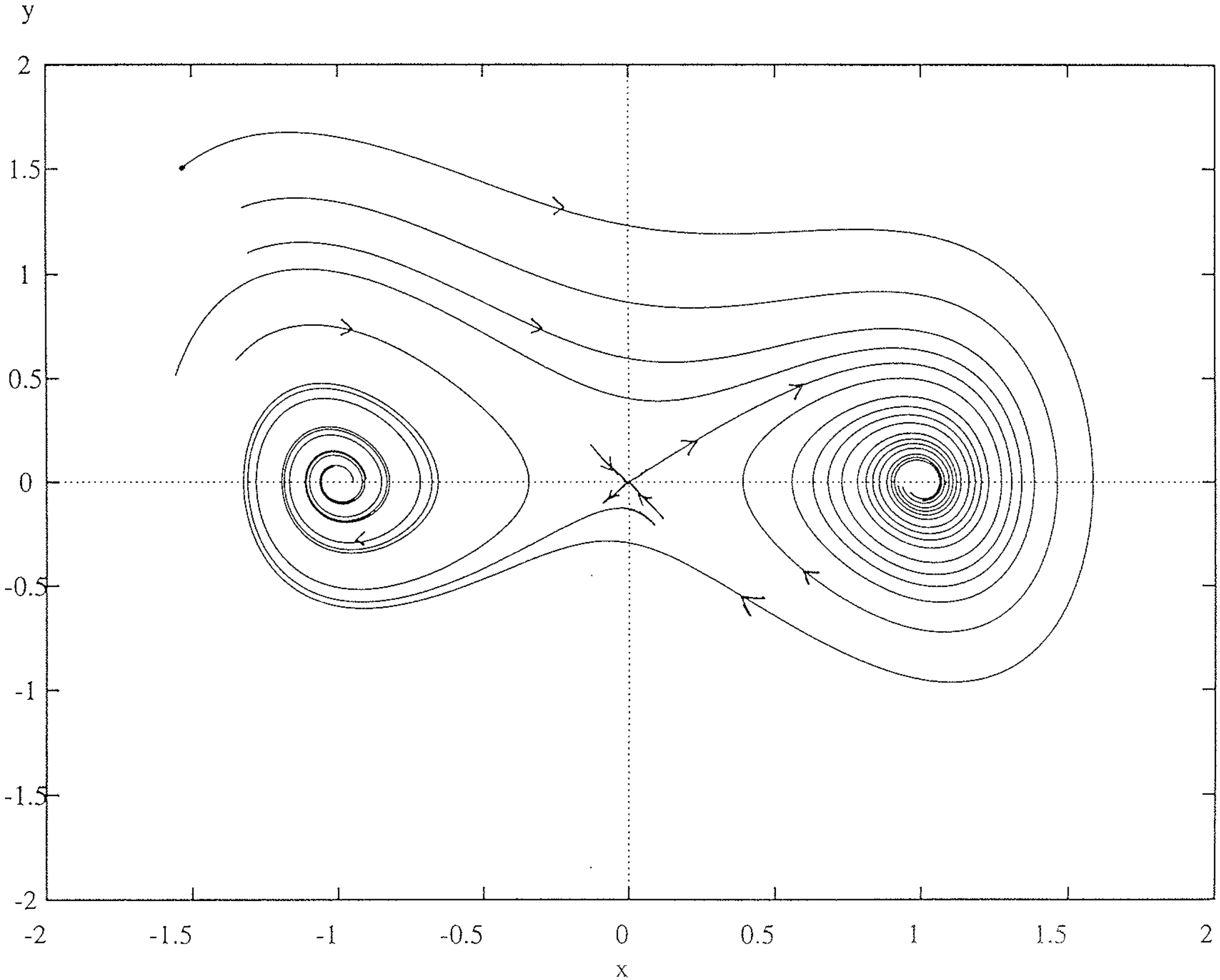
$(X(0), y(0)) = (0,0)$ CHAOTIC ATTRACTOR.

PLOT 5 $\quad \delta=.25,\ F=.25$ POINCARE MAP CORRESPONDING TO TRANSIENT CHAOS.

$x + \delta x - x + x' : 0$ phase plane

PLOT 1

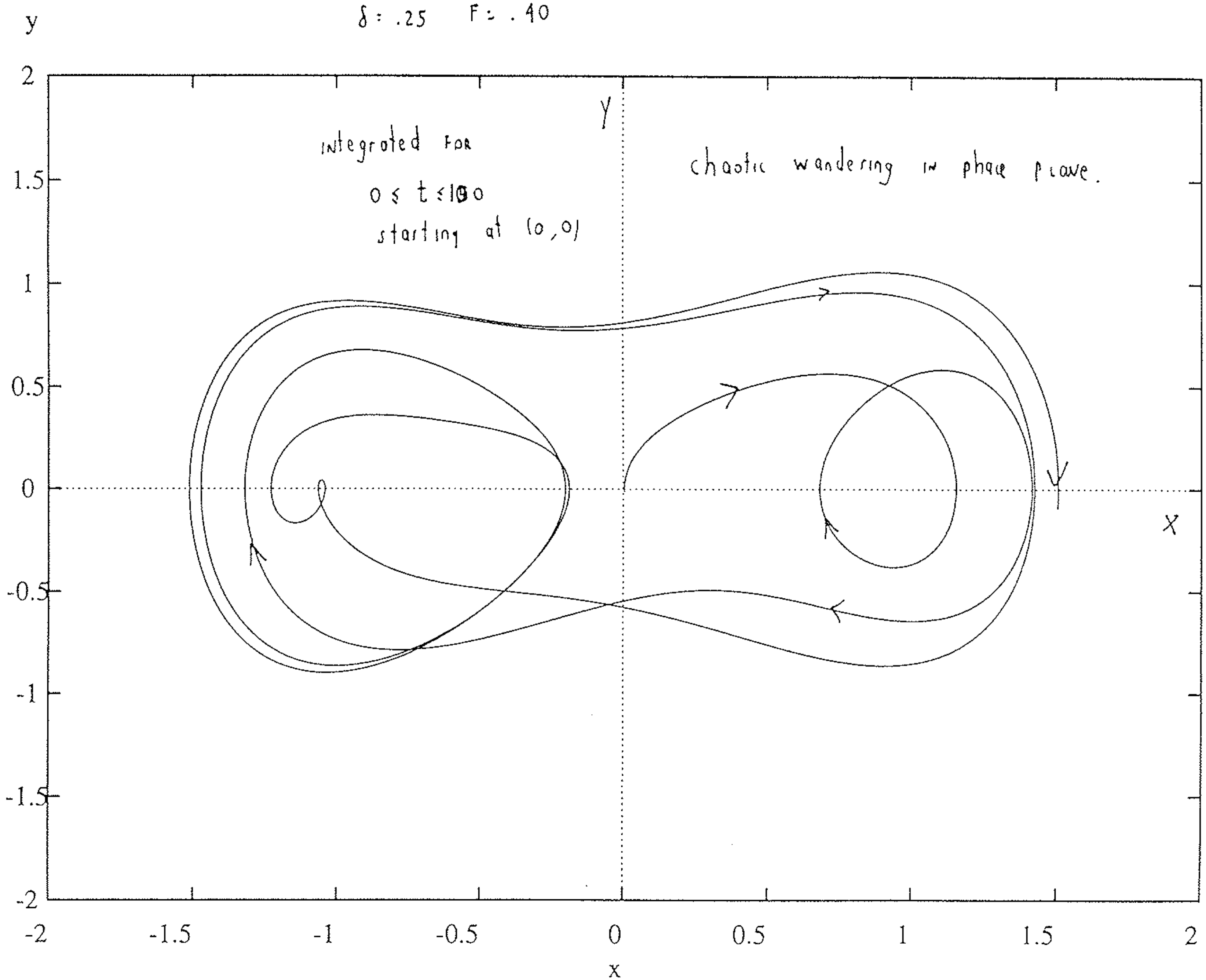
$\delta = .25$



PLOT 2a

$\ddot{x} + \delta\dot{x} + -x + x^3 = F\cos t$ "Phase" plane

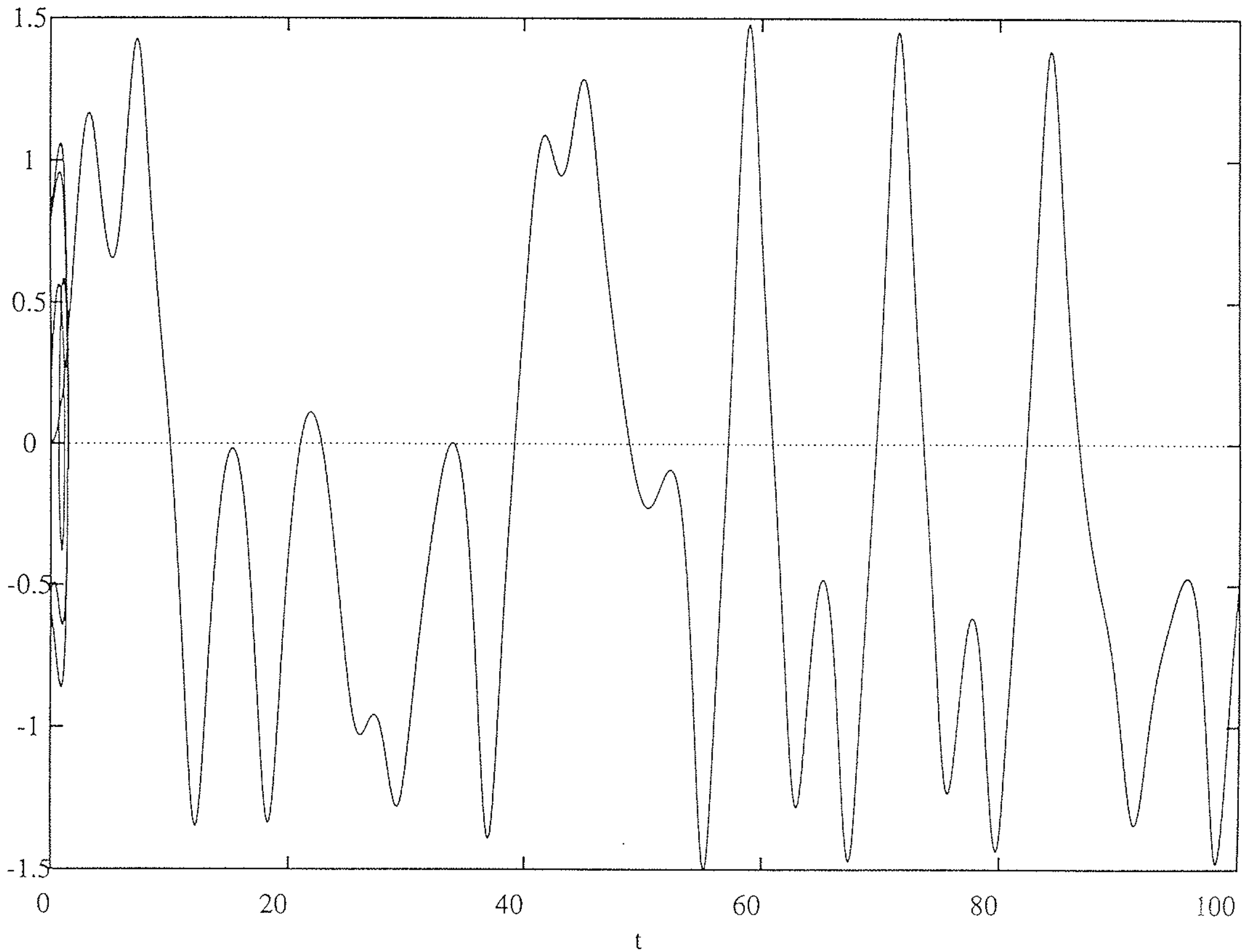
$\delta = .25 \quad F = .40$



Plot 2b

$$\ddot{x} + \delta \dot{x} - x + x^3 = F \cos \omega t$$

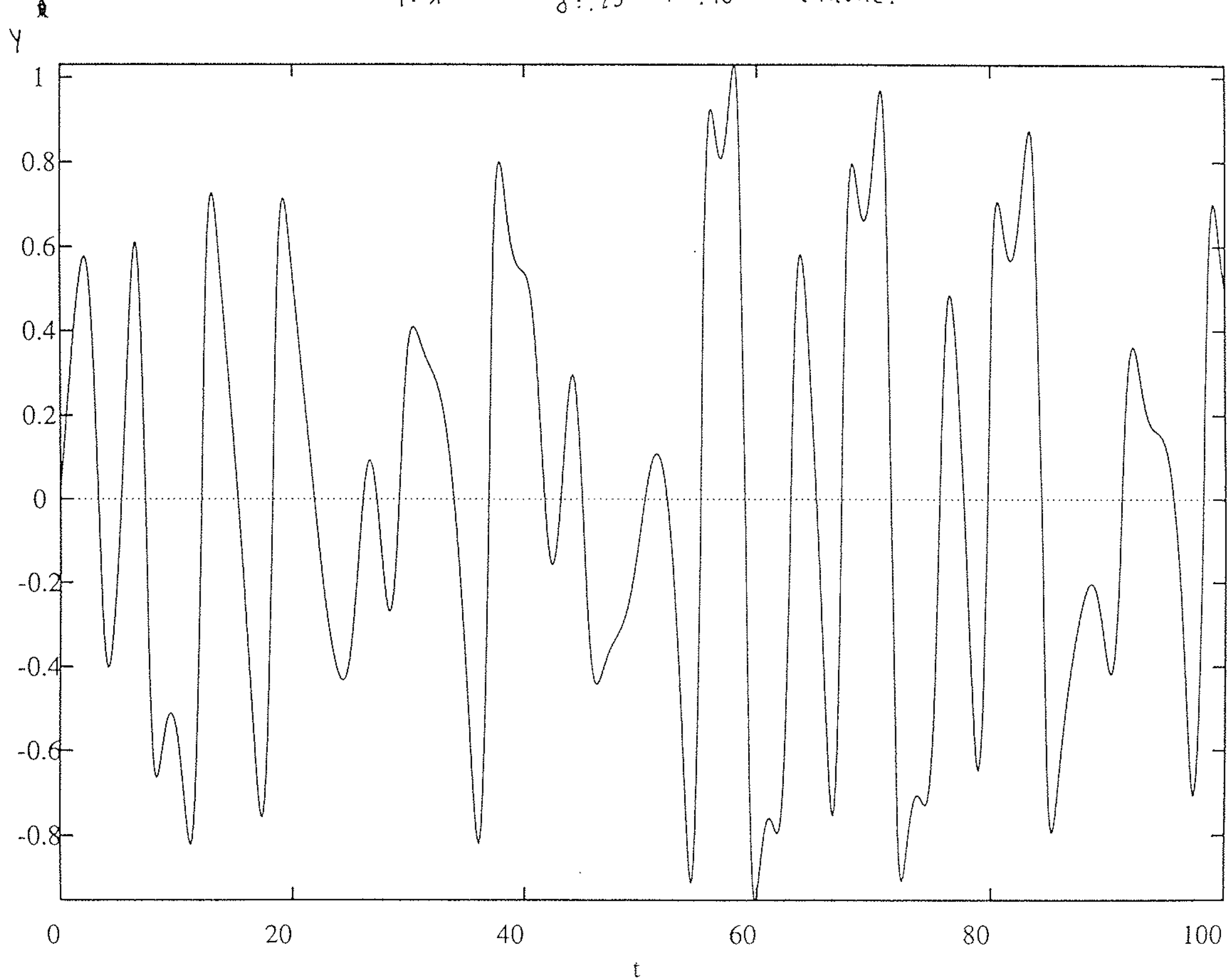
$\delta = .25$ $F = .40$ No apparent pattern in time.



PLOT 2C

Y vs t FOR $X + \delta X - X + X^3 = F \cos t$

$Y = \dot{X}$ $\delta = .25$ $F = .40$ chaotic.



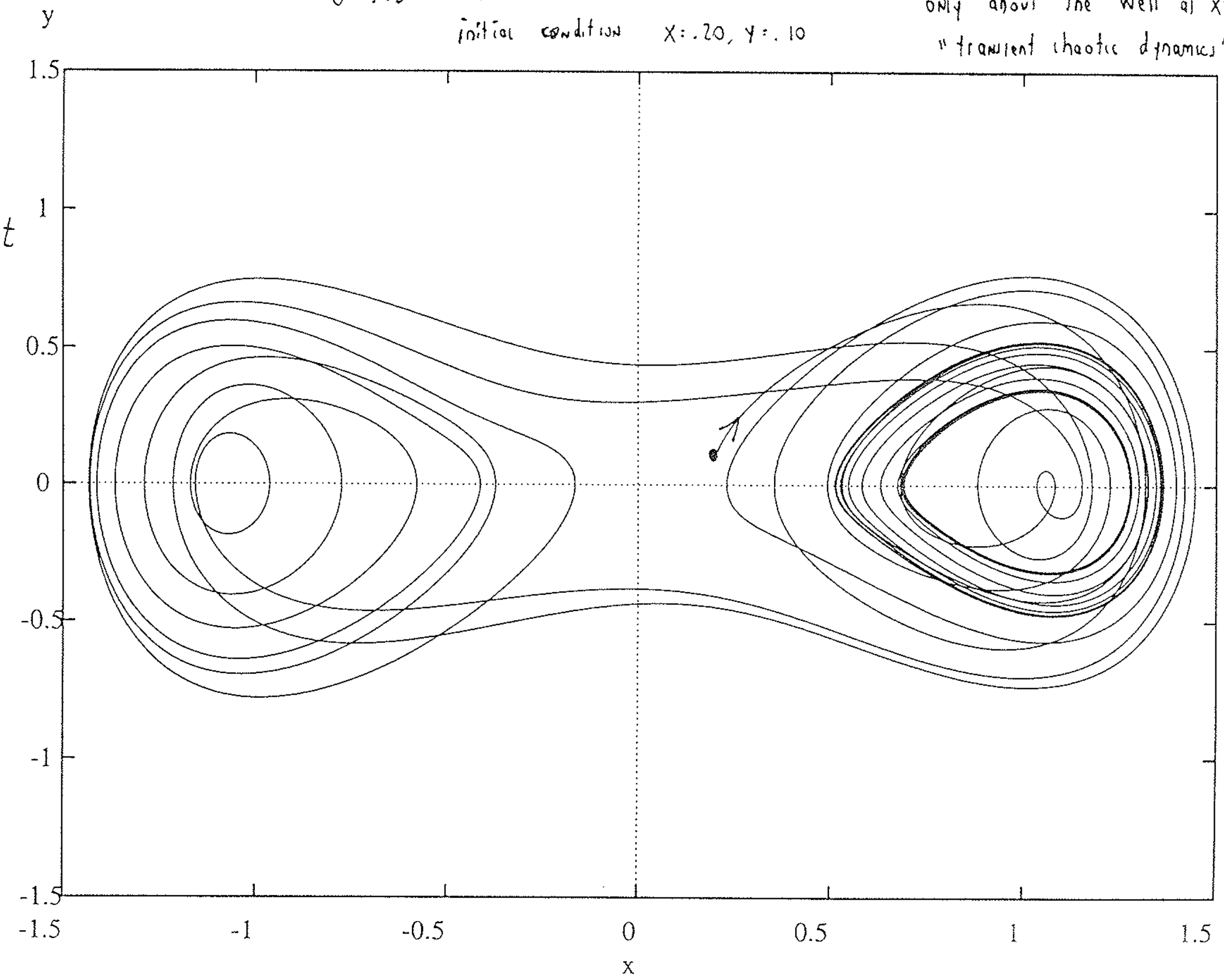
PLOT 3a

$\delta = .25 \quad F = .25$

initial condition $x = .20,\ y = .10$

eventually particle oscillates only about the well at $x = 1$ "transient chaotic dynamics"

$'' + \delta x' - x + x^3 = F \cos t$



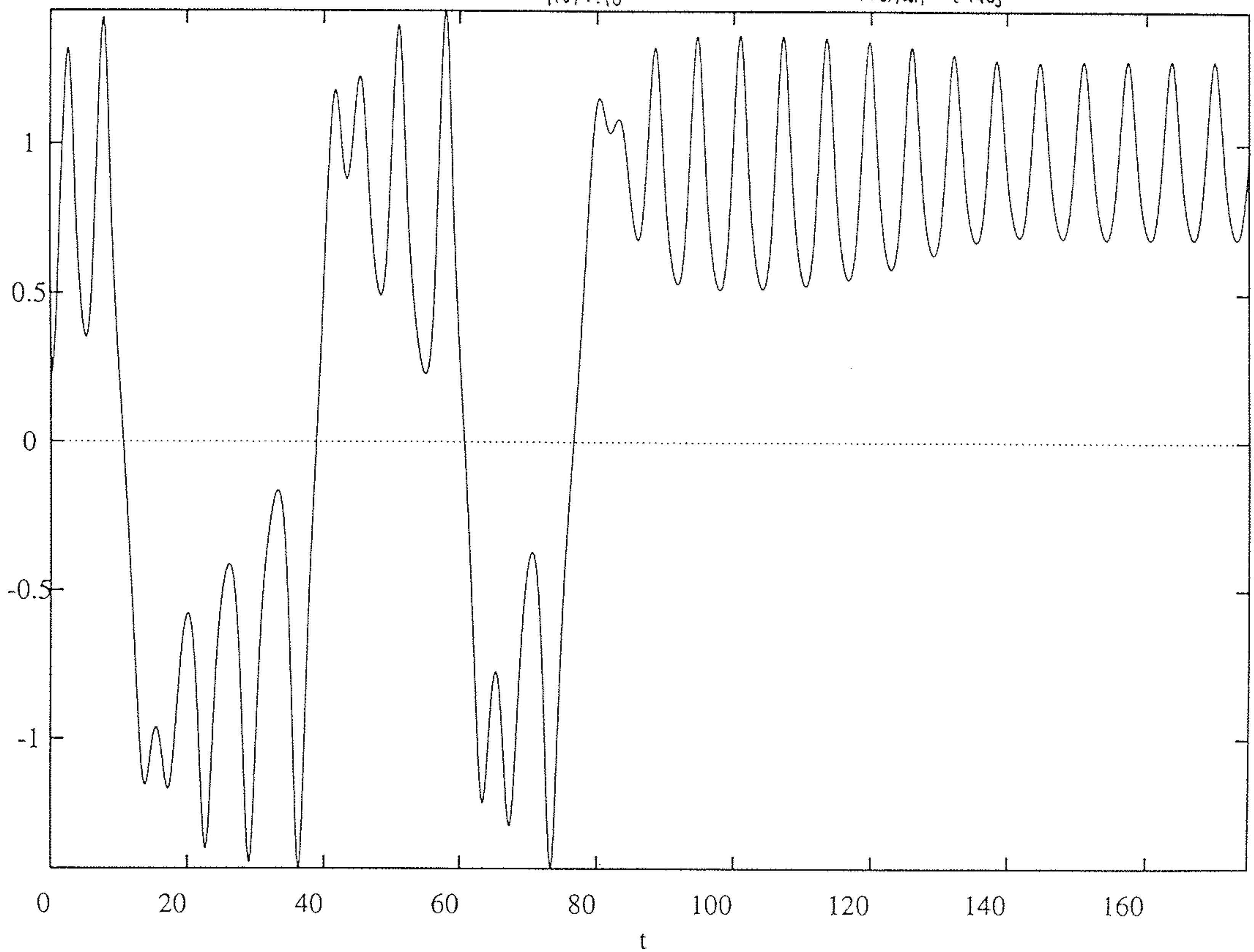
PLOT 3b

$\ddot{x} + \delta\dot{x} - x + x^3 = F\cos t$

$\delta = .25$ $F = .25$ $x(0) = .20$ $y(0) = .10$

eventually particle oscillates only about $x = 1$

transient chaos



PLOT 3C

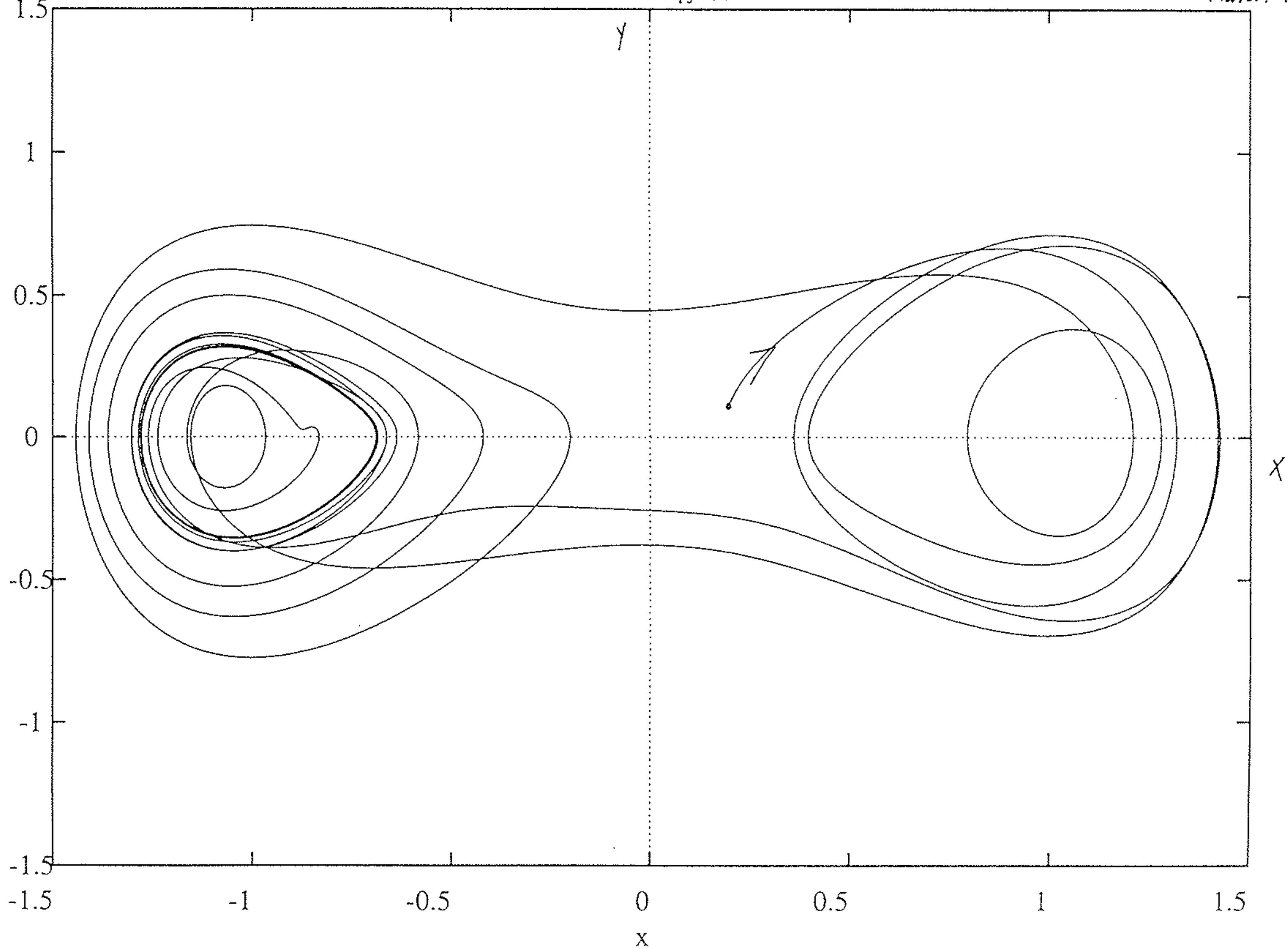
$\ddot{x} + \delta\dot{x} - x + x^3 = F\cos t$

$\delta = .25$ $F = .25$

initial condition $x_0 = .195$, $y_0 = .10$

eventually particle oscillates only about the left well at $x = -1$

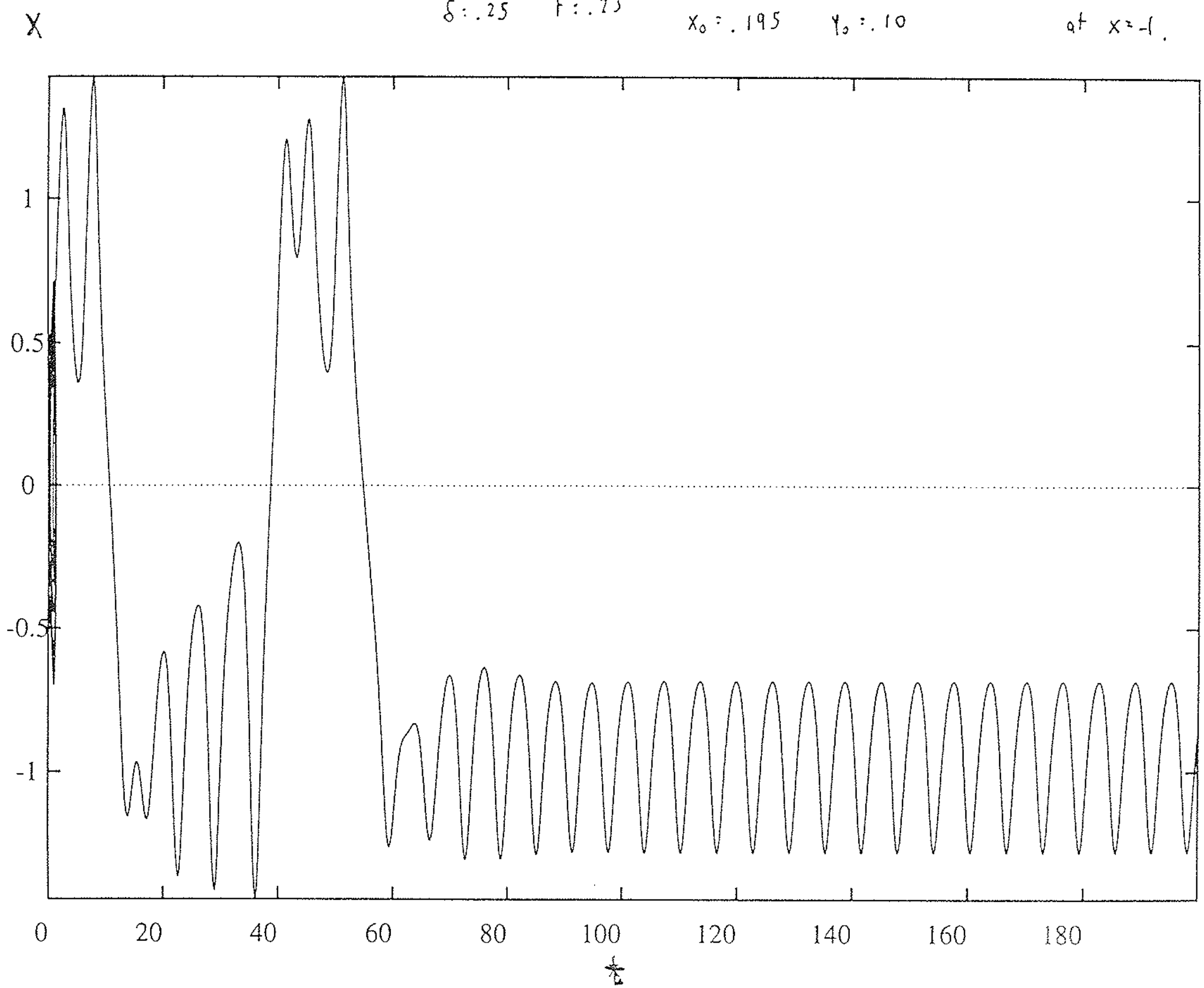
transient chaos



$\ddot{x} + \delta\dot{x} - x + x' = F\cos\omega t$

$\delta = .25 \quad F = .25 \quad x_0 = .195 \quad y_0 = .10$

about the well
at $x = -1$.



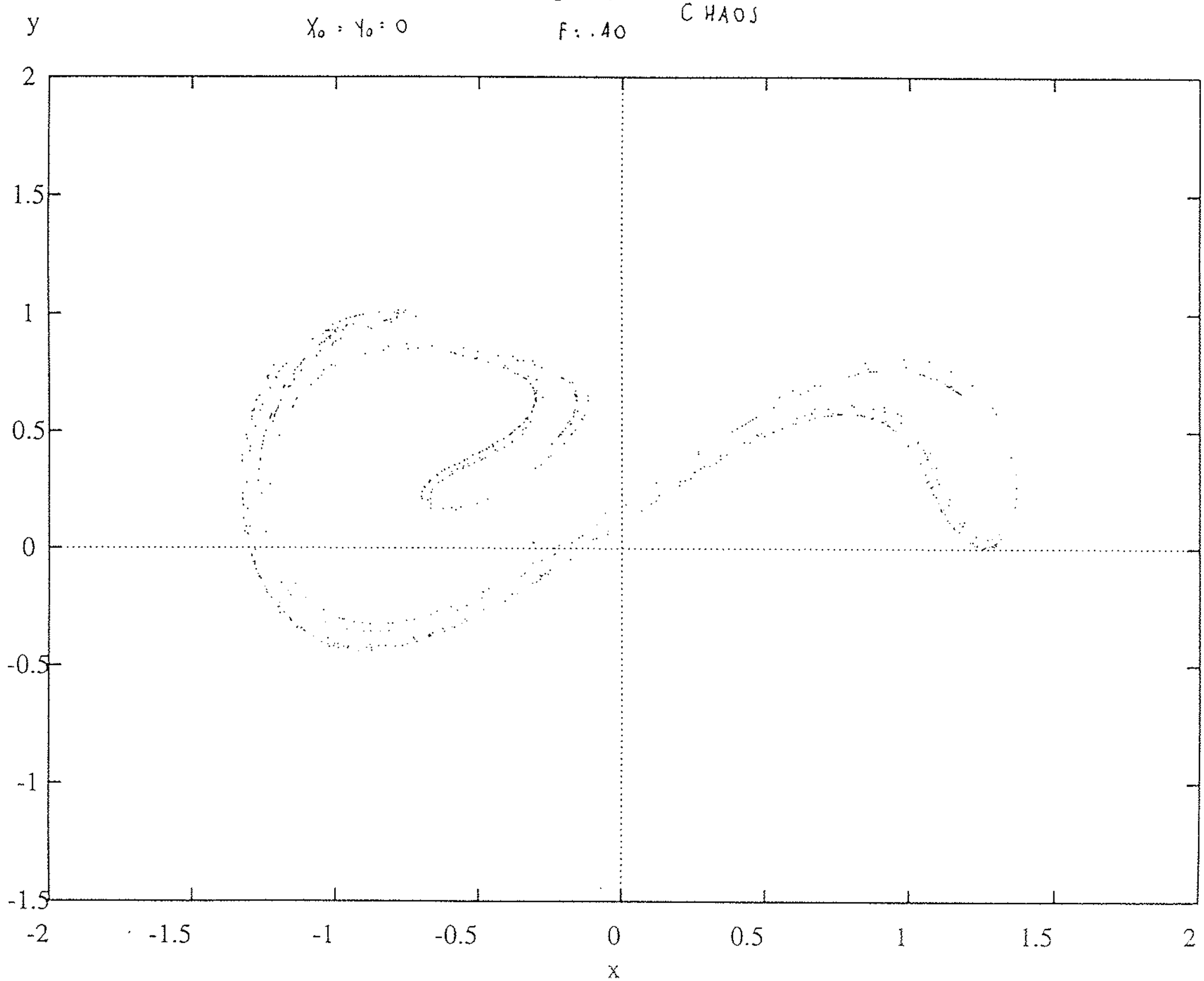
PLOT 4

DOUBLE-WELL POTENTIAL

POINCARE SECTION $\delta = .25$

$X_0 = Y_0 = 0$ $F = .40$

CHAOS

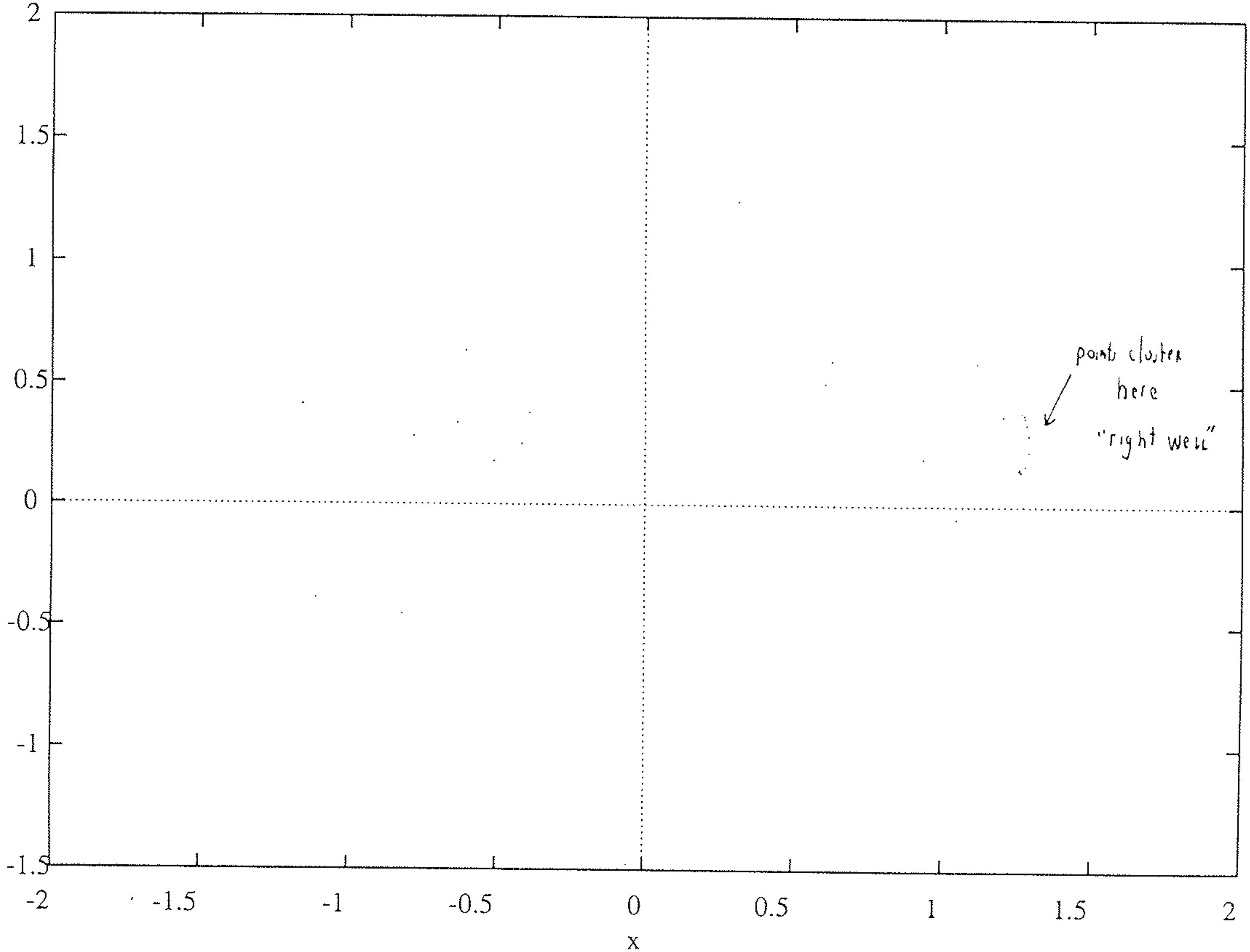


PLOT 5

DOUBLE WELL POTENTIAL

POINCARE SECTION

X_0 : .20 Y_0 : .10 δ : .25 F : .20



STABILITY THEORY (MAPS)

SUPPOSE $\underline{x}' = f(\underline{x})$ $\quad x \in \mathbb{R}^N$. SUPPOSE $\exists$ A PERIODIC ORBIT Γ.

CONSIDER A LOCAL CROSS SECTION $\Sigma \subset \mathbb{R}^N$ OF DIMENSION $N-1$ THAT INTERSECTS Γ IN A SINGLE POINT Z.

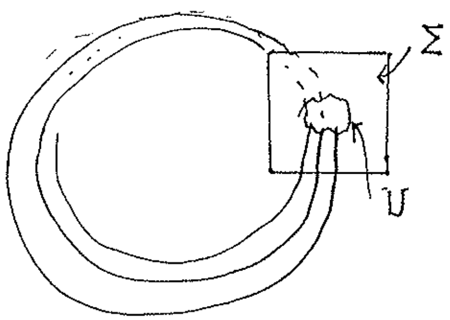
NOW FOR SOME SET $U \subset \Sigma$ WITH $z \in U$ WE DEFINE $P : U \rightarrow \Sigma$ BY

$$P(\underline{x}) = \varphi(\underline{x}, \tau(\underline{x})) \quad \text{WITH} \quad \phi(\underline{x}, \tau(\underline{x})) \in \Sigma \quad \text{AND} \quad \phi(\underline{x}, t) \notin \Sigma \quad \text{FOR}$$

$0 < t < \tau(\underline{x})$. IF THE PERIOD OF Γ IS T THEN $\tau(\underline{z}) = T$ AND THE STABILITY OF THE PERIODIC SOLUTION IS THE STABILITY OF THE POINT Z ON THE MAP.

$\tau(x)$ is the return ti

P IS CALLED A POINCARE RETURN MAP.



$$\Longrightarrow \quad \underline{X}_{n+1} = P(\underline{X}_n)$$

$$\underline{Z} = P(\underline{Z})$$

EXAMPLE 1 CONSIDER

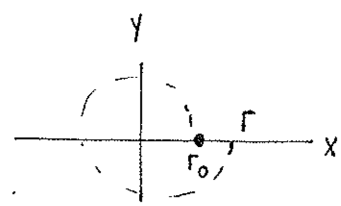
$$x' = x - y - x(x^2 + y^2)$$
$$y' = x + y - y(x^2 + y^2)$$

IN POLAR COORDINATES

$$r' = r(1 - r^2) \qquad \varphi' = 0$$

CHOOSE

$$\Sigma = \{ (r, \varphi) \mid r > 0, \ \varphi = 0 \}$$



SUPPOSE NOW we start at $r = r_0$, $\varphi = 0$. THEN we get $\tau(x) = 2\pi$ FOR ALL $x > 0$. IT TAKES 2π UNITS TO RETURN TO Σ.

THEN,

$$\int_{r_0}^{r} \frac{dr}{r(1 - r^2)} = \int_0^{2\pi} dt$$

NOW IF you integrate this one gets $\quad r = \left[1 + e^{-4\pi} (r_0^{-2} - 1) \right]^{-1/2}$

THUS $\quad r = P(r) \qquad$ NOTICE $\quad P(1) = 1 \longrightarrow r = 1$ IS A FIXED POINT O

NOW DEFINE r_n TO BE THE TIME OF n^{th} return TO Σ.

THEN $$r_{n+1} = P(r_n) \qquad P(r) = \left[1 + e^{-4\pi}(r_0^{-2} - 1)\right]^{-1/2}$$

NOW TO show that $r=1$ is stable we linearize:

$$r_n = 1 + V_n \qquad P(1+V_n) = P(1) + P'(1)\,V_n + \dots$$

THEN $$V_{n+1} = P'(1)\,V_n$$

$$P'(1) = -\frac{1}{2}[\quad]^{-3/2}\left(e^{-4\pi}\right)\left(-2r^{-3}\right) = -\frac{1}{2}(-2)\,e^{-4\pi} = e^{-4\pi}$$

AT $r=1$.

THUS SINCE $|P'(1)| < 1$ we have stability.

<u>REMARK</u> suppose that $\underline{X}_{n+1} = B\,\underline{X}_n$ B CONSTANT MATRIX.

LET $B = S\Lambda S^{-1}$.

$$\underline{X}_{n+1} = S\Lambda S^{-1}\,\underline{X}_n$$

$$S^{-1}\underline{X}_{n+1} = \Lambda S^{-1}\,\underline{X}_n$$

DEFINE $\underline{\xi}_n = S^{-1}\underline{X}_n \longrightarrow \underline{\xi}_{n+1} = \Lambda\,\underline{\xi}_n \qquad \underline{\xi}_n = \Lambda^n\,\underline{\xi}_0$

$\longrightarrow \underline{\xi}_n \longrightarrow 0$ AS $n \to \infty$ IF eigenvalues $\rho_1, \dots, \rho_N$ OF Λ satisfy $|\rho_j| < 1$.