M400 SOLUTIONS

SOLUTION 1

$$u_t = u_{xx} + e^{-t}\sin(3x), \quad 0<x<\pi, \quad t>0$$

$$u(x,0) = f(x), \quad 0 \le x \le \pi$$

$$u(0,t) = 0, \quad u(\pi,t) = 1.$$

(a) OBSERVE WE HAVE INHOMOGENEOUS BC. MUST FIRST GET HOMOGENEOUS BC.

LET (x) SOLVE $\left.\begin{array}{l} w_{xx} = 0, \quad 0<x<\pi \\ w(0) = 0, \quad w(\pi) = 1 \end{array}\right\} \rightarrow w = x/\pi.$

LET $u(x,t) = \dfrac{x}{\pi} + v(x,t).$

THEN
$$\left\{\begin{array}{l} v_t = v_{xx} + e^{-t}\sin(3x), \quad 0<x<\pi, \quad t>0 \\ v(x,0) = f(x) - x/\pi \\ v(0,t) = 0, \quad v(\pi,t) = 0. \end{array}\right.$$

THE HOMOGENEOUS PROBLEM HAS SL EIGENPROBLEM:

$$\left.\begin{array}{l} \phi_{xx} + \lambda\phi = 0 \\ \phi(0) = \phi(\pi) = 0 \end{array}\right\} \text{ SL PROBLEM}$$

SO $\phi_n(x) = \sin(nx), \quad n = 1,2,\ldots$ WEIGHT FUNCTION IS $w = 1$.

WE PUT $v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx).$

THEN THE PDE FOR v BECOMES

$$\sum_{n=1}^{\infty} a_n' \sin(nx) = -n^2\sum_{n=1}^{\infty} a_n \sin(nx) + e^{-t}\sum_{n=1}^{\infty}\delta_{n3}\sin(nx)$$

$$\delta_{n3} = \left\{\begin{array}{ll} 0 & n\neq 3 \\ 1 & n=3 \end{array}\right.$$

BY ORTHOGONALITY WE GET THAT

$$\sum_{n=1}^{\infty} \left(a_n' + n^2 a_n - \delta_{n3} e^{-t} \right) \sin(nx) = 0$$

IMPLIES $\quad a_n' + n^2 a_n - \delta_{n3} e^{-t} = 0.$

THIS GIVES $\quad a_n' = -n^2 a_n$ FOR $n = 1, 2, \ldots$ BUT $n \neq 3$

$\qquad a_n' = -9 a_n + e^{-t}$ FOR $n = 3.$

IN TERMS OF AN INITIAL VALUE $a_n(0)$ WE SOLVE TO GET

$$a_n(t) = \begin{cases} a_n(0) e^{-n^2 t} & n \neq 3 \\ a_3(0) e^{-9t} + \frac{1}{8} e^{-t} - \frac{1}{8} e^{-9t} & \text{FOR } n = 3. \end{cases} \qquad (1)$$

TO FIND INITIAL CONDITIONS $a_n(0)$ WE USE

$$v(x,0) = f(x) - x/\pi = x^2/\pi^2 - x/\pi = \sum_{n=1}^{\infty} a_n(0) \sin(nx)$$

BY ORTHOGONALITY $\quad a_n(0) = \dfrac{\int_0^\pi \left[x^2/\pi^2 - x/\pi \right] \sin(nx)\, dx}{\int_0^\pi \sin^2(nx)\, dx} = \dfrac{2}{\pi} \left[\dfrac{1}{\pi^2} \int_0^\pi x^2 \sin(nx)\, dx - \dfrac{1}{\pi} \int_0^\pi x \sin(nx)\, dx \right]$

FROM THE INTEGRALS PROVIDED

$\int_0^\pi x \sin(nx)\, dx = \dfrac{\pi (-1)^{n-1}}{n}, \quad \int_0^\pi x^2 \sin(nx) = \dfrac{2}{n^3} \left[(-1)^n - 1 \right] + \dfrac{\pi^2}{n} (-1)^{n-1}$ WE

CALCULATE $\quad a_n(0) = \dfrac{2}{\pi} \left[\dfrac{1}{n} (-1)^{n-1} + \dfrac{2}{\pi^2 n^3} \left[(-1)^n - 1 \right] - \dfrac{(-1)^{n-1}}{n} \right] = \dfrac{-4}{\pi^3 n^3} \left[1 - (-1)^n \right].$

THEREFORE,

$$a_n(0) = \begin{cases} -\dfrac{8}{\pi^3 n^3} & \text{IF } n \text{ IS ODD } (\text{i.e. } n = 2k+1,\ k = 0, 1, 2, \ldots) \\ 0 & \text{IF } n \text{ IS EVEN. } (\text{i.e. } n = 2k,\ k = 1, 2, \ldots) \end{cases} \qquad (2)$$

WE THEN WRITE $\quad u(x,t) = \dfrac{x}{\pi} + \sum_{n=1}^{\infty} a_n(t) \sin(nx) \qquad (3)$

NOW BY PUTTING (2) INTO (1) AND USING (3) WE GET EXACT SOLUTION:

$$u(x,t) = \frac{x}{\pi} + \frac{\sin(3x)}{8}\left(e^{-t} - e^{-9t}\right) - \sum_{k=0}^{\infty} \frac{8}{\pi^3} \frac{\sin((2k+1)x)}{[2k+1]^3} e^{-(2k+1)^2 t}$$

b) NOW FOR $t \to \infty$ WE WANT TO FIND e^{-t} TERMS. IT COMES FROM $k=0$ IN SUM AND $\frac{1}{8}\sin(3x)e^{-t}$. SO FOR t large we have

$$u(x,t) \simeq \frac{x}{\pi} + e^{-t}\left[\frac{\sin(3x)}{8} - \frac{8\sin x}{\pi^3}\right] + \cdots$$

COMPARING WITH $u(x,t) \simeq C_0(x) + C_1(x)e^{-t}$ WE IDENTIFY

$$C_0(x) = x/\pi \qquad C_1(x) = \frac{\sin(3x)}{8} - \frac{8\sin x}{\pi^3}.$$

PROBLEM 2

$$u_{tt} = c^2 u_{xx} + \delta(x - L/2)\cos(wt), \quad 0 \le x \le L,\ t \ge 0$$

$$u(0,t) = 0, \quad u(L,t) = 0$$

$$u(x,0) = u_t(x,0) = 0$$

HERE $c > 0$ AND $\delta(x - L/2)$.

SETTING $u = \Phi(x)T(t)$ FOR HOMOGENEOUS problem we get

$$\begin{aligned} &\Phi'' + \lambda \Phi = 0 \\ &\Phi(0) = \Phi(L) = 0 \end{aligned} \longrightarrow \lambda_n = \frac{n^2\pi^2}{L^2}, \quad \Phi_n(x) = A\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2 \ldots$$

$W(x) = 1$ FOR weight FUNCTION

NOW TO DERIVE A SOLUTION FOR HOMOGENEOUS problem WE WILL USE THE FIRST APPROACH IN THE NOTES. WE FIRST NORMALIZE THE EIGENFUNCTIONS SO THAT

$$\int_0^L \Phi_n^2\, dx = 1 \longrightarrow A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1 \longrightarrow A^2 \frac{L}{2} = 1$$

SO $A = \sqrt{\frac{2}{L}}$. WE WORK WITH EIGENFUNCTIONS

$$\left.\lambda_n = \frac{n^2\pi^2}{L^2}, \quad \Phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right).\right\} \quad (1)$$

FOR INHOMOGENEOUS problem we write

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \Phi_n(x) \qquad (2)_1$$

BY ORTHOGONALITY $\quad b_n(t) = \int_0^L u\,\Phi_n\, dx. \ \Big\} \ (2)_2$

WE NOW MULTIPLY PDE BY Φ_n AND INTEGRATE:

$$\int_0^L u_{tt}\,\Phi_n\, dx = c^2 \int_0^L u_{xx}\,\Phi_n\, dx + \int_0^L \delta(x - L/2)\cos(wt)\,\Phi_n(x)\, dx$$

NOW INTEGRATING BY PARTS TWICE AND USING $u = 0$ AND $\Phi_n = 0$ AT $x = 0, L$ WE GET $\frac{d^2}{dt^2}\int_0^L u\,\Phi_n\, dx = c^2 \int_0^L u\,\Phi_{n\,xx}\, dx + \int_0^L \delta(x - L/2)\cos wt\,\Phi_n\, dx.$

USING $\Phi_{nxx} = -\lambda_n \Phi_n$ WE GET USING ALSO THE SIFTING PROPERTY OF THE DELTA FUNCTION THAT

$$\frac{d^2}{dt^2}\left(\int_0^L u\,\Phi_n\,dx\right) = -c^2\lambda_n\left(\int_0^L u\,\Phi_n\,dx\right) + \cos(\omega t)\,\Phi_n(L/2) \qquad (3)$$

NOW USING $(2)_2$ WE OBTAIN $b_n'' = -c^2\lambda_n b_n + \cos(\omega t)\,\Phi_n(L/2)$

NOW PUTTING

$$\omega_n^2 \equiv c^2\lambda_n = \frac{n^2\pi^2c^2}{L^2} \rightarrow \omega_n = \frac{n\pi c}{L} \qquad (4)$$

WE CONCLUDE THAT FOR $n = 1, 2, 3, \ldots$

$$\left.\begin{aligned} & b_n'' + \omega_n^2 b_n = \Phi_n(L/2)\cos(\omega t) \\ \text{WITH}\quad & b_n(0) = 0, \quad b_n'(0) = 0 \\ \text{WHERE}\quad & \Phi_n(L/2) = \sqrt{\tfrac{2}{L}}\sin\left(\tfrac{n\pi}{2}\right). \end{aligned}\right\} \qquad (5)$$

<u>REMARKS</u> (i) THE ODE'S CAN ALSO BE DERIVED BY METHOD 2 IN THE NOTES WHERE WE PUT $u = \sum b_n \Phi_n$ INTO THE PDE, DIFFERENTIATING UNDER THE SUMMATION SIGN

(ii) THE VALUES $\omega_n = \frac{n\pi c}{L}$ ARE THE "NATURAL FREQUENCIES" OF VIBRATION OF THE STRING.

(iii) THE INITIAL CONDITIONS $b_n(0) = 0$ AND $b_n'(0) = 0$ FOLLOW FROM USING $u(x,0) = 0 = \sum_{n=1}^{\infty} b_n(0)\,\Phi_n(x)$ AND $u_t(x,0) = 0 = \sum_{n=1}^{\infty} b_n'(0)\,\Phi_n(x)$

$\rightarrow b_n(0) = 0,\ b_n'(0) = 0.$

(iv)
$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{IF } n \text{ IS EVEN} \\ 1 & \text{IF } n = 1, 5, 9, \ldots \\ -1 & \text{IF } n = 3, 7, 11, \ldots \end{cases}$$

• THEREFORE, IF $n = 2k$ FOR $k = 1, 2, \ldots$ WE HAVE $\Phi_{2k}(L/2) = 0$ AND

$$\left.\begin{aligned} b_{2k}'' + \omega_{2k}^2\, b_{2k} &= 0 \\ b_{2k}(0) = b_{2k}'(0) &= 0 \end{aligned}\right\} \longrightarrow b_{2k}(t) \equiv 0 \quad \forall t > 0. \qquad (6)$$

• NOW IF $n = 2k+1$ FOR $k = 0, 1, 2, \ldots$ WE HAVE $\Phi_{2k+1}(L/2) = (-1)^k \sqrt{\frac{2}{L}}$.

THEN FOR $k = 0, 1, 2, \ldots$

$$\left.\begin{aligned} b_{2k+1}'' + \omega_{2k+1}^2\, b_{2k+1} &= (-1)^k \sqrt{\frac{2}{L}} \cos(\omega t) \\ b_{2k+1}(0) = 0, \quad b_{2k+1}'(0) &= 0. \end{aligned}\right\} \qquad (7)$$

IN THIS WAY THE SOLUTION $u(x,t)$ IS SIMPLY A SUM OVER THE ODD INTEGERS,

$$\left. u(x,t) = \sum_{k=0}^{\infty} b_{2k+1}(t)\, \Phi_{2k+1}(x) = \sqrt{\frac{2}{L}} \sum_{k=0}^{\infty} b_{2k+1}(t) \sin\left(\frac{(2k+1)\pi x}{L}\right) \right\} \quad (8)$$

IT REMAINS TO SOLVE FOR $b_{2k+1}(t)$ IN (7) FOR THE TWO CASES IN THE PROBLEM STATEMENT.

(a) SUPPOSE $\omega \neq \omega_n$ FOR ANY $n = 1, 2, 3 \ldots$ THEN THE HOMOGENEOUS SOLUTION FOR (7) IS

$$b_{H\,2k+1}(t) = A \sin(\omega_{2k+1} t) + B \cos(\omega_{2k+1} t)$$

THE PARTICULAR SOLUTION HAS THE FORM

$$b_{P\,2k+1} = D \cos(\omega t)$$

$$\longrightarrow \qquad D\left[\omega_{2k+1}^2 - \omega^2\right] = (-1)^k \sqrt{\frac{2}{L}}$$

THIS GIVES $$D = \frac{(-1)^K \sqrt{2/L}}{\omega_{2K+1}^2 - \omega^2}.$$

NOW THE SOLUTION IS

$$b_{2K+1}(t) = A\sin(\omega_{2K+1} t) + B\cos(\omega_{2K+1} t) + \frac{(-1)^K\sqrt{2/L}}{\omega_{2K+1}^2 - \omega^2}\cos(\omega t)$$

NOW SETTING $\quad b_{2K+1}'(0) = 0 \rightarrow A = 0$

$$b_{2K+1}(0) = 0 \longrightarrow \quad B = -\frac{(-1)^K\sqrt{2/L}}{\omega_{2K+1}^2 - \omega^2}.$$

THEREFORE
$$\left.\begin{aligned} b_{2K+1}(t) &= \sqrt{\frac{2}{L}}\,\frac{(-1)^K}{\omega_{2K+1}^2 - \omega^2}\left[\cos(\omega t) - \cos(\omega_{2K+1} t)\right] \\ \text{WHERE} \quad \omega_{2K+1} &= \pi(2K+1)\pi c/L. \end{aligned}\right\} \quad (9)$$

PUTTING (9) INTO (8) GIVES THE SOLUTION.

(b) NOW SUPPOSE THAT $\omega = \omega_N$ FOR SOME INTEGER N. IF N IS EVEN THEN NO RESONANCE WILL OCCUR SINCE $\Phi_N(L/2) = 0$ AND FROM (6) WE HAVE $b_N(t) \equiv 0$.

SO ASSUME THAT $\omega = \omega_N$ FOR SOME <u>ODD</u> INTEGER N. FOR THIS INTEGER N LET $K^* = (N-1)/2$. WE HAVE THAT FOR $K \neq K^*$

$$b_{2K+1}(t) = \sqrt{\frac{2}{L}}\,\frac{(-1)^K}{\omega_{2K+1}^2 - \omega^2}\left[\cos(\omega t) - \cos(\omega_{2K+1} t)\right], \quad K = 0, 1, 2, \ldots \; K \neq K^*$$

AS BEFORE. HOWEVER FOR $K = K^*$ WE MUST SOLVE THE

RESONANT ODE PROBLEM:

$$b_{2N^*+1}'' + \omega_{2N^*+1}^2 b_{2N^*+1} = (-1)^{2N^*+1}\sqrt{\frac{2}{L}}\cos(\omega\ \ )$$

$$b_{2N^*+1}(0) = 0, \quad b_{2N^*+1}'(0) = 0.$$

THE HOMOGENEOUS SOLUTION IS

$$b_{2N^*+1}(t) = A\sin(\omega_{2N^*+1}t) + B\cos(\omega_{2N^*+1}t)$$

FOR PARTICULAR SOLUTION CONSIDER

$$\hat{b}_{2N^*+1}'' + \omega_{2N^*+1}^2 \hat{b}_{2N^*+1} = (-1)^{2N^*+1}\sqrt{\frac{2}{L}}\, e^{i\omega_{2N^*+1}t}$$

PUT $\quad \hat{b}_{2N^*+1}(t) = D\,t\,e^{i\omega_{2N^*+1}t}$

$\rightarrow$ SUBSTITUTE TO OBTAIN

$$D\left[\,2i\omega_{2N^*+1} e^{i\omega_{2N^*+1}t} - \omega_{2N^*+1}^2\, t\, e^{i\omega_{2N^*+1}t} + \omega_{2N^*+1}^2\, t\, e^{i\omega_{2N^*+1}t}\right] = (-1)^{2N^*+1}\sqrt{\frac{2}{L}}\, e^{i\omega_{2N^*+1}t}$$

THIS YIELDS

$$D = \frac{(-1)^{2N^*+1}}{(2i\omega_{2N^*+1})}\sqrt{\frac{2}{L}} = \frac{-i(-1)^{2N^*+1}}{2\omega_{2N^*+1}}\sqrt{\frac{2}{L}}.$$

SO $\quad b_{2N^*+1} = RE\left[\frac{-i(-1)^{2N^*+1}}{2\omega_{2N^*+1}}\sqrt{\frac{2}{L}}\,t\left(\cos(\omega_{2N^*+1}t) + i\sin(\omega_{2N^*+1}t)\right)\right]$

THE PARTICULAR SOLUTION IS

$$b_{P\,2N^*+1}(t) = \sqrt{\frac{2}{L}}\,\frac{(-1)^{2N^*+1}}{2\omega_{2N^*+1}}\, t\sin(\omega_{2N^*+1}t)$$

THE GENERAL SOLUTION IS

$$b_{2k^*+1} = A \sin\left(\omega_{2k^*+1} t \right) + B \cos\left(\omega_{2k^*+1} t \right) + \sqrt{\frac{2}{L}}\, \frac{(-1)^{2k^*+1}}{2\omega_{2k^*+1}}\, t \sin\left(\omega_{2k^*+1} t \right)$$

IMPOSING $\quad b_{2k^*+1}(0) = 0 \quad \rightarrow \quad B = 0$

$$b'_{2k^*+1}(0) = 0 \quad \rightarrow \quad A = 0.$$

THEREFORE, WE HAVE FOR $\omega = \omega_N$ AND N ODD, WITH $k^* = (N-1)/2$ THAT

$$b_{2k^*+1}(t) = \sqrt{\frac{2}{L}}\, \frac{(-1)^{2k^*+1}}{2\omega_{2k^*+1}}\, t \sin\left(\omega_{2k^*+1} t \right) \Bigg\} \quad (10)$$

IN CONCLUSION,

$$u(x,t) = \sqrt{\frac{2}{L}}\, b_{2k^*+1}(t) \sin\left(\frac{(2k^*+1)\pi x}{L} \right) + \sum_{\substack{k=0 \\ k \neq k^*}}^{\infty} \sqrt{\frac{2}{L}}\, b_{2k+1}(t) \sin\left(\frac{(2k+1)\pi x}{L} \right)$$

WHERE $b_{2k+1}(t)$ FOR $k \neq k^*$ IS GIVEN IN (9) AND

$$b_{2k^*+1}(t) = \sqrt{\frac{2}{L}}\, \frac{(-1)^{2k^*+1}}{2\omega_{2k^*+1}}\, t \sin\left(\omega_{2k^*+1} t \right)$$

WHICH CAN BE WRITTEN ALSO AS

$$b_N(t) = \sqrt{\frac{2}{L}}\, \frac{(-1)^N}{2\omega_N}\, t \sin\left(\omega_N t \right) \qquad \text{FOR } N \text{ ODD.}$$

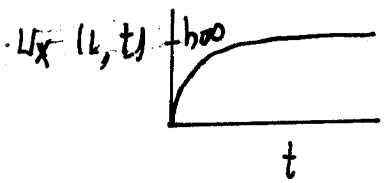
PROBLEM 3

$$u_t = D\,u_{xx}, \quad 0 \le x \le L,\ t \ge 0$$

$$u_x(0,t) = 0, \quad u_x(L,t) = h(t) \quad \text{WITH} \quad h(t) = h_\infty (1 - e^{-t})$$

$$u(x,0) = 0.$$

(a) OBSERVE THAT



SO THAT THE HEAT FLUX ENTERING WILL SATURATE AS $t \to \infty$.

INTEGRATING: $$\int_0^L u_t\,dx = D\int_0^L u_{xx}\,dx$$

$$\rightarrow \quad \frac{d}{dt}\int_0^L u\,dx = D\,u_x\Big|_0^L = D\,u_x(L,t) - D\,u_x(0,t)$$

SO $$\frac{d}{dt}\left(\int_0^L u\,dx\right) = D\,h(t)$$

NOW DIVIDE BY L AND DEFINE $u_{ave}(t) = \frac{1}{L}\int_0^L u\,dx.$

WE GET
$$\left.\begin{aligned} &\frac{d}{dt}u_{ave} = \frac{D}{L}h(t) \\ &u_{ave}(0) = \frac{1}{L}\int_0^L u(x,0)\,dx = \frac{1}{L}\int_0^L 0\,dx = 0. \end{aligned}\right\}$$

THE SOLUTION IS AFTER INTEGRATING AND SETTING $t=0$ WITH $u_{ave}(0)=0$ IS

$$u_{ave}(t) = \frac{D h_\infty}{L}\left(t + e^{-t} - 1\right).$$

FOR t large $u_{ave} \cong \frac{D\,t}{L} h_\infty$ SO THAT THE SAMPLE IS heating up UNIFORMLY AT EACH x IN $0 \le x \le L$.

(b) NOW SET $u(x,t) = \dfrac{h(t)\,x^2}{2L} + v(x,t)$ INTO THE PDE:

$$v_x(0,t) = u_x(0,t) - h(t)\,x/L\,\Big|_{x=0} = u_x(0,t) = 0.$$

$$V_x(L,t) = u_x(L,t) - \frac{h(t)x}{L}\Big|_{x=L} = h(t) - h(t) = 0.$$

SO $\quad V_x(0,t) = 0, \quad V_x(L,t) = 0.$

NOW PUTTING INTO INITIAL CONDITION, WE USE $h(0)=0$ TO GET

$$V(x,0) = u(x,0) - \frac{h(t)x^2}{2L}\Big|_{t=0} = u(x,0) = 0$$

NOW PUTTING THIS INTO THE PDE

$$\frac{h'(t)x^2}{2L} + V_t = \frac{D}{2L}h(t)\,(x^2)'' + DV_{xx} = \frac{Dh(t)}{L} + DV_{xx}$$

THIS GIVES

$$\left.\begin{aligned} &V_t = DV_{xx} + \overbrace{\frac{Dh(t)}{L} - \frac{h'(t)x^2}{2L}}^{\equiv G} \\ &V_x(0,t) = 0, \quad V_x(L,t) = 0 \\ &V(x,0) = 0 \end{aligned}\right\} \quad (*)$$

(c) DEFINE

$$G(x,t) \equiv \frac{Dh(t)}{L} - \frac{h'(t)x^2}{2L}. \qquad (D)$$

SETTING $G=0$ THE UNDERLYING SL EIGENVALUE PROBLEM IS

$$\Phi'' + \lambda\Phi = 0, \quad \Phi'(0) = \Phi'(L) = 0$$

SO $\quad \lambda_0 = 0, \; \Phi_0 = 1 \; ; \; \lambda_n = \frac{n^2\pi^2}{L^2}, \; \Phi_n = \cos\left(\frac{n\pi x}{L}\right) \qquad (1)$

THE weight FUNCTION IS $W=1$ AND ORTHOGONALITY RELATION IS $\int_0^L \Phi_n\Phi_m\,dx = 0$ FOR $n \neq m$.

NOW LOOK FOR $\quad V(x,t) = \sum_{n=0}^{\infty} b_n(t)\,\Phi_n(x)$

WHERE $\quad V(x,0) = 0 = \sum_{n=0}^{\infty} b_n(0)\,\Phi_n(x) \rightarrow b_n(0) = 0.$

PUTTING THIS INTO THE PDE:

$$\sum_{n=0}^{\infty} b_n' \Phi_n = \sum_{n=0}^{\infty} D b_n \Phi_n'' + G(x,t); \quad \text{NOW USE } \Phi_n'' = -\lambda_n \Phi_n \text{ TO GET}$$

$$\sum_{n=0}^{\infty} b_n' \Phi_n = -\sum_{n=0}^{\infty} \lambda_n D b_n \Phi_n + G(x,t) \qquad (2)$$

NOW EXPAND $\quad G(x,t) = \sum_{n=0}^{\infty} g_n(t) \Phi_n(x)$

WHERE BY ORTHOGONALITY WITH WEIGHT $w=1$

$$g_n(t) = \frac{\int_0^L G(x,t)\Phi_n(x)\,dx}{\int_0^L \Phi_n^2\,dx}. \qquad (3)$$

THEN (2) BECOMES

$$\sum_{n=0}^{\infty} (b_n' + D\lambda_n b_n - g_n)\Phi_n = 0$$

BY ORTHOGONALITY we get the ODE's

$$\left.\begin{aligned} & b_n' + D\lambda_n b_n = g_n(t) \qquad \text{FOR } n=0,1,2,\ldots \\ & b_n(0) = 0 \end{aligned}\right\} \qquad (4)$$

NOW CALCULATE $g_n(t)$. FOR $n=0$ WE HAVE USING $G(x,t) = Dh/_L - h'x^2/_{2L}$ THAT

$$g_0(t) = \frac{\int_0^L \left[\frac{Dh}{L} - \frac{h'x^2}{2L}\right]dx}{\int_0^L 1\,dx} = \frac{1}{L}\left[\left(\frac{Dh}{L}\right)L - \frac{h'L^3}{6L}\right] = \frac{Dh}{L} - \left(\frac{h'}{6}\right)L \qquad (5)$$

NOW FOR $n = 1,2,3,\ldots$

$$g_n(t) = \frac{\int_0^L \left(\frac{Dh}{L} - \frac{h'x^2}{2L}\right)\cos(n\pi x/_L)\,dx}{\underbrace{\int_0^L \cos^2(n\pi x/_L)}_{=L/2}} = \frac{2}{L}\int_0^L \left[\frac{Dh}{L} - \frac{h'x^2}{2L}\right]\cos\left(\frac{n\pi x}{L}\right)dx$$

NOW USING THE GIVEN INTEGRAL

$$\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2L^3}{\pi^2 n^2}(-1)^n \quad \text{FOR } n=1,2,\ldots$$

WE GET
$$g_n(t) = \frac{2Dh}{L^2}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx - \frac{h'}{L^2}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \underbrace{\frac{2Dh}{L^2}\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L}_{=0} - \left(\frac{h'}{L^2}\right)\frac{2L^3}{\pi^2 n^2}(-1)^n$$

SO WITH $h' = h_\infty e^{-t}$ WE GET

$$\left. g_n(t) = \frac{2L h_\infty}{\pi^2 n^2}(-1)^{n-1} e^{-t} \right\} \quad (6)$$

NOW PUT (5) AND (6) INTO (4):

$$\left.\begin{aligned} &\text{FOR } n=0: \quad b_0' = g_0(t) = \frac{Db}{L} - \frac{b'}{6}L = \frac{Dh_\infty}{L}(1-e^{-t}) - \frac{h_\infty e^{-t}}{6}L \\ &b_0(0) = 0 \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} &\text{FOR } n=1,2,\ldots \rightarrow b_n' + \frac{Dn^2\pi^2}{L^2}b_n = g_{n\infty} e^{-t} \\ &b_n(0) = 0 \end{aligned}\right\} \quad (8)$$

WITH
$$g_{n\infty} \equiv \frac{2L h_\infty}{\pi^2 n^2}(-1)^{n-1}.$$

NOW WE INTEGRATE (7) AND IMPOSE THAT $b_0(0)=0$, AND $b_n(0)=0$, $n=1,2,\ldots$

THIS GIVES
$$\left. b_0(t) = \frac{Dh_\infty}{L}(t + e^{-t} - 1) - \frac{h_\infty}{6}(-e^{-t}+1)L \right\} \quad (9)$$

AND
$$b_n(t) = \frac{g_{n\infty} e^{-t}}{\left(\frac{Dn^2\pi^2}{L^2} - 1\right)} - \frac{g_{n\infty}}{\left(\frac{Dn^2\pi^2}{L^2} - 1\right)} e^{-Dn^2\pi^2 t/L^2} \quad (10)$$

PROVIDED THAT $Dn^2\pi^2/L^2 \neq 1$ FOR <u>ANY</u> $n = 1, 2, \ldots$. IF FOR SOME $n = N$ WE HAVE $DN^2\pi^2/L^2 = 1$ WE MUST CALCULATE $b_N(t)$ IN THE FORM

$$b_N(t) = \underbrace{C_1 e^{-DN^2\pi^2 t/L^2}}_{\text{HOMOG. SOL'N}} + \underbrace{C_2\, t\, e^{-t}}_{\text{PARTICULAR SOLUTION}}$$

AND FIND C_1. I WILL IGNORE THIS COMPLICATION.

SO IN CONCLUSION IF $Dn^2\pi^2/L^2 \neq 1$ FOR <u>ANY</u> $n = 1, 2, \ldots$

THE SOLUTION TO ORIGINAL PROBLEM IS

$$u(x,t) = \frac{h(t)x^2}{2L} + v(x,t) = \frac{h x^2}{2L} + b_0(t) + \sum_{n=1}^{\infty} b_n(t)\cos\left(\frac{n\pi x}{L}\right) \qquad (11)_1$$

WITH

$$b_0(t) = \frac{D h_\infty}{L}\left(t + e^{-t} - 1\right) - \frac{hL}{6} \qquad (11)_2$$

AND FOR $n = 1, 2, \ldots$

$$b_n(t) = \frac{g_{n\infty}}{\left(\frac{Dn^2\pi^2}{L^2} - 1\right)}\left(e^{-t} - e^{-Dn^2\pi^2 t/L^2}\right)$$

WITH

$$g_{n\infty} = \frac{2L h_\infty}{\pi^2 n^2}(-1)^{n-1} \qquad (11)_3$$

NOW WE CALCULATE $\int_0^L u(x,t)\,dx$. SINCE $\int_0^L \cos(n\pi x/L)\,dx = 0$ WE GET

$$\int_0^L u(x,t)\,dx = \int_0^L \frac{hx^2}{2L}\,dx + \int_0^L b_0(t)\,dx = \frac{hL^2}{6} + L\, b_0$$

USING $(11)_2$ FOR b_0,

$$\int_0^L u(x,t)\,dx = \cancel{\frac{hL^2}{6}} + L\left[\frac{Dh_\infty}{L}\left(t + e^{-t} - 1\right) - \cancel{\frac{hL}{6}}\right]$$

THIS GIVES

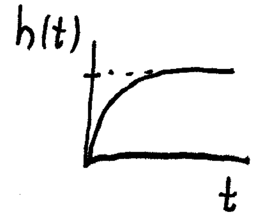
$$\int_0^L u(x,t)\,dx = Dh_\infty\left(t + e^{-t} - 1\right)$$

FINALLY, DIVIDING BY L WE GET $u_{ave}(t) \equiv \frac{1}{L}\int_0^L u\,dx = \frac{Dh_\infty}{L}\left(t + e^{-t} - 1\right)$.

THIS IS IDENTICAL TO THAT DERIVED IN (a) BY INTEGRATING PDE.

(d) NOW CONSIDER THE PROBLEM WITH BULK LOSS

$$u_t = D u_{xx} - k u, \quad 0 \le x \le L, \quad t \ge 0$$
$$u_x(0,t) = 0, \quad u_x(L,t) = h(t)$$
$$u(x,0) = 0. \qquad h(t) = h_\infty(1 - e^{-t})$$



INTEGRATING THE PDE:

$$\int_0^L u_t \, dx = D\int_0^L u_{xx}\, dx - k\int_0^L u\, dx.$$

$$\frac{d}{dt}\left(\int_0^L u\, dx\right) = D u_x \Big|_0^L - k\left(\int_0^L u\, dx\right)$$
$$= D u_x(L,t) - k\int_0^L u\, dx = D h(t) - k\int_0^L u\, dx$$

SO DIVIDING BY L AND DEFINING $u_{ave}(t) \equiv \frac{1}{L}\int_0^L u\, dx$ WE GET

$$\left.\begin{aligned} &\frac{d}{dt} u_{ave} + k u_{ave} = \frac{D}{L} h(t) = \frac{D}{L}[1 - e^{-t}]h_\infty \\ &u_{ave}(0) = 0 \end{aligned}\right\}$$

THIS IS A FIRST ORDER ODE WITH INTEGRATING FACTOR e^{kt}.

$$(u_{ave}\, e^{kt})' = h_\infty D\,[e^{kt} - e^{(k-1)t}]/L$$

$$u_{ave}\, e^{kt} = \begin{cases} \frac{h_\infty D}{L}\left[\frac{1}{k} e^{kt} - \frac{1}{k-1} e^{(k-1)t}\right] + C & \text{IF } k \ne 1 \\ h_\infty \frac{D}{L}\left[\frac{1}{k} e^{kt} - t\right] + C & \text{IF } k = 1. \end{cases}$$

SO $$u_{ave}(t) = \begin{cases} h_\infty \frac{D}{L}\left[\frac{1}{k} - \frac{1}{k-1} e^{-t}\right] + c e^{-kt} & \text{IF } k \ne 1 \\ h_\infty \frac{D}{L}\left[1 - t e^{-t}\right] + c e^{-t} & \text{IF } k = 1 \end{cases}$$

NOW SET $u_{ave}(0) = 0$ TO GET C.

$$u_{ave}(t) = \begin{cases} h_\infty \frac{D}{L}\left[\frac{1}{\kappa} - \frac{1}{\kappa-1}e^{-t}\right] + \frac{h_\infty D}{L}\left(\frac{1}{\kappa-1} - \frac{1}{\kappa}\right)e^{-\kappa t} & \text{IF } \kappa \neq 1 \\ h_\infty \frac{D}{L}\left[1 - te^{-t}\right] - \frac{D h_\infty}{L} e^{-t} & \text{IF } \kappa = 1. \end{cases}$$

NOW $$\lim_{t\to\infty} u_{ave}(t) = \frac{h_\infty D}{L\kappa} \qquad \forall \kappa > 0.$$

• WE CONCLUDE THAT WITH A BULK LOSS TERM (i.e. THE $-\kappa u$ TERM IN PDE) THE AVERAGE TEMPERATURE ACROSS THE DOMAIN IS BOUNDED AS $t \to \infty$. THE INCOMING FLUX FROM $x = L$ "BALANCES" THE BULK LOSS SO THAT ULTIMATELY THE AVERAGE TEMPERATURE REMAINS CONSTANT. ORIGINALLY WITH NO BULK LOSS (NO $-\kappa u$ TERM) THE AVERAGE TEMPERATURE CONTINUALLY INCREASES WITH TIME AND $u_{ave}(t) \cong \dfrac{D h_\infty t}{L}$ FOR large t.

PROBLEM 4

(a) SOLVE $u'' + \frac{1}{r} u' = 0$, $a \le r \le b$, $u(a) = 1$ AND $u(b) = 2$.

SOL'N MULTIPLY BY r: $r u'' + u' = (r u')' = 0$ SO $r u' = c_1 \rightarrow u' = c_1/r$

SO $u = c_1 \log r + c_2$; $u(a) = 1 \rightarrow c_1 \log a + c_2 = 1$
$u(b) = 2 \rightarrow c_1 \log b + c_2 = 2$ $\rightarrow c_1(\log b - \log a) = 1$

SO $c_1 = \dfrac{1}{\log(b/a)}$ $\quad c_2 = 1 - c_1 \log a = 1 - \dfrac{\log a}{\log(b/a)}$

$$\rightarrow u = \frac{1}{\log(b/a)} \log r + 1 - \frac{\log a}{\log(b/a)}$$

(b) SOLVE $u'' + \frac{2}{r} u' = M$ $\quad 0 \le r \le a$; $u'(a) = -h[u(a) - 1]$
u, u' BOUNDED AS $r \to 0$.

SOL'N MULTIPLY BY r^2:

$$r^2 u'' + 2 r u' = r^2 M \rightarrow (r^2 u')' = r^2 M$$

SO $r^2 u' = \frac{1}{3} r^3 M + c_1 \rightarrow u' = \frac{1}{3} r M + \frac{c_1}{r^2}$.

CHOOSE $c_1 = 0 \rightarrow$ SO BOUNDED AS $r \to 0$.

$$u' = \frac{1}{3} r M \rightarrow u = \frac{1}{6} r^2 M + c_2.$$

TO FIND c_2 PUT INTO $u'(a) = -h[u(a) - 1]$

$$\rightarrow \frac{1}{3} a M = -h\left[\frac{1}{6} a^2 M - 1\right] - h c_2.$$

SO $$c_2 = 1 - \frac{M}{6}\left(a^2 + 2a/h\right)$$

$$\rightarrow u = \frac{M}{6}(r^2 - a^2) + 1 - \frac{aM}{3h}.$$

OBSERVE, AS EXPECTED THERE IS NO SOLUTION IF $h = 0$.

(c) SOLVE $\bar{u}'' + \frac{2}{r}\bar{u}' - B^2\bar{u} = 0, \quad 0 \le r \le a$

$\bar{u}(a) = 1, \quad \bar{u}, \bar{u}'$ BOUNDED AS $r \to 0$.

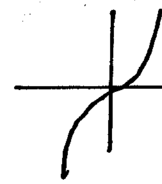
<u>SOL'N</u> LET $\bar{u}(r) = \frac{1}{r}\psi(r)$ AND ONE GETS

$$\psi'' - B^2\psi = 0.$$

THE LINEAR COMBINATION OF SOLUTIONS CAN BE CONSTRUCTED WITH EITHER $\psi = \text{SPAN}\{\sinh(Br), \cosh(Br)\}$ <u>OR</u> $\psi = \text{SPAN}\{e^{-Br}, e^{Br}\}$.

SINCE WE REQUIRE $\bar{u}$ BOUNDED AS $r \to 0$ WE MUST HAVE $\psi(0) = 0$.

THUS TAKE $\psi = A\sinh(Br)$ SINCE $\sinh 0 = 0$



$\sinh z = \frac{e^z - e^{-z}}{2}$

SO $$\bar{u}(r) = \frac{A\sinh(Br)}{r}$$

NOW $\bar{u}(a) = 1 \longrightarrow 1 = A\sinh(Ba)/a \to A = a/\sinh(Ba)$

$$\bar{u}(r) = \left(\frac{a}{r}\right)\frac{\sinh(Br)}{\sinh(Ba)}$$

(d) NOW SOLVE $\bar{u}'' + \frac{2}{r}\bar{u}' - B^2\bar{u} = 0$ IN $r > a$

$\bar{u}(a) = 1, \bar{u}, \bar{u}'$ BOUNDED AS $r \to \infty$.

<u>SOL'N</u> WE NEED $\bar{u} = \psi/r$ $\quad \psi = \text{SPAN}\{e^{-Br}, e^{Br}\}$ WITH $B > 0$.

THE ONE THAT <u>DECAYS</u> AS $r \to \infty$, I.E. $\psi = e^{-Br}$, GIVES

$\bar{u} = A\frac{e^{-Br}}{r}$. NOW $\bar{u}(a) = 1 \longrightarrow 1 = Ae^{-Ba}/a$

SO $A = ae^{Ba}$.

$$\longrightarrow \bar{u} = \left(\frac{a}{r}\right)e^{-B(r-a)}$$

SOLUTION 5

(*) $x^2 y'' + x y' - [x^2 + \nu^2] y = 0$

(a) PUT $y = x^{\alpha} + ..$ FOR $x \to 0$. THEN

$$\alpha(\alpha-1) x^{\alpha} + \alpha x^{\alpha} - [\nu^2 x^{\alpha} + x^{2+\alpha} + ..] = 0.$$

SO $$x^{\alpha}[\alpha^2 - \nu^2] + x^{\alpha+2}[..] = 0.$$

THE INDICIAL EQUATION IS $\alpha^2 - \nu^2 = 0$ SO $\alpha = \pm\nu$ WITH $\nu > 0$.

THUS FOR $\nu > 0$ WE HAVE $y_1 \sim C_0 x^{\nu}$ AND $y_2 \sim C_1 x^{-\nu}$ AS $x \to 0^+$

FOR $\nu = 0$ WE HAVE $y_1 \sim C_0$ AND $y_2 \sim (\log x) C_1$ AS $x \to 0$

BY FROBENIUS theory.

(b) NOW TO ELIMINATE y' PROCEED AS IN THE NOTES.

WE PUT $y = p\psi$ TO GET

$$x^2[p\psi'' + 2p'\psi' + p''\psi] + x[p\psi' + p'\psi] - (x^2 + \nu^2)p\psi = 0.$$

SO $$x^2 p\psi'' + \psi'[2p' x^2 + xp] + \psi[x^2 p'' + xp' - (x^2 + \nu^2)p] = 0.$$

DIVIDE BY $x^2 p$:

$$\psi'' + \psi'\left[\frac{2p'}{p} + \frac{1}{x}\right] + \psi\left[\frac{p''}{p} + \frac{p'}{xp} - \left(1 + \frac{\nu^2}{x^2}\right)\right] = 0.$$

WE CHOOSE $p(x)$ SO THAT $\frac{2p'}{p} + \frac{1}{x} = 0$ OR $p' + \frac{1}{2x} p = 0$.

(WHICH gives $p = x^{-1/2}$). THEN

$$\frac{\left(p'' + p'/x\right)}{p} = \frac{\left(\frac{3}{4} x^{-5/2} - \frac{1}{2} x^{-5/2}\right)}{x^{-1/2}} = \frac{1}{4x^2}.$$

(C) WE CONCLUDE THAT IF $y = x^{-1/2}\psi$ THEN (*) BECOMES EXACTLY.

$$\psi'' - \left[1 + \left(\nu^2 - 1/4\right)/x^2\right]\psi = 0.$$

NOW FOR $x \to \infty$, $\psi'' - \psi \,\hat{=}\, 0$ SO $\psi \,\hat{\approx}\, e^{\pm x}$.

WE CONCLUDE FOR $x \to +\infty$ THAT

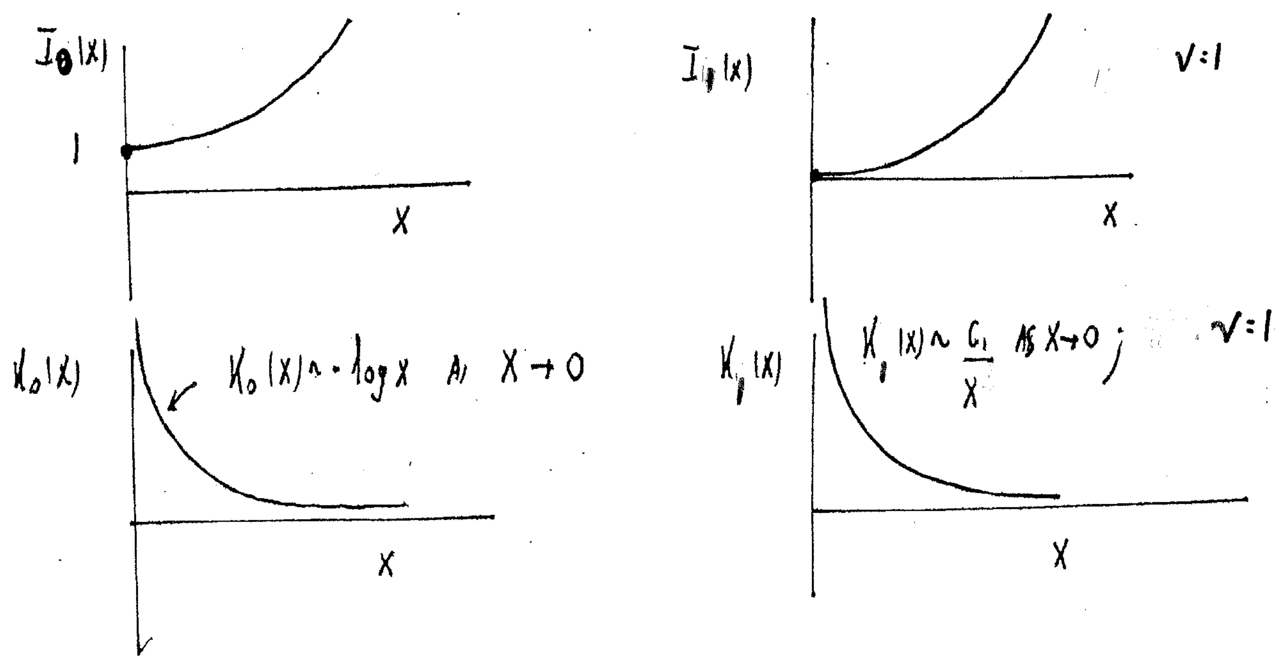
$$y = \frac{\psi}{x^{1/2}} \,\hat{\approx}\, \text{SPAN}\left\{ \frac{e^{x}}{\sqrt{x}}, \frac{e^{-x}}{\sqrt{x}} \right\}$$

grows decays.

THE general solution to (*) is

$$y = C_1 I_\nu(x) + C_2 K_\nu(x)$$

where $I_\nu(z)$, $K_\nu(z)$ ARE CALLED MODIFIED BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND OF ORDER ν. THE PLOTS ARE



(d) THE GENERAL SOLUTION TO

$$u'' + \frac{1}{r} u' - 4u = 0 \quad \text{IS} \quad u = C_1 I_0(2r) + C_2 K_0(2r).$$

(A) IF $u(a) = 1$ AND u, u' BOUNDED AS $r \to 0$ WE MUST SET $C_2 = 0$ SINCE $K_0(2r)$ UNBOUNDED AS $r \to 0$. THUS $u = C_1 I_0(2r)$.

SINCE $u(a) = 1$ WE GET $u = \dfrac{I_0(2r)}{I_0(2a)}$

(B) IF $u(a) = 1$ BUT $u \to 0$ AS $r \to \infty$ WE SET $C_1 = 0$ SINCE $I_0(2r)$ UNBOUNDED AS $r \to \infty$. THEN $u(a) = 1$ YIELDS

$$u = \frac{K_0(2r)}{K_0(2a)}$$