SYSTEMS OF EQUATIONS (AUTONOMOUS)

$$dx/dt = f(x,y)$$

$$dy/dt = g(x,y)$$

Let (x_0, y_0) be an eq. point where $f(x_0,y_0) = g(x_0,y_0) = 0$.

let $x = x_0 + \tilde{x}$, $y = y_0 + \tilde{y}$ $\quad \tilde{x} \ll 1$, $\tilde{y} \ll 1$

$$\begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = J \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \qquad J = \begin{pmatrix} f_x^0 & f_y^0 \\ g_x^0 & g_y^0 \end{pmatrix} \qquad \hat{x} = e^{\lambda t} \underline{v} \rightarrow (J - \lambda I)\underline{v} = 0.$$

OUTLINE
(i) classify the stability and type of eq. point
(ii) does classification persist under small nonlinear terms?
(iii) draw a local phase portrait.

CONSIDER $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det(J - \lambda I) = 0 \qquad (a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

LET $p = a + d = \operatorname{trace} J$
$q = ad - bc = \det J$

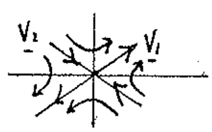
$$\lambda_\pm = \frac{p \pm \sqrt{p^2 - 4q}}{2}$$

trace $J = \lambda_+ + \lambda_-$
det $J = \lambda_+ \lambda_-$

IF λ's complex $\lambda_+ = \overline{\lambda_-}$
det $J = |\lambda_+|^2 > 0$
tr $J = 2\,\mathrm{RE}(\lambda_+)$

(i) IF $\lambda_\pm$ real $\lambda_+ > 0$, $\lambda_- < 0$ $\longrightarrow$ saddle point

$$\tilde{x} = c_1 \underline{v}_1 e^{\lambda_+ t} + c_2 \underline{v}_2 e^{\lambda_- t} \longrightarrow c_1 \underline{v}_1 e^{\lambda_+ t} \text{ as } t \to \infty.$$

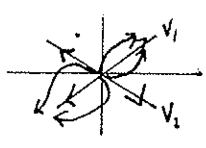


NEED $q < 0$ $\quad$ det $J < 0$ only

(ii) $\lambda_+ > 0$, $\lambda_- > 0$ $\quad \lambda_+ > \lambda_- > 0$ (λ's real) unstable node

$$\hat{x} = c_1 \underline{v}_1 e^{\lambda_+ t} + c_2 \underline{v}_2 e^{\lambda_- t}$$

$$\hat{x} \to c_1 v_1 e^{\lambda_+ t} \text{ as } t \to \infty$$

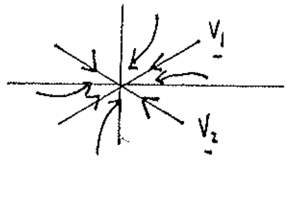


$p > 0$, $p^2 - 4q > 0$
$q > 0$.

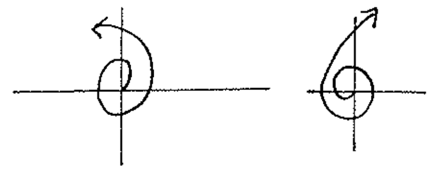
(iii) $\lambda_+ < 0$, $\lambda_- < 0$ $\quad \lambda_- < \lambda_+ < 0$ (λ's real) stable node

$$\hat{x} = c_1 \underline{v}_1 e^{\lambda_+ t} + c_2 \underline{v}_2 e^{\lambda_- t} \longrightarrow c_1 \underline{v}_1 e^{\lambda_+ t} \text{ as } t \to \infty$$

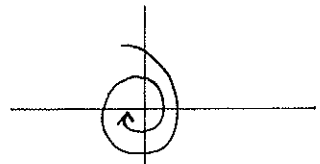
need $p^2 - 4q > 0$, $p < 0$, $q > 0$.



(iv) λ_+, λ_- complex $RE(\lambda_+) > 0$. $\longrightarrow$ $\det J > 0$, $p > 0$. $p^2 - 4q < 0$.
unstable spiral $q > 0$, $p > 0$, $p^2 - 4q < 0$.



(v) λ_+, λ_- complex $RE(\lambda_+) < 0$ $\longrightarrow$ $q > 0$, $p < 0$, $p^2 - 4q < 0$
stable spiral



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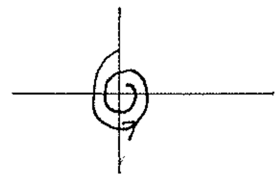
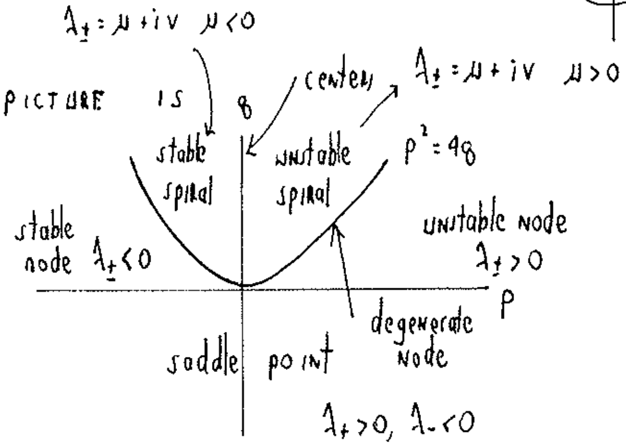


PICTURE IS



$q = \det J$
$p = tr\, J$

SPECIAL CASES

(vi) $p = 0$ $tr\, J = 0$, $\det J > 0$ closed orbits
$\longrightarrow$ $\lambda_\pm = \pm i\mu$ center

(vii) degenerate node $p^2 = 4q$.

NOTE: if $\det J < 0$ unstable
if $\det J > 0$ and $trace\, J > 0$ $\longrightarrow$ unstable
if $\det J > 0$ AND $trace\, J < 0$ $\longrightarrow$ stable

THEOREM THE CLASSIFICATION OF AN EQUILIBRIUM POINT FOR NONLINEAR SYSTEM IS EXACTLY THE SAME AS FOR THE LINEARIZED PROBLEM EXCEPT FOR borderline case of a center or a degenerate node, or when an eigenvalue is zero.

(i) center for linearized problem may become a stable or an unstable spiral depending on small nonlinear terms, or it may remain a center.

(ii) degenerate node may become a spiral or a node.

THEOREM consider a conservative system $x'' = f(x)$. A center for the linearized problem remain a center for nonlinear problem.
$\longrightarrow$ closed orbits occur near a center.

REMARK HYPERBOLIC FIXED POINT: $RE(\lambda) \neq 0$ FOR BOTH eigenvalues. No repeated eigenvalues. LOCAL PHASE PORTRAIT near a hyperbolic fixed point is similar to linearization.

EXAMPLES OF THEORY

EXAMPLE 1 (ROTATION OF A SPIRAL)

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

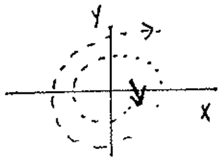
$\det(J-\lambda I)=0 \to (2-\lambda)^2+9=0$

$\lambda = 2 \pm 3i$

$\det J = 13$

$\text{trace } J > 0.$

NOW



let $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ $\quad x' = 2x_0 > 0$, $\quad y' = -3x_0 < 0$

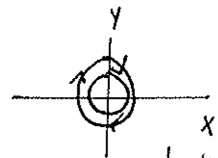
clockwise rotation.

$$\underline{x} = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \bar{\underline{v}}_1 e^{\bar{\lambda}_1 t}$$

EXAMPLE 2 (A center)

$$\alpha > 0 \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues $\lambda^2+\alpha^2=0$ so $\lambda = \pm i\alpha$.



NOW $\quad dx/dt = \alpha y \quad dy/dt = -\alpha x$

so $\quad \dfrac{dy}{dx} = -\dfrac{x}{y} \quad$ so $\quad y\,dy = -x\,dx \to y^2 + x^2 = E.$

NOTE IF $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ then $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha x_0 \end{pmatrix} \to y' < 0$ when $\alpha > 0$ and $x_0 > 0$.

EXAMPLE 3 (~~stable spiral~~ saddle)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -8 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\det(J-\lambda I)=0 \to (5-\lambda)(1-\lambda)-32=0$

$\lambda^2 - 6\lambda - 27 = (\lambda-9)(\lambda+3) = 0$

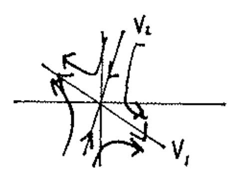
so $\lambda = -3, 9$ saddle point

$\det J = -27 < 0.$

$\lambda_1 = 9 \quad (J-9I)\underline{v} = \begin{pmatrix} -4 & -4 \\ -8 & -8 \end{pmatrix}\underline{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = -3 \quad (J+3I)\underline{v} = \begin{pmatrix} 8 & -4 \\ -8 & 4 \end{pmatrix}\underline{v} = \underline{0} \to \underline{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\underline{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{9t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-3t}$$

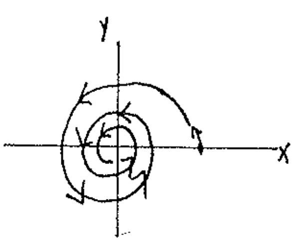


EXAMPLE (stable spiral)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\longrightarrow (-1-\lambda)^2 + 2 = 0$ so $\lambda^2 - 2\lambda + 3 = 0 \quad \lambda = -1 \pm \sqrt{2}\, i$

IF $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x_0 \\ 2x_0 \end{pmatrix}$ $\quad x' < 0$, $\quad y' > 0$



EXAMPLE 4 (CONSERVATIVE SYSTEM)

$$\ddot{X} + V'(X) = 0$$

let $\dot{x} = y \qquad \dot{y} = -V'(x)$

$$x' = y = f(x,y)$$
$$y' = -V'(x) = g(x,y)$$

let $y=0,\ x_0 = x_0$ be an equilibrium point. THEN linearize:

$$x = x_0 + \tilde{x}, \quad y = y_0 + \tilde{y} \longrightarrow \begin{pmatrix} \hat{x}' \\ \hat{y}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -V''(x_0) & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

eigenvalues satisfy $\lambda^2 + V''(x_0) = 0$

IF $V''(x_0) < 0 \rightarrow$ MAXIMUM $\longrightarrow \lambda_\pm = \pm\sqrt{-V''(x_0)} \longrightarrow$ saddle point

IF $V''(x_0) < 0 \rightarrow$ MINIMA $\longrightarrow \lambda_\pm = \pm i\sqrt{-V''(x_0)}$ center.

Linearized theory predicts a center (closed orbits near $(x_0,0)$). Is there really closed orbits for nonlinear problem?

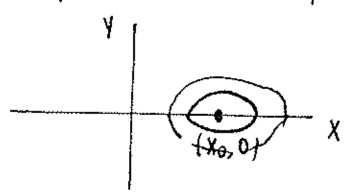
FIND conserved quantity.

$$y'x' = y^2 y' \qquad \text{subtract} \qquad y^2 y' + x V'(x) = 0$$
$$x'y' = -V'(x)x$$

THUS $E = \frac{1}{2}y^2 + V(x)$ CLAIM $\nabla E \cdot \underline{x}' = 0$ MOTION IS ON level curves OF E. $\nabla E = (V'(x), y)$ $\qquad \nabla E \cdot \underline{x}' = y V'(x) - y V'(x) = 0.$

SINCE V HAS A MINIMA NEAR $x = x_0$ E is locally convex.

$\longrightarrow$ periodic orbits persist FOR nonlinear problem.



EXAMPLE 5 (NONLINEAR TERMS MAY CHANGE A CENTER)

$$x' = y + Bx(x^2 + y^2) \qquad B \text{ constant.}$$
$$y' = -x + By(x^2 + y^2)$$

Linearized problem near $(0,0)$ $\quad \begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \tilde{y} \end{pmatrix}$

THUS Linearized theory predicts a center. Do the nonlinear terms perturb the center to become a spiral? OR do closed orbits remain near $(0,0)$ despite the nonlinear terms.

For spiral problems it is often convenient to change variables

$$x = r\cos\varphi \qquad y = r\sin\varphi .$$

THEN $\quad r^2 = x^2 + y^2 . \quad \varphi = \tan^{-1}(y/x)$

$$r r' = x x' + y y' \qquad \varphi' = \frac{\dfrac{y'}{x} - \dfrac{y x'}{x^2}}{1 + y^2/x^2} = \frac{x y' - x' y}{x^2 + y^2} .$$

THUS $\quad r' = x\dfrac{x'}{r} + \dfrac{y y'}{r} \qquad \varphi' = \dfrac{x y' - x' y}{x^2 + y^2} .$

NOW let $\quad x' = y + B x r^2$

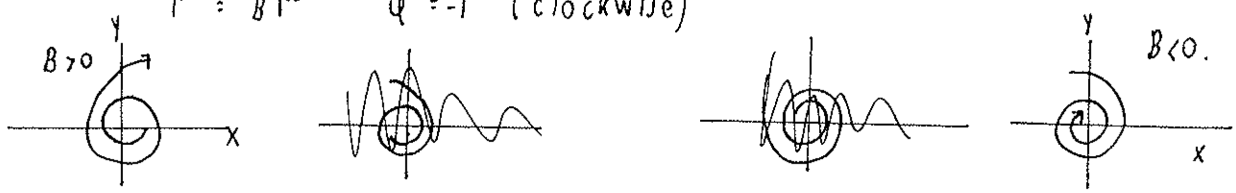
$\qquad\qquad y' = -x + B y r^2$

THEN we get $\quad r' = \dfrac{1}{r}\left[xy + Bx^2 r^2 - xy + By^2 r^2 \right] = B r^3$

$$\varphi' = \frac{1}{r^2}\left[-x^2 + Bxyr^2 - y^2 - Bxyr^2 \right] = -1$$

THE system transforms exactly to

$$r' = B r^3 \qquad \varphi' = -1 \quad \text{(clockwise)}$$



THUS A center gets perturbed to stable spiral IF $B < 0$ and an unstable spiral IF $B > 0$.

<u>EXAMPLE 6</u> (HAMILTONIAN SYSTEM)

SUPPOSE $\exists$ A FUNCTION $H(x,y)$ SUCH THAT

$$x' = -H_y \qquad \underset{\sim}{F} = \begin{pmatrix} -H_y \\ H_x \end{pmatrix} \qquad \underset{\sim}{x}' = \underset{\sim}{F}$$

$$y' = H_x$$

THEN A CONSTANT OF THE MOTION IS $H(x,y) = H_0$. NOW LET

$\dfrac{dH}{dt} = H_x x' + H_y y' = 0$ FROM THE ODE. SO MOTION IS ON level SURFACE $H(x,y) = \text{CONSTANT}$. ALSO NOTE $\nabla \cdot \underset{\sim}{F} = 0$ WHICH MEANS MEASURING PRESERVING MAPS. NOW IF H HAS A MINIMA AT $\underset{\sim}{x}_0$ $\quad H^0_{xx} > 0, \; H^0_{yy} > 0, \; H^0_{xx} H^0_{yy} - (H^0_{xy}) > 0$

THEN $\quad J = \begin{pmatrix} -H_{xy} & -H_{yy} \\ H_{xx} & H_{xy} \end{pmatrix} \qquad \text{trace } J = 0 \quad \det J = H_{xx}H_{yy} - H_{xy}^2 > 0$

$$\lambda^2 + H_{xx}H_{yy} - H_{xy}^2 = 0$$

THUS IF H HAS A MINIMA AT $\underset{\sim}{x}_0 \longrightarrow \underset{\sim}{x}_0$ IS A CENTER. $\qquad$ IF H HAS A SADDLE THEN SO DOES $\underset{\sim}{x}_0$.

IF H HAS A MAXIMUM AT $\underset{\sim}{x}_0 \longrightarrow \underset{\sim}{x}_0$ IS A CENTER STILL.

EXAMPLE 7 (SMALL NONLINEAR TERMS AGAIN)

$$x'' + f(x)\,x' + V'(x) = 0$$

$f(x)$ is a nonlinear damping type term. Suppose $f(0) = 0$ but that $f(x) \approx x^2$ as $x \to 0$. $\longrightarrow$ expect instability.

LET $E = \frac{x'^2}{2} + V(x)$

THEN $\frac{dE}{dt} = x'x'' + V'(x)\,x' = -f(x)\,x'^2$

NEAR A local minima OF $V(x)$, $E \geq 0$ BUT $dE/dt \leq 0$ IF $f(x) > 0$

$dE/dt \geq 0$ IF $f(x) < 0$

NOW IF $f(x) > 0$ FOR $x > x_0$, $f(x) < 0$ FOR $x < 0$, this suggests that

$$\frac{dE}{dt} = \begin{cases} < 0 & x > x_0 \\ > 0 & x < x_0 \end{cases}$$

possibly $E \to E_0$ as $t \to \infty$ corresponding to a periodic solution.

EXAMPLE 8 (DAMPING AND CONSERVATIVE SYSTEMS)

CONSIDER $x'' + bx' - x + x^3 = 0$ $(b \geq 0)$ IF $b = 0$ $x'' + V'(x) = 0$ $V = \frac{1}{4}(1 - x^2)^2$

MINIMA AT $x = \pm 1$.

NOW WITH DAMPING: $x' = y$

$y' = -by + x - x^3$

NEAR $(0,0)$ $J = \begin{pmatrix} 0 & 1 \\ 1 & -b \end{pmatrix}$ $-\lambda(-b-\lambda) - 1 = 0$

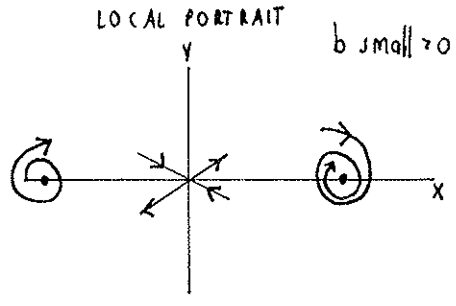
$\lambda^2 + \lambda b - 1 = 0$

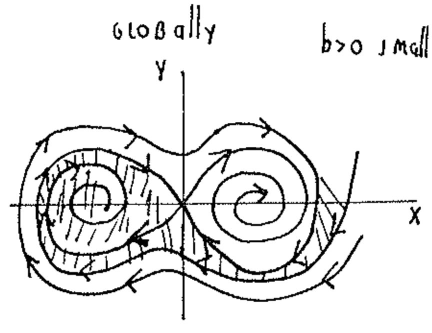
$\lambda_\pm = \frac{-b \pm \sqrt{b+4}}{2}$

THUS IF $b > 0$ get a saddle point at ORIGIN $(0,0)$

AT $(\pm 1, 0)$ $J = \begin{pmatrix} 0 & 1 \\ -2 & -b \end{pmatrix}$ $\lambda(\lambda + b) + 2 = 0$ $\longrightarrow$ $\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 8}}{2}$

THUS IF $0 \leq b < \sqrt{8}$ $\longrightarrow$ stable spirals.





Anything in shaded region goes to left well.

Basin of attraction is complicated

BASIN OF attraction bounded by stable manifold of the saddle.

GLOBAL PHASE PORTRAITS

EXAMPLE 1 $\quad x' = y$

$y' = x + 2x^3$

THE only equilibrium point is at $(0,0)$. Near $(0,0)$ $\quad \tilde{x}' = \tilde{y}$, $\tilde{y}' = \tilde{x}$

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \lambda^2 = 1$$

$\lambda = \pm 1$ saddle point.

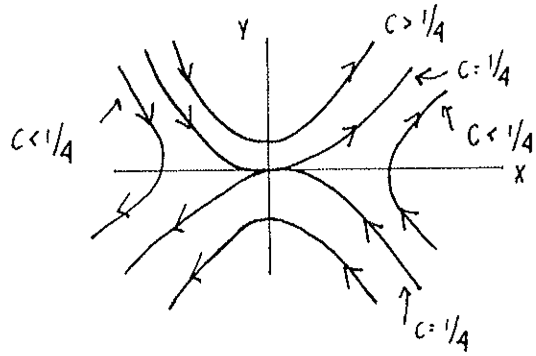
$\lambda_1 = 1 \rightarrow (J - I)\underline{v}_1 = \underline{0} \longrightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \underline{v}_1 = \underline{0} \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \rightarrow (J + I)\underline{v}_2 = \underline{0} \longrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \underline{v}_2 = \underline{0} \quad \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

} orthogonal since J symmetric.

NOW $\dfrac{dy}{dx} = \dfrac{x + 2x^3}{y}$ so $y^2 = x^2 + x^4/2 + C$ $\qquad z = x^2$

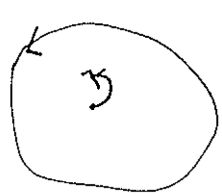
set $y = 0$ $\quad z + \dfrac{z^2}{2} + C = 0 \quad z = -1 \pm \sqrt{1-4C}$



IF $C > \frac{1}{4} \rightarrow$ NO values of x where $y = 0$

EXAMPLE 2 (GENERAL RESULT)

ASSUME THAT $\dfrac{dx}{dt} = f(x,y)$ $\dfrac{dy}{dt} = g(x,y)$ ARE sufficiently smooth. Then solution trajectories cannot cross. This implies trajectories do not cross at some finite value of t. Otherwise uniqueness would be violated. Trajectories can meet at a fixed point as $t \to \infty$ to reach such a point if it is stable. Thus consider any trajectory starting inside a closed curve C, where C is a trajectory. Then, it remains in C for all time



(i) could approach fixed point as $t \to \infty$

(ii) could approach periodic solution

(iii) wander to fill up the set? (NOT IN 2-D).

$\longrightarrow$ Ruled out by Poincare-Bendixson Theorem.

PREDATOR-PREY PROBLEMS

x = predator, y: prey
$\alpha, \beta, a, b, c, d > 0$

THERE ARE 3 TYPES OF MODELS

$x \geq 0, y \geq 0$

(1) P-PREY WITH OVERCROWDING

$$x' = (-\alpha - ax + by)x$$
$$y' = (\beta - cx - dy)y$$

$a=0, d=0 \rightarrow$ infinite carrying capacity.

(2) COMPETITION between two species

$$x' = (\alpha - ax - by)x$$
$$y' = (\beta - cx - dy)y$$

(3) CO-OPERATION between two species

$$x' = (\alpha - ax + by)x$$
$$y' = (\beta + cx - dy)y$$

<u>OUTLINE</u> EACH PROBLEM HAS THE FORM

$$x' = f(x,y)$$
$$y' = g(x,y)$$

(i) FIND EQUILIBRIA: THEY SATISFY $f(x,y)=0,\ g(x,y)=0$

(ii) let (x_0, y_0) BE AN EQUILIBRIA. LET

$$x = x_0 + \hat{x},\quad y = y_0 + \hat{y}.$$

LINEARIZING we get

$$\begin{pmatrix} \hat{x}' \\ \tilde{y}' \end{pmatrix} = \begin{pmatrix} f_x^0 & f_y^0 \\ g_x^0 & g_y^0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \qquad J = \begin{pmatrix} f_x^0 & f_y^0 \\ g_x^0 & g_y^0 \end{pmatrix}$$

(iii) FOR each eq. point calculate eigenvalues of J AND classify stability (saddle, spiral, center ...). DRAW local trajectories.

(iv) plot nullclines to get vector field. plot $f(x,y)=0,\ g(x,y)=0$ AND find regions where $x'>0,\ x'<0,\ y'>0,\ y'<0$.

(v) DRAW the entire phase-plane and interpret biologically.

EXAMPLE I (LOTTKA-VOLTERRA)

$$x' = (-4+y)x = f(x,y) \qquad x: \text{predator},\ y: \text{prey}$$

$$y' = (4-x)y = g(x,y) \qquad \text{infinite carrying capacity}$$

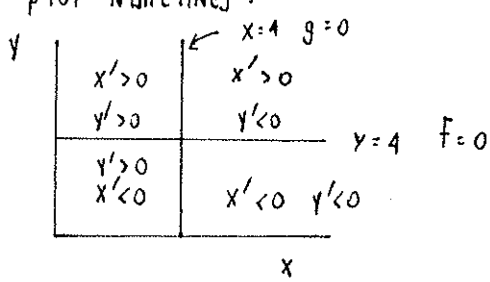
EQUILIBRIA AT $\quad x(y-4)=0,\quad y(4-x)=0 \longrightarrow (4,4),\ (0,0)$

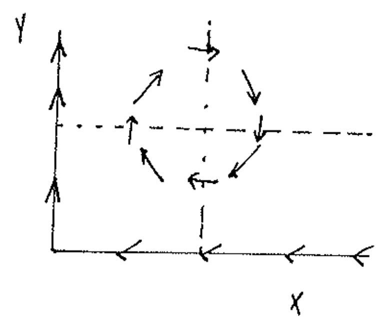
NOW $\quad J = \begin{pmatrix} f_x^0 & f_y^0 \\ g_x^0 & g_y^0 \end{pmatrix} = \begin{pmatrix} -4+y_0 & x_0 \\ -y_0 & 4-x_0 \end{pmatrix}$

AT $(0,0)$ $\quad J = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$ eigenvalues are $\lambda_1 = 4,\ \lambda_2 = -4 \longrightarrow$ saddle point

AT $(4,4)$ $\quad J = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}$ eigenvalues are $\lambda = \pm 4i$ center.

NOW we plot nullclines:





DO WE HAVE closed orbits? centers are not robust to small perturbations.
FIND a conserved integral.

$$\frac{dy}{dx} = \frac{(4-x)y}{(y-4)x} \quad \text{THUS} \quad \left(\frac{y-4}{y}\right)dy = \left(\frac{4-x}{x}\right)dx \longrightarrow y - 4\ln|y| = 4\ln|x| - x + E$$

SO THAT $\quad E = y + x - 4\ln(xy) \qquad$ E is a constant independent of t.

WE CAN WRITE THIS AS $\quad E = \ln\left[\dfrac{e^{x+y}}{x^4 y^4}\right]$

HENCE $\quad (x^4 e^{-x})(y^4 e^{-y}) = C \qquad$ FOR EACH $x > 0 \longrightarrow \exists$ two values of y

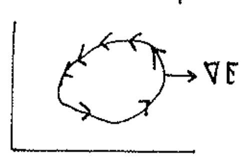
FOR EACH $y > 0 \longrightarrow \exists$ two values of x.

NOTICE $\max\limits_{x \geq 0} x^4 e^{-x} = 4^4 e^{-4}$ which occurs when $x = 4$.

HENCE $\quad x^4 y^4 e^{-(x+y)} = C \quad$ WITH $\quad C \leq 2 \cdot 4^4 (e^{-4})^2 \longrightarrow$ PERIOD SOLUTIO.

AND we have closed orbits in the phase plane.

NOW NOTICE $\quad \nabla E \cdot \underline{x}' = 0$



$\nabla E = (1 - 4/x,\ 1 - 4/y)$

$\nabla E \cdot \underline{x}' = \underline{0}$ CAN BE checked.

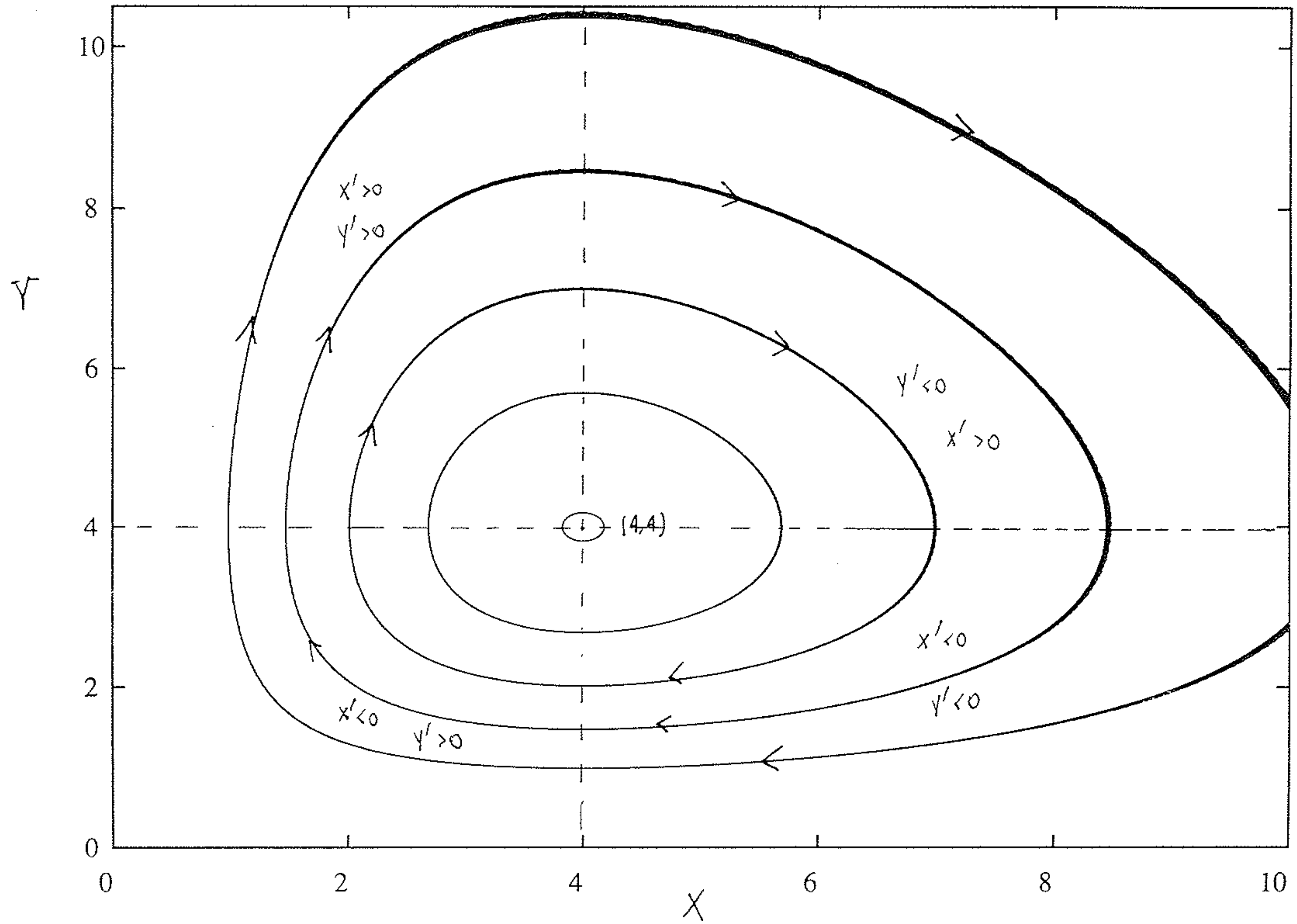
LOTTKA- VOLTERRA SYSTEM
(NO OVERCROWDING EFFECT)

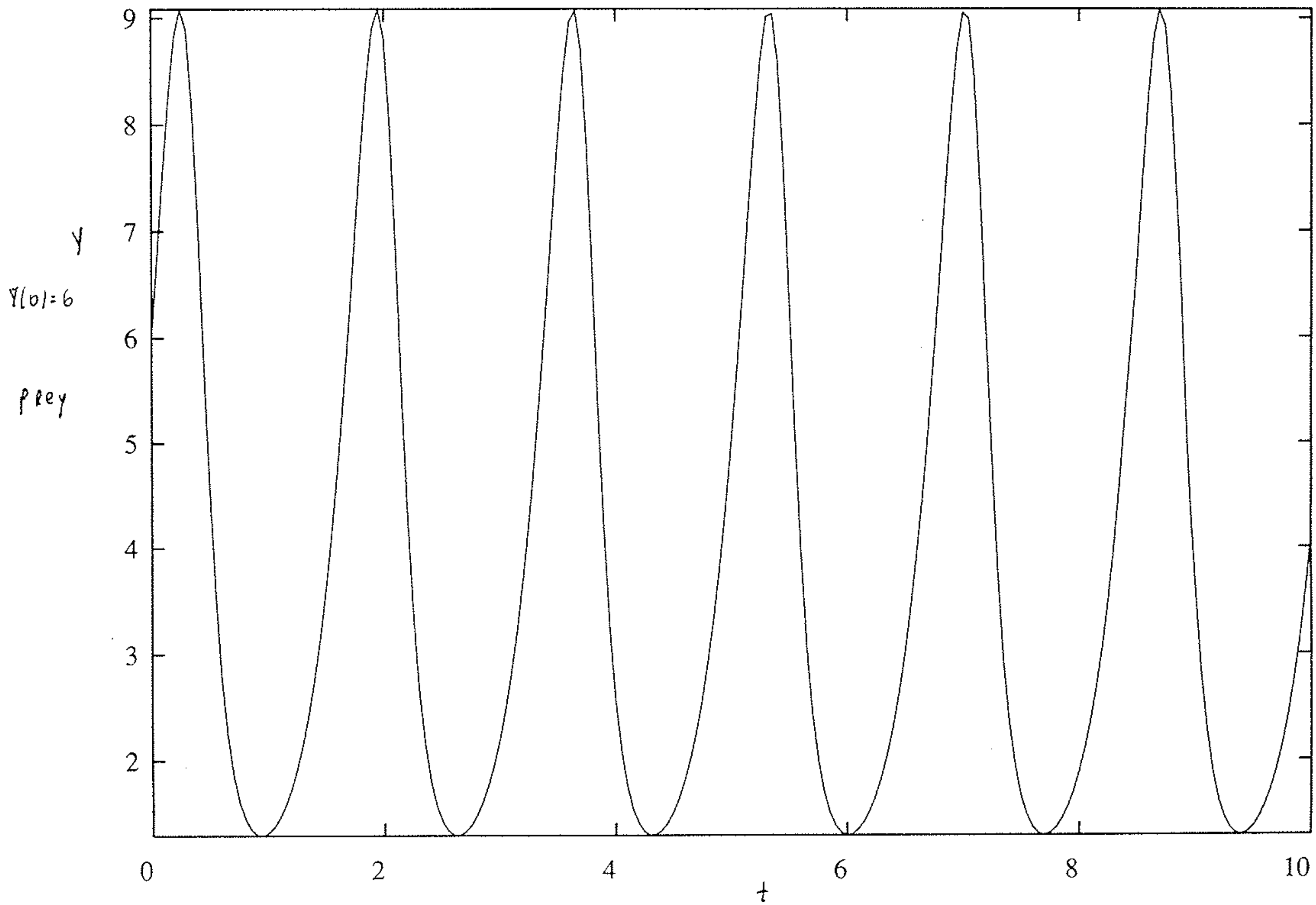
$x' = (-4 + y)x$

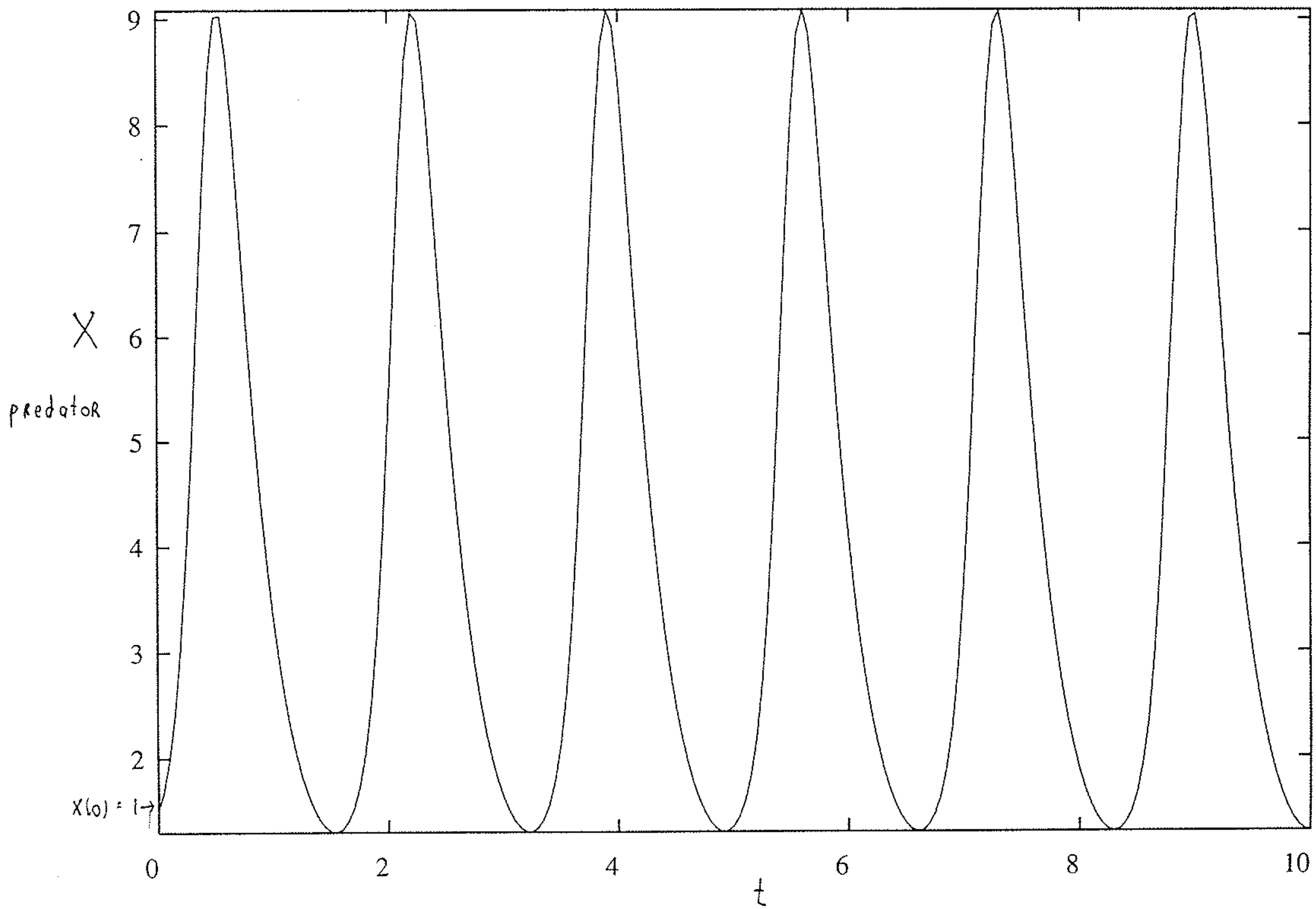
$y' = (4 - x)y$

PERIODIC SOLUTIONS

x: predator y: prey.




y
Y(0)=6
prey
9
8
7
6
5
4
3
2
0
2
4
6
8
10
t



X
predator
X(0) = 1→
9
8
7
6
5
4
3
2
0
2
4
6
8
10
t

REMARK

$$\frac{dx}{dt} = (-a + by)x \qquad x = \text{predator}$$

$$\frac{dy}{dt} = (c - dx)y \qquad y = \text{prey}$$

$$J = \begin{pmatrix} -a + by_0 & bx_0 \\ -dy_0 & c - dx_0 \end{pmatrix}$$

equilibria is $x_0 = c/d$ AND $y_0 = a/b$. $\rightarrow$ $J = \begin{pmatrix} 0 & bc/d \\ -ad/b & 0 \end{pmatrix}$

eigenvalues are $\lambda = \pm i (ac)^{1/2}$ frequency near (x_0, y_0) is $(ac)^{1/2}$.

EXAMPLE 2 (~~COMPETITION~~ COOPERATION model)

$$x' = (4 - 2x + y)x = f(x,y) \qquad (4 - 2x + y)x = 0$$

$$y' = (4 + x - 2y)y = g(x,y) \qquad \text{AND} \quad (4 + x - 2y)y = 0$$

} eq. equations

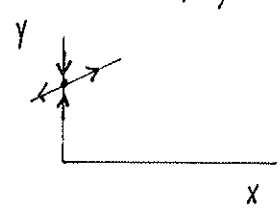
EQ. POINTS $(0,0)$, $(0,2)$, $(2,0)$, $(4,4)$

$$J = \begin{pmatrix} 4 - 4x_0 + y_0 & x_0 \\ y_0 & 4 + x_0 - 4y_0 \end{pmatrix}$$

NEAR $(0,2)$ $\quad J = \begin{pmatrix} 6 & 0 \\ 2 & -4 \end{pmatrix}$ eigenvalues $\lambda_1 = -4$, $\lambda_2 = 6$ saddle point

$$\lambda_1 = -4 \qquad (J - \lambda_1 I)\,\underline{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & 0 \\ 20 & 0 \end{pmatrix} \underline{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

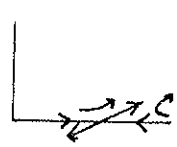
$$\lambda_2 = 6 \qquad (J - \lambda_2 I)\,\underline{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ 20 & -10 \end{pmatrix} \underline{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \underline{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$



NEAR $(2,0)$ $\quad J = \begin{pmatrix} -4 & 2 \\ 0 & 6 \end{pmatrix}$ eigenvalues $\lambda_1 = -4$, $\lambda_2 = 6$ saddle point

$$\lambda_1 = -4 \rightarrow (J - \lambda_1 I)\,\underline{v}_1 = \underline{0} \rightarrow \begin{pmatrix} 0 & 2 \\ 0 & 10 \end{pmatrix} \underline{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 6 \rightarrow (J - \lambda_2 I)\,\underline{v}_2 = \underline{0} \rightarrow \begin{pmatrix} -10 & 2 \\ 0 & 0 \end{pmatrix} \underline{v}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$



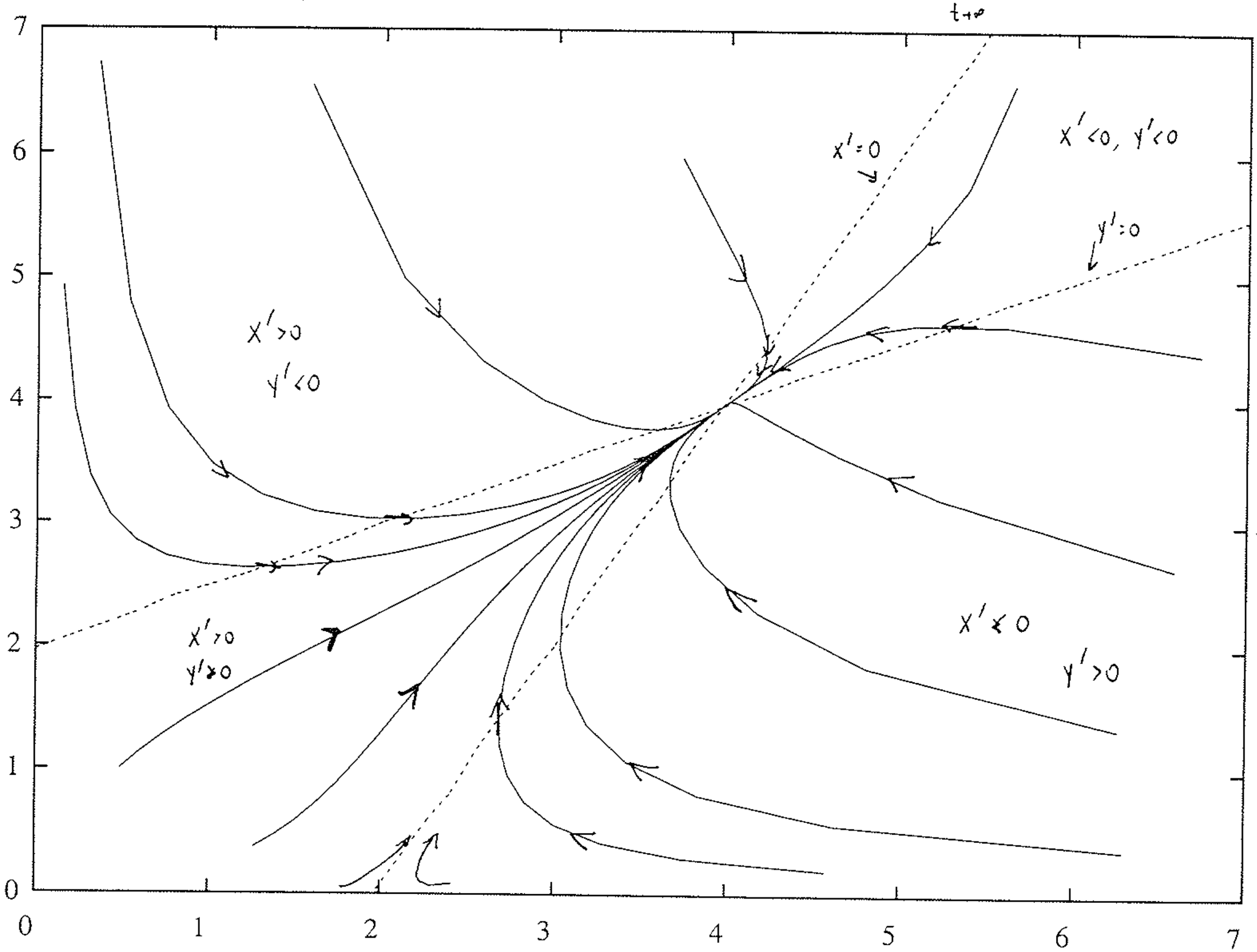
COOPERATION MODEL

$$x' = (4-2x+y)x$$
$$y' = (4+x-2y)y$$

EQ. POINTS $(0,0)$, $(4,4)$, $(0,2)$, $(2,0)$

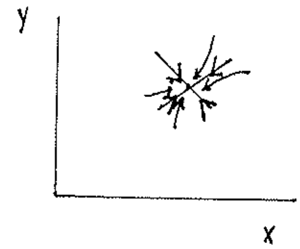
$(4,4)$ is a stable node.

$\lim_{t\to\infty} (x(t), y(t)) = (4,4)$.



NEAR (4,4) $\quad J = \begin{pmatrix} -8 & 4 \\ 4 & -8 \end{pmatrix}$ $\quad (-8-\lambda)^2 - 16 = 0$

$-8-\lambda = \pm 4 \quad \lambda_2 = -12,\ \lambda_1 = -4$ stable node



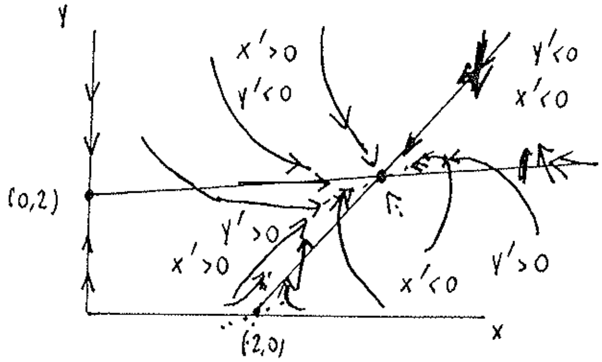
$\lambda_1 = -4: \ (J - \lambda_1 I)\, \underline{v}_1 = \underline{0} \longrightarrow \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \underline{v}_1 = \underline{0} \longrightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -12: \ (J - \lambda_2 I)\, \underline{v}_1 = \underline{0} \longrightarrow \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \underline{v}_2 = \underline{0} \longrightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

THUS $\quad \underline{X}(t) = c_1 e^{-4t} \underline{v}_1 + c_2 e^{-12t} \underline{v}_2$ NEAR (4,4)

$\underline{x}(t) \to c_1 e^{-4t} \underline{v}_1$ as $t \to \infty$ unless $c_1 = 0$ (smallest eigenvalue gives deca

NULLCLINES

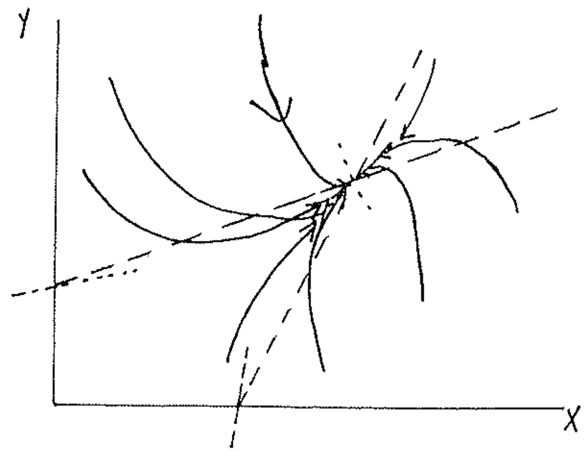


$f = 0$ on $x = 0, \quad y = 2x - 4$

$g = 0$ on $y = 0, \quad y = 2 + x/2$

THE FLOW is computed on next page.

Biologically as $t \to \infty$ then $(x, y) \to (4,4)$ as $t \to \infty$ both species can co-exist.



stable and unstable manifolds do not follow the nullcline but are between nullclines.

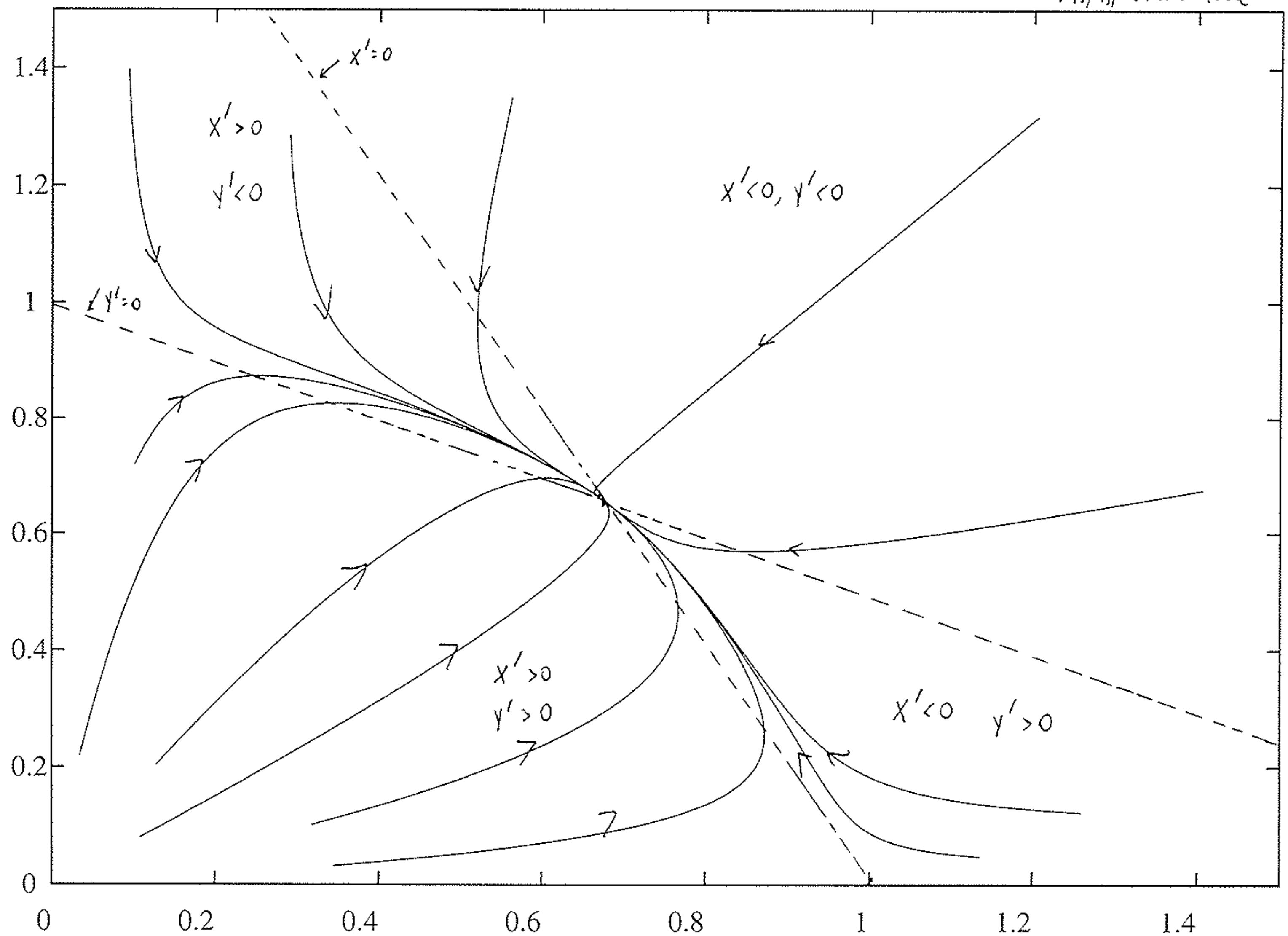
stable competition
(co-existence
is possible)

$x' = (2 - 2x - y)\,x$

$y' = (2 - x - 2y)\,y$

$\lim_{t \to \infty} (x, y) = (2/3, 2/3)$ for almost all initial conds.

$(2/3, 2/3)$ stable node



competitive exclusion:

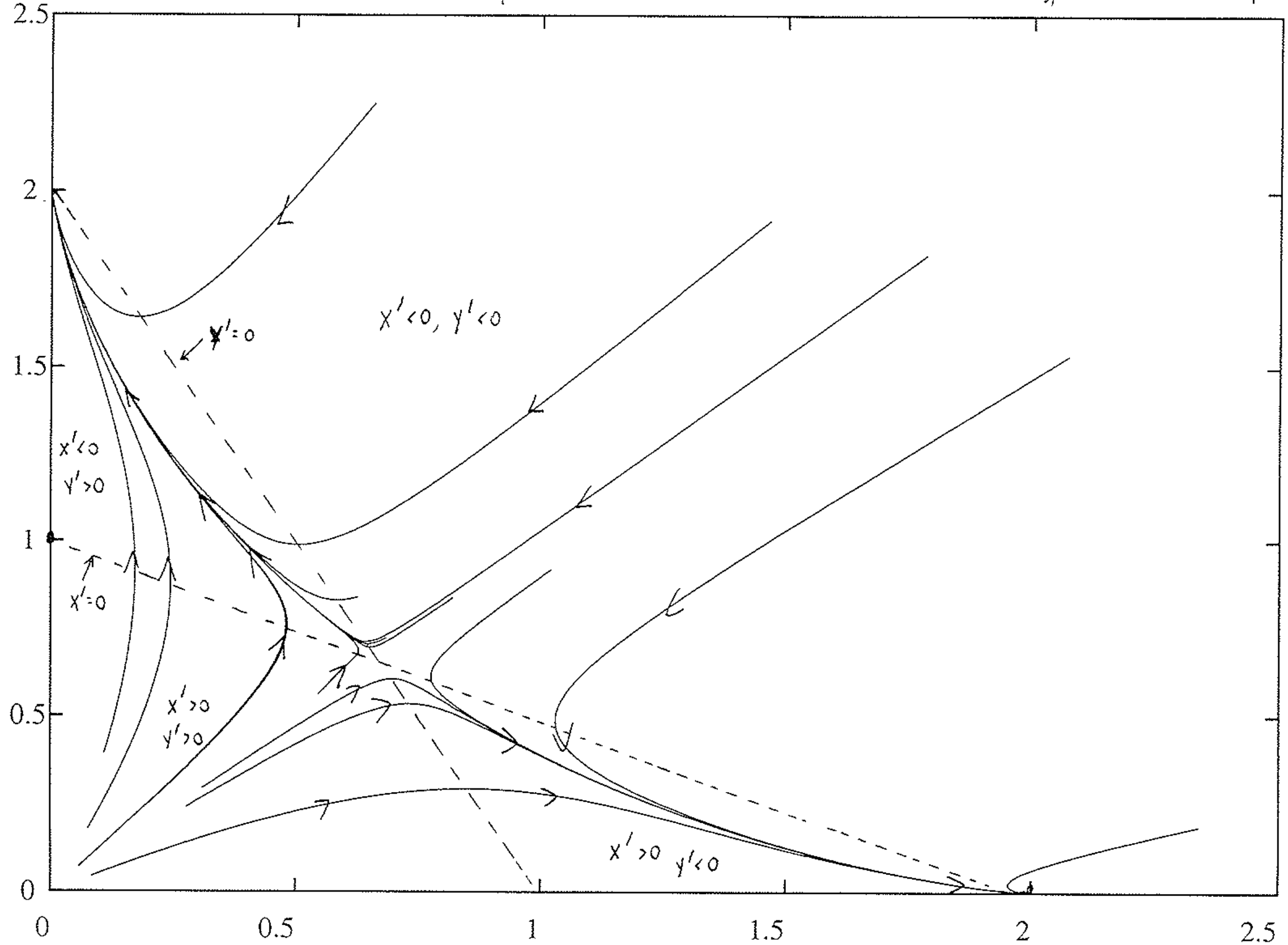
stable-coexistence unlikely.

$x' = (2 - x - 2y)x$

$y' = (2 - y - 2x)y$

$\lim_{t\to\infty} (x, y) = (2, 0)$ or $(0, 2)$ depending on initial cond.

$(2/3, 2/3)$ is a saddle point.



EXAMPLE 3 (STABLE COMPETITION)

$$x' = (2-2x-y)x = f(x,y)$$
$$y' = (2-x-2y)y = g(x,y)$$

EQUILIBRIUM POINTS ARE AT $(1,0), (0,1), (0,0), (2/3, 2/3)$

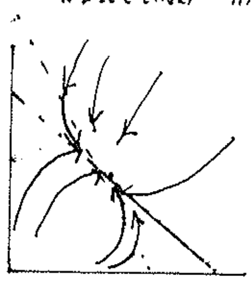
$$J = \begin{pmatrix} 2-4x_0-y_0 & -x_0 \\ -y_0 & 2-x_0-4y_0 \end{pmatrix}$$

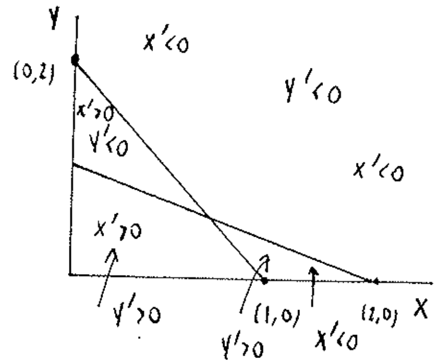
AT $(x_0, y_0) = (2/3, 2/3)$ $\quad J = \begin{pmatrix} -4/3 & -2/3 \\ -2/3 & -4/3 \end{pmatrix}$ $\quad \mathrm{tr}\, J = -8/3 < 0 \quad \mathrm{RE}[\lambda] < 0$

$\det J = \frac{16}{9} - \frac{4}{9} = \frac{12}{9} > 0$

THUS SINCE $\mathrm{tr}\, J = \lambda_1 + \lambda_2$ $\det J = \lambda_1 \lambda_2$ $\rightarrow$ either eigenvalues are both real negative or have negative real parts. BUT J is symmetric $\rightarrow$ both eigenvalues real negative. stable node

NULLCLINES AS SHOWN





$f = 0 \quad x = 0, \ y = 2 - 2x$

$g = 0 \quad y = 0 \quad y = 1 - x/2$

$\lim_{t\to\infty} (x,y) = (2/3, 2/3)$ for almost all initial conditions.

EXAMPLE (UNSTABLE COMPETITION)

$$x' = (2-x-2y)x = f(x,y)$$
$$y' = (2-y-2x)y = g(x,y)$$

HERE BOTH species will NOT co-exist as $t \to \infty$. either $\lim_{t\to\infty} (x,y) = (2,0)$ OR $(0,2)$ depending ON initial condition

EQUILIBRIA AT $(2,0), (0,2), (0,0), (2/3, 2/3)$.

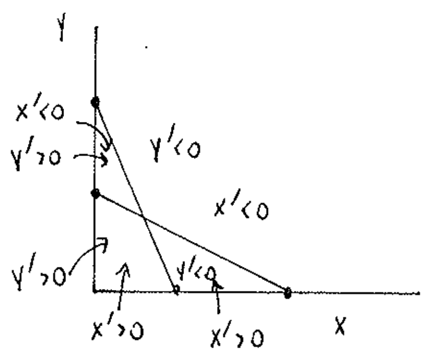
$$J = \begin{pmatrix} 2-2x_0-2y_0 & -2x_0 \\ -2y_0 & 2-2y_0-2x_0 \end{pmatrix}$$

AT $(2/3, 2/3)$ $\quad J = \begin{pmatrix} -2/3 & -4/3 \\ -4/3 & -2/3 \end{pmatrix}$

trace $J = -4/3$ $\quad \det J = -12/9 < 0$.

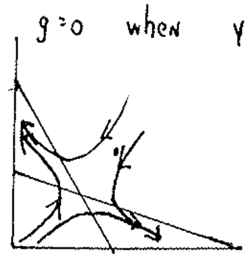
THUS $\lambda_1 > 0, \lambda_2 < 0$ Real. we have a saddle point (unstable).

NULLCLINES



$f = 0$ when $x = 0, \ y = 1 - x/2$

$g = 0$ when $y = 0, \ y = 2 - 2x$



PICTURE IS ON next page

EXAMPLE (PREDICTION OF BEHAVIOR)

$$N_1' = r N_1 (1 - N_1/K_1) - b_1 N_1 N_2 \qquad N_1' = dN_1/dt$$
$$N_2' = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$$

FOR $r_i > 0$, $K_i > 0$, $b_i > 0$ this is a competition model. we non-dimensionalize by setting $N_1 = K_1 X$ $\quad t = T\tau$ $\quad N_2 = \gamma y$

$$\frac{K_1}{T} X' = r_1 K_1 X (1 - X) - b_1 K_1 \gamma X y \longrightarrow \frac{X'}{r_1 T} = X(1-X) - \frac{b_1 \gamma}{r_1} X y$$
$$\frac{\gamma}{T} y' = r_2 \gamma y \left(1 - y/(K_2/\gamma)\right) - b_2 K_1 \gamma X y \longrightarrow y' = r_2 T y \left(1 - y/(K_2/\gamma)\right) - b_2 K_1 T X y.$$

choose $T = 1/r_1$ AND $\gamma = (b_1/r_1)^{-1} = r_1/b_1$

let's get

$$X' = X(1-X) - Xy \qquad a = r_2/r_1$$
$$y' = a y (1 - b y) - c X y \qquad b = \frac{\gamma}{K_2} = \frac{b_1^{-1}}{r_1^{-1} K_2} = \frac{r_1}{K_2 b_1}$$
$$c = b_2 K_1 / r_1$$

EQUILIBRIA ARE:
$$y = 1 - X$$
$$a(1 - by) = cX \longrightarrow y = \frac{1}{b} - \frac{c}{ab} X$$

$$X' = X[1 - X - y]$$
$$y' = y[a - aby - cX]$$

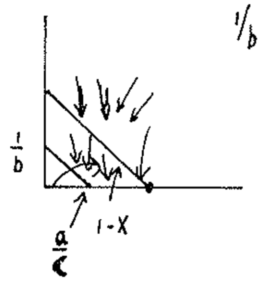
equilibria at $y = 1/b, X = 0$

$y = 0, X = 1$

AT $(0,0)$ AND maybe one more

FOUR relevant pictures:

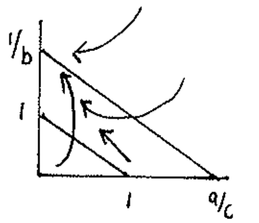
① X WINS



$1/b < 1$, $\frac{a}{c} < 1$

$\lim_{t \to \infty} (X, y) = (1, 0)$

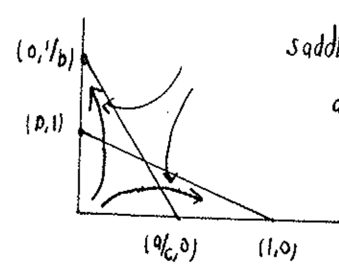
② y WINS



$1/b > 1$, $a/c > 1$

$\lim_{t \to \infty} (X, y) = (0, 1/b)$

③ either X OR y WINS



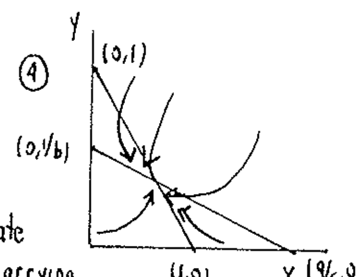
saddle point

$a/c < 1$, $\frac{1}{b} >$

co-existence IF $b > 1$; $a/c > 1$

$\longrightarrow r_1/K_2 b_1 > 1$, $\quad r_2/b_2 K_2 > 1$

④



growth rate exceeds carrying capacity for co-existence

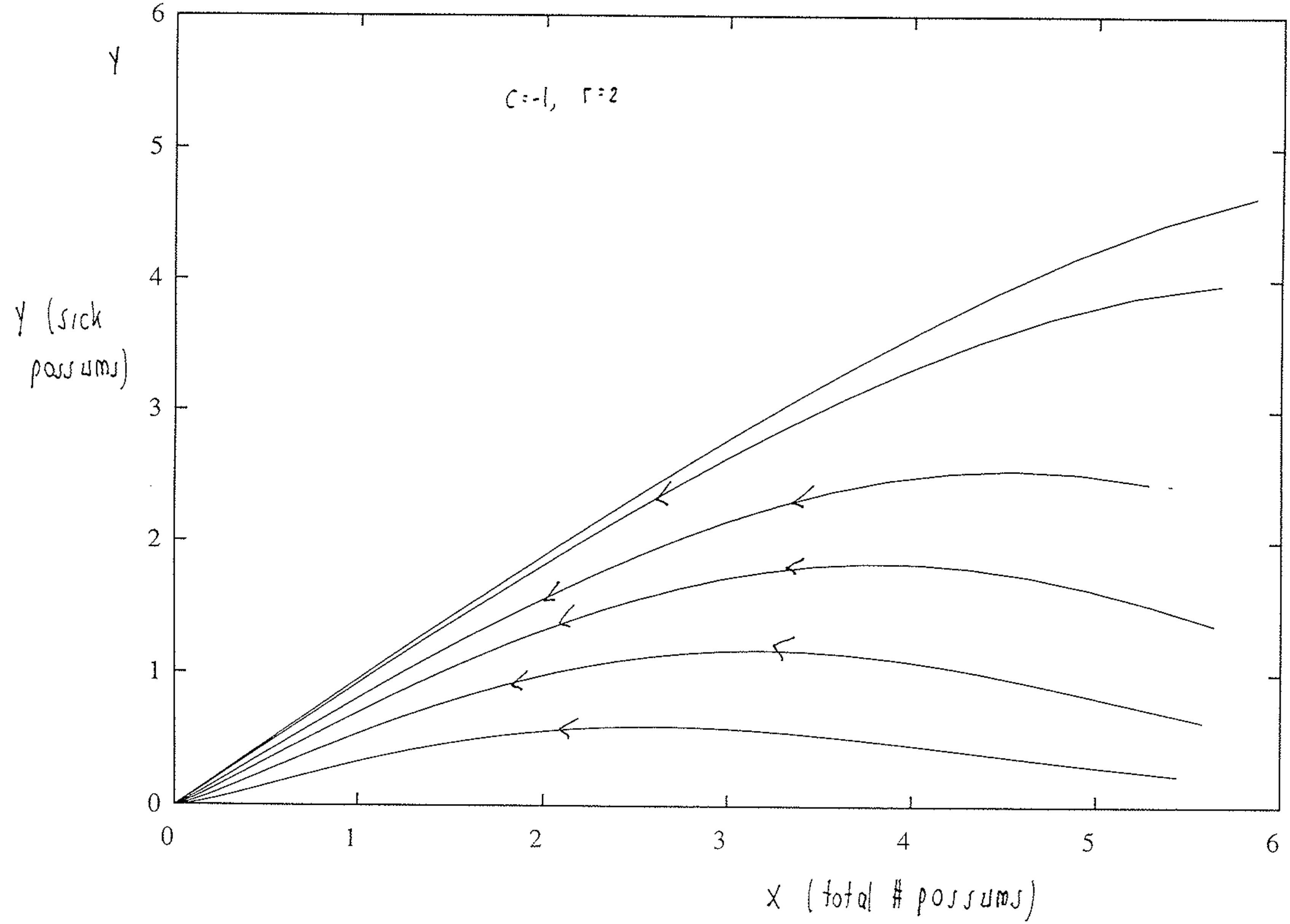
co-existence $1/b < 1$

$a/c > 1.$

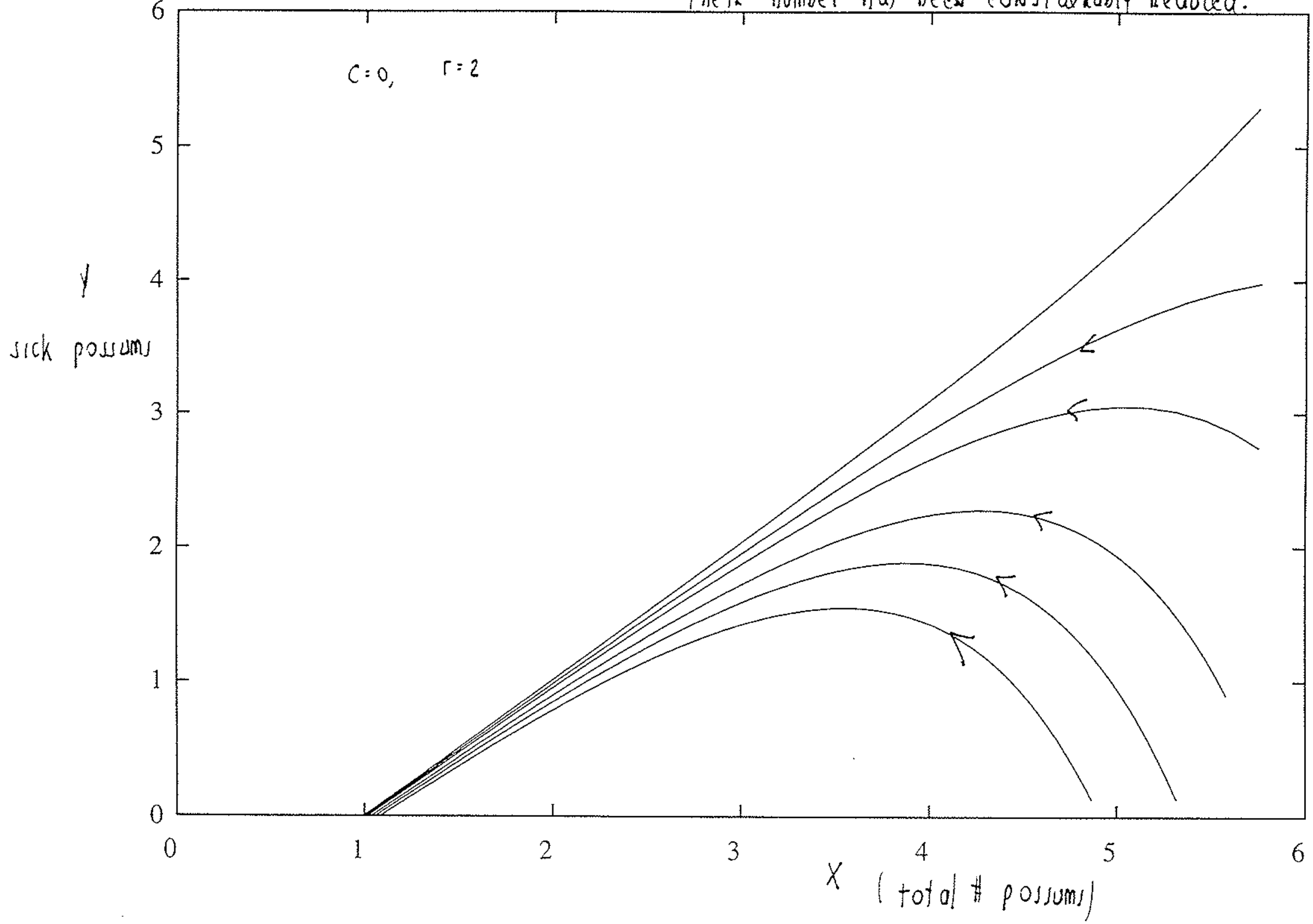
EPIDEMIC MODEL

$dx/d\tau = cx - y \qquad dy/d\tau = y(x-y) - ry$

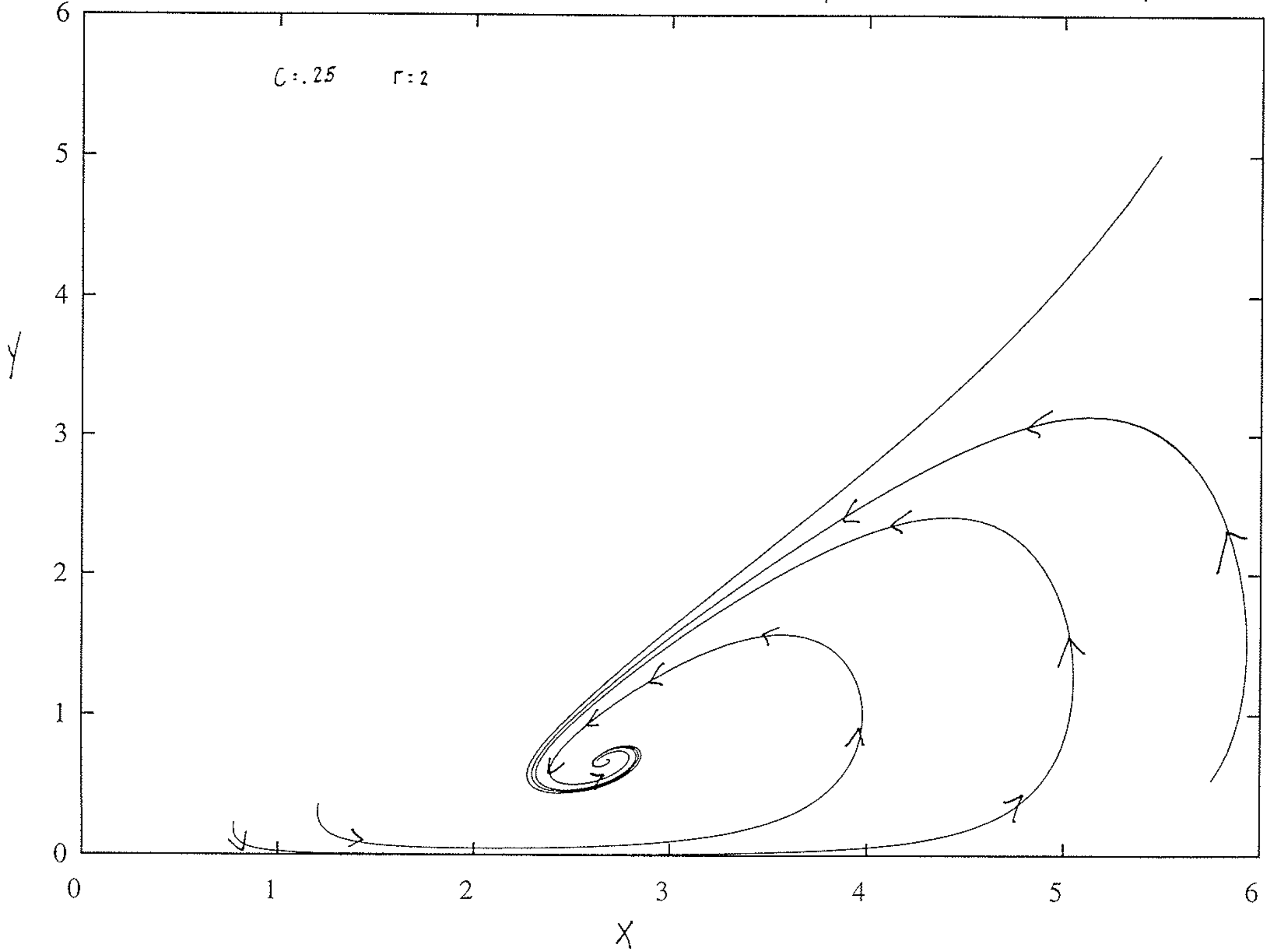
all possums die out



total # sick possums die out but
when epidemic ends there are still healthy possums left.
Their number has been considerably reduced.



there are still some sick possums left as $t \to \infty$
but it is cyclical. healthy and sick reach steady state



EPIDEMIC MODEL

ECOLOGICAL BALANCE IN NEW ZEALAND DISTURBED BY INTRODUCTION OF AUSTRALIAN POSSUM. IN FORESTS OF NEW ZEALAND, THERE ARE NO NATURAL PREDATORS AND POSSUMS HAVE MULTIPLIED. POSSUMS HAVE TUBERCULOSIS WITH ROUGHLY HALF THE POPULATION INFECTED.

P: POSSUMS AT TIME t $\qquad P = I + H$. $\quad H$: healthy possums

I: Infected possums, H: Healthy possums

$$(*) \quad \begin{cases} P' = (a-b)P - \alpha I \\ I' = B I H - (\alpha + b) I \end{cases}$$

b: natural death rate
a: natural growth rate
α: sick die at rate α

THEN $\quad (P-I)' = -BIH + (a-b)P + bI$

THUS $\quad H' = -BIH + (a-b)P + bI.$

NOW THIS CAN BE WRITTEN AS $\quad H' = -BIH + aP - bH$

IN $(*)$ LET $\quad P = P_0 X, \quad I = I_0 Y, \quad t = T\tau.$

CHOOSE $P_0 = I_0$. THEN,

$$\frac{X'}{\alpha T} = \left(\frac{a-b}{\alpha}\right) X - Y \qquad \text{choose } \alpha = 1/T$$

$$\frac{Y'}{T} = B P_0 Y(X-Y) - (\alpha+b) Y \qquad B P_0 T = 1 \longrightarrow P_0 = \alpha/B.$$

THEN $\quad (+) \quad \begin{cases} X' = cX - Y \\ Y' = Y(X-Y) - rY \end{cases}$ $\qquad r = 1 + b/\alpha > 1$, $\quad c = \dfrac{a-b}{\alpha}$

Need $X \geq Y$ only.

<u>NOTE</u>: IF c is small $\longrightarrow$ AND $c < 0$ THEN Natural death rate > birth rate and the sick die at a large rate.
might expect everyone to die out.
(both $X, Y \to 0$ as $t \to \infty$ which implies healthy and sick possums both die out)

IF $c > 0$ then perhaps as $t \to \infty$ there can be an equilibrium that is stable with $X > 0$ AND $Y > 0$

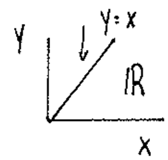
IF $c = 0$ Natural death rate = birth rate, Then perhaps the epidemic will die out. some possums Remain after $Y \to 0$. Hence not all possums are killed off.

ANALYSIS WE SET $cx = y$

$$y[\,x - y - r\,] = 0$$

THE EQUILIBRIA ARE $(0,0)$ AND $(x_E, y_E) = \left(r/_{1-c},\ cr/_{1-c} \right)$

NOW IF $c = 0$ THEN AS A special case, $y = 0$ is an equilibrium point FOR any x.

NOTE: (i) IF $c < 0$ only eq point in 

IR is at $(0,0)$

(ii) IF $c = 0$ the whole line $x \geq 0$, $y = 0$ is an eq. point.

(iii) IF $0 < c < 1$ THEN $0 < y < x$, AND EQ. POINT IS IN IR.

(iv) IF $c > 1$ THEN EQ. POINT HAS GONE TO THIRD QUADRANT.

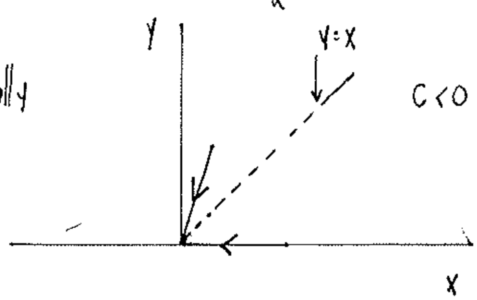
LINEARIZATION AT $(0,0)$

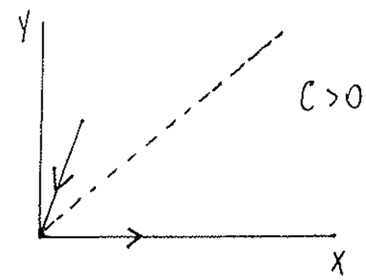
$$J = \begin{pmatrix} c & -1 \\ y & x - 2y - r \end{pmatrix} \qquad \text{AT } (0,0): \quad J = \begin{pmatrix} c & -1 \\ 0 & -r \end{pmatrix}$$

eigenvalues are $\lambda = +c$, $\lambda = -r$

eigenvectors: $\lambda = c \longrightarrow \underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad \lambda = -r \quad v = \begin{pmatrix} 1 \\ c+r \end{pmatrix}$

BUT $c + r = 1 + a/_{\alpha} > 1$ which lies out of IR.

locally 



NOTICE THAT globally on line $y = 0$ we have $x' = cx$. so flow is out if $c > 0$ AND IN IF $c < 0$.

LINEARIZATION AT (x_E, y_E) SUPPOSE $0 < c < 1$ THEN (x_E, y_E) IS IN R.

$$J = \begin{pmatrix} c & -1 \\ \frac{cr}{1-c} & \frac{r}{1-c} - \frac{2cr}{1-c} - r \end{pmatrix} = \begin{pmatrix} c & -1 \\ \frac{cr}{1-c} & -\frac{cr}{1-c} \end{pmatrix} = \begin{pmatrix} c & -1 \\ \mu & -\mu \end{pmatrix} \qquad \mu = \frac{cr}{1-c}$$

NOW $\operatorname{trace} J = c - \mu = c - \dfrac{c r}{1-c} = c\left(1 - \dfrac{r}{1-c}\right) = \dfrac{c}{1-c}\left(1-(r+c)\right)$

BUT SINCE $c + r = 1 + a/\alpha > 1 \longrightarrow \operatorname{trace} J < 0$

NOW $\det J = -c\mu + \mu = \mu(1-c)$

THUS IF $0 < c < 1 \longrightarrow \det J > 0$.

thus we either have a stable node OR a stable spiral point

the eigenvalues satisfy $\lambda = \dfrac{\operatorname{tr} J \pm \sqrt{(\operatorname{tr} J)^2 - 4\det J}}{2}$

THUS IF $(\operatorname{tr} J)^2 < 4 \det J \longrightarrow$ spiral point

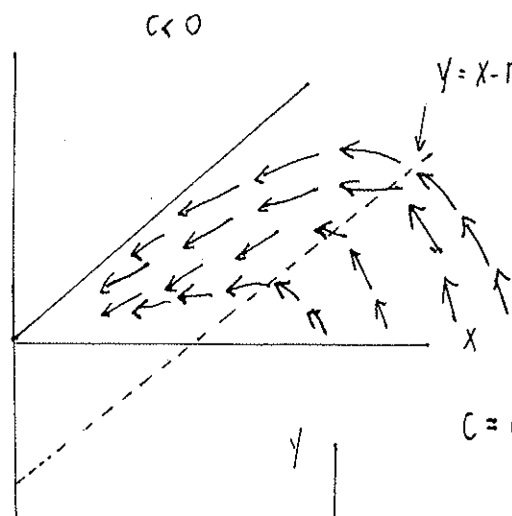
$\longrightarrow \left(\dfrac{c}{1-c}\right)^2 \left[1-(r+c)\right]^2 < \dfrac{4cr}{(1-c)}(1-c)$

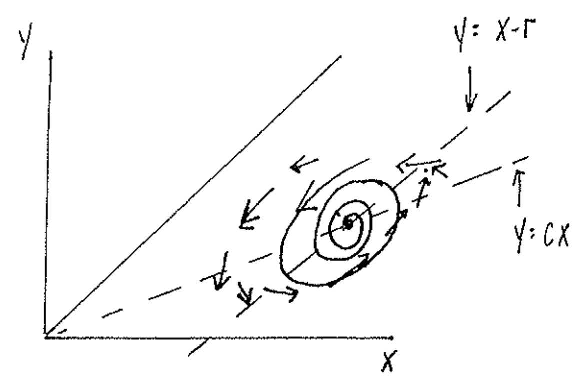
THUS $\dfrac{c}{(1-c)^2}\left[1-(r+c)\right]^2 < 4r$ which will always be true if c is small enough.

THIS Reduces to: IF $2r(1+c^2) - c(1+3r) - c^3 > 0 \rightarrow$ spiral point.

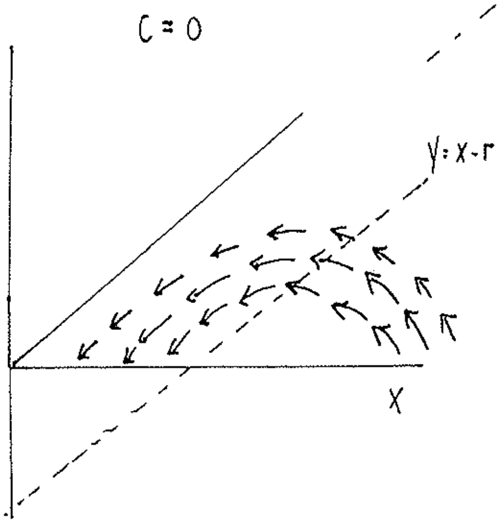
NULLCLINE ANALYSIS $\quad x' < 0$ always

A few sick initially, more get sick and then sick and healthy die out





$y = 0$ FOR $x > 0$ is an equilibrium point. so people get infected but the infected die out leaving some healthy ones left.



$\dfrac{dy}{dx} = y + r - x$

$y = x - r + 1 + ce^{x}$

so $\dfrac{dx}{dt} = -y = r - 1 - x - ce^{x}$

NOW AS $t \to \infty$

$r - x - 1 = ce^{x}$

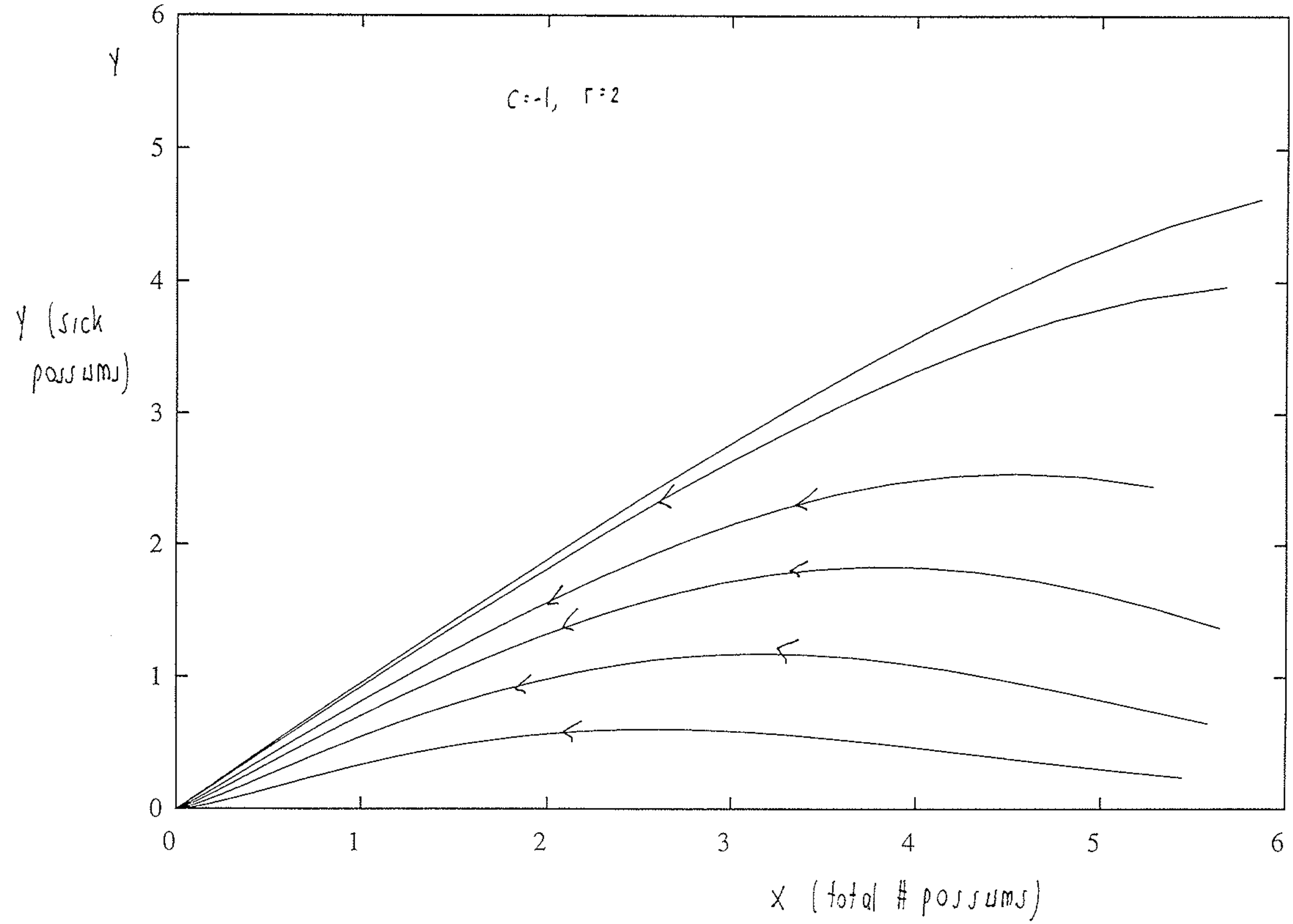
IF $c = 0$

then $x_E = -(1 - r)$,

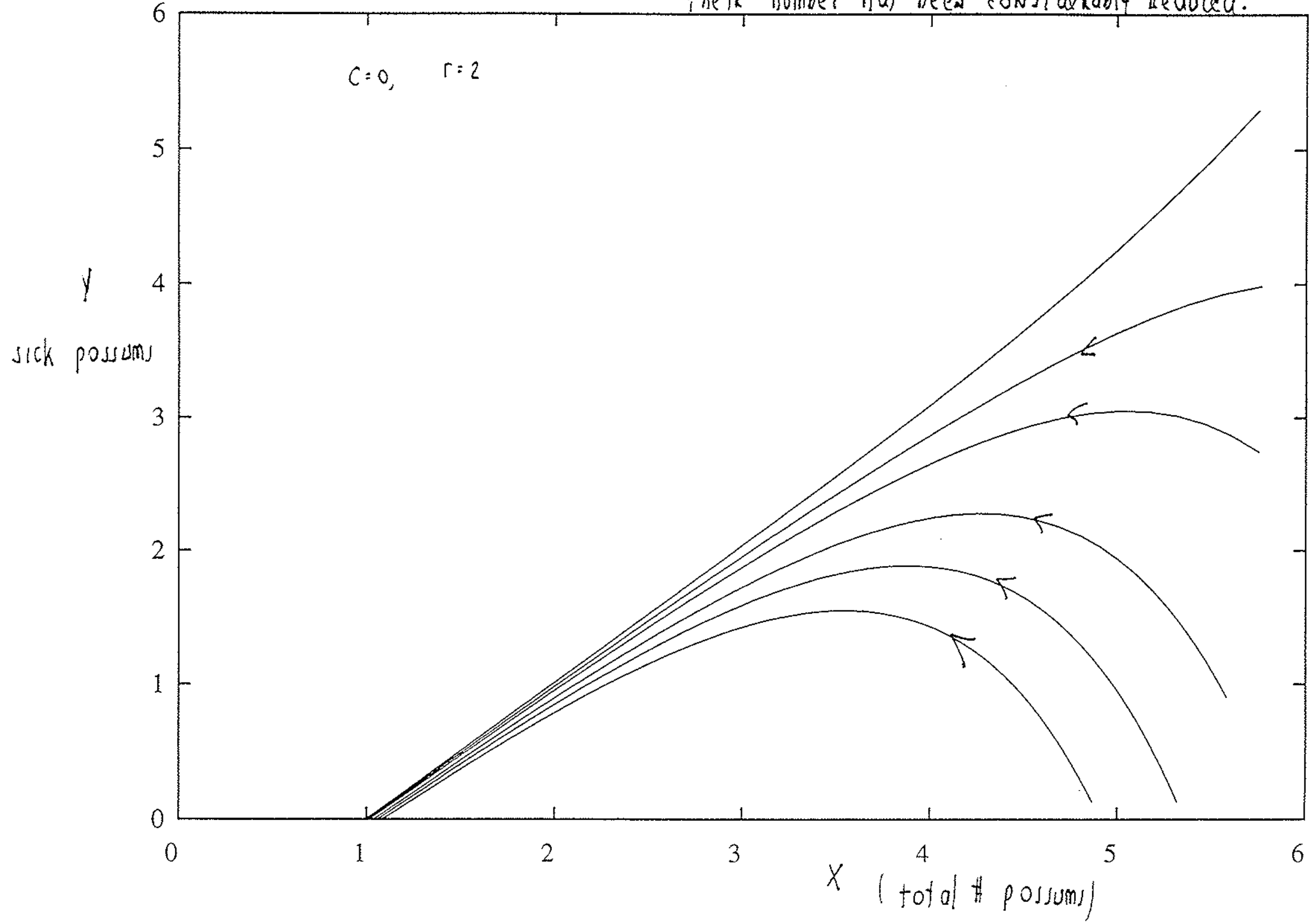
EPIDEMIC MODEL

$dx/dt = cx - y \qquad dy/dt = y(x-y) - ry$

all possums die out



total # sick possums die out but
when epidemic ends there are still healthy possums left.
Their number has been considerably reduced.



there are still some sick possums left as $t \to \infty$
but it is cyclical. healthy and sick reach steady state