FULLY NONLINEAR EQUATIONS

WE CONSIDER $F(x, y, u, p, q) = 0$ WITH $p = \partial u/\partial x$, $q = \partial u/\partial y$.

FOR INSTANCE $F = p^2 + q^2 - 1$ IS the EIKONAL EQUATION.

WE DIFFERENTIATE WRT x:

$$F_x + F_u u_x + F_p p_x + F_q q_x = 0$$

BUT $u_x = p$ AND $q_x = p_y$. HENCE

$$\frac{\partial p}{\partial x} F_p + \frac{\partial p}{\partial y} F_q = -F_x - p F_u$$

THIS IS A QUASI-LINEAR EQUATION FOR p.

SO ON (1) $\frac{dx}{ds} = F_p$, $\frac{dy}{ds} = F_q$, THEN $\frac{dp}{ds} = -F_x - p F_u$.

NOW REPEAT THE PROCEDURE BY DIFFERENTIATING WRT y:

$$F_y + F_u u_y + F_p p_y + F_q q_y = 0$$

BUT $u_y = q$ AND $p_y = q_x$. SO

$$F_p \frac{\partial q}{\partial x} + F_q \frac{\partial q}{\partial y} = -F_y - F_u q.$$

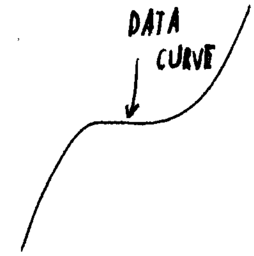
THIS IS QUASI-LINEAR EQUATION FOR q:

HENCE ON (2) $\frac{dx}{ds} = F_p$, $\frac{dy}{ds} = F_q$, THEN $\frac{dq}{ds} = -F_y - F_u q$.

COMBINING (1) AND (2) WE ARE LEFT WITH A 5×5 SYSTEM OF ODE'S:

(CHARPIT'S EQUATIONS)
$$\left\{ \begin{array}{l} dx/ds = F_p \qquad dy/ds = F_q \\ dp/ds = -F_x - pF_u, \quad dq/ds = -F_y - qF_u \\ du/ds = pF_p + qF_q \end{array} \right.$$

THEY ARE SOLVED WITH THE INITIAL DATA



$$x = x_0(\tau), \quad y = y_0(\tau), \quad u = u_0(\tau) \quad \text{AT } s = 0$$

THE INITIAL CONDITION FOR p_0, q_0 (i.e. $p(0), q(0)$)

ARE OBTAINED FROM

$$F(x_0, y_0, u_0, p_0, q_0) = 0$$

together WITH $\dfrac{du_0}{ds} = p_0 \dfrac{dx_0}{ds} + q_0 \dfrac{dy_0}{ds}$.

BECAUSE F IS NON-LINEAR IT MAY BE POSSIBLE THAT MORE THAN ONE CHARACTERISTIC PASSES THROUGH EACH POINT ON THE DATA CURVE.

<u>EXAMPLE 1</u> CONSIDER $u_y + u_x^2 = 0$ WITH INITIAL DATA

$$u(x, 0) = ax \quad \text{ON } -\infty < x < \infty.$$

<u>SOLUTION</u> HERE $F(x, y, u, u_x, u_y) = u_y^2 + u_x^2 = p^2 + q$.

SO CHARPIT'S EQUATION BECOME (WITH $F_p = 2p$, $F_q = 1$)

$$\frac{dx}{ds} = 2p, \quad x(0) = \tau \qquad \frac{dp}{ds} = 0, \quad p(0) = a$$

$$\frac{dy}{ds} = 1, \quad y(0) = 0 \qquad \frac{dq}{ds} = 0, \quad q(0) = -a^2$$

$$\frac{du}{ds} = 2p^2 + q, \quad u(0) = a\tau$$

WE SOLVE THIS ODE SYSTEM:

$$p = a, \quad q = -a^2$$

SO

$$\frac{dx}{ds} = 2a, \quad x(0) = \tau \longrightarrow x = 2as + \tau$$

$$\frac{dy}{ds} = 1, \quad y(0) = 0 \longrightarrow y = s$$

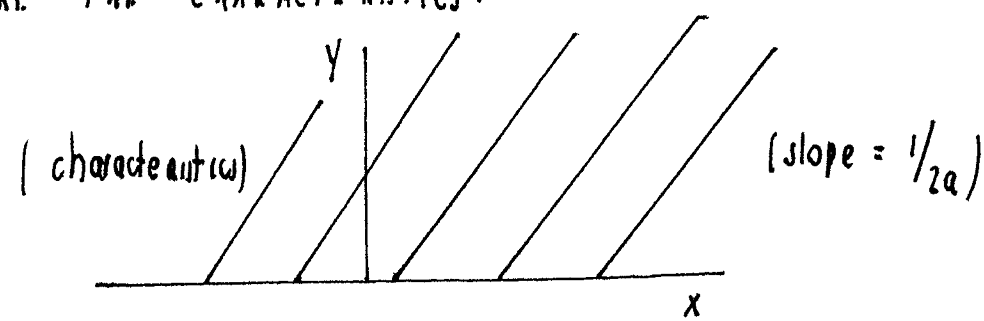
$$\frac{du}{ds} = 2a^2 - a^2, \quad u(0) = a\tau \longrightarrow u = a^2 s + a\tau.$$

NOW ELIMINATE s AND τ: $s = y$, $\tau = x - 2ay$.

SO $u = a^2(y) + a(x - 2ay) \longrightarrow u = a(x - ay)$

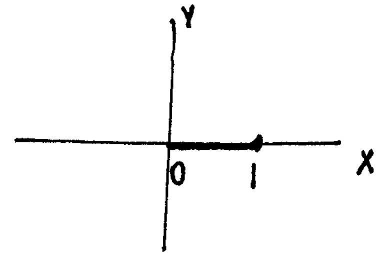
$\longrightarrow$ $u = a(x - ay)$ IS THE SOLUTION AND $x = 2ay + \tau$,

ARE THE CHARACTERISTICS:



so if y is time, this is a wave moving to the right.

EXAMPLE 2



SOLVE $U - X U_X - \frac{1}{2} U_y^2 + X^2 = 0$

WITH $U(X,0) = X^2 - \frac{1}{6} X^4$ FOR $0 < X < 1$.

NOW WE WRITE $F(X, Y, U, U_X, U_Y) = U - pX - \frac{1}{2} q^2 + X^2 = 0$ $\quad p = U_X, \; q = U_y$

CHARPIT'S SYSTEM BECOMES:

$$\frac{dx}{ds} = F_p = -X$$

$$\frac{dy}{ds} = F_q = -q$$

$$\frac{dU}{ds} = p F_p + q F_q = -pX - q^2$$

$$\frac{dp}{ds} = -F_X - F_U\, p = -(-p + 2X) - p = -2X$$

$$\frac{dq}{dx} = -F_Y - F_U\, q = -q$$

NOW parameterize data CURVE BY $X = \tau$, $y = 0$, $U = \tau^2 - \frac{1}{6}\tau^4$, $p = 2\tau - \frac{2}{3}\tau^3$. WE THEN SET $F = 0$ TO FIND q AT $s = 0$:

$$U - pX - \frac{1}{2} q^2 + X^2 = \tau^2 - \frac{1}{6}\tau^4 - \left(2\tau - \frac{2}{3}\tau^3\right)\tau - \frac{1}{2} q^2 + \tau^2 = 0.$$

THIS YIELDS $-\frac{1}{6}\tau^4 + \frac{2}{3}\tau^4 - \frac{1}{2} q^2 = 0 \longrightarrow \frac{\tau^4}{2} - \frac{1}{2} q^2 = 0.$

THIS GIVES $q = \pm \tau^2$

LET'S TAKE THE + ROOT SO THAT (THE MINUS ROOT GIVES ANOTHER SOLUTION)

$$\frac{dx}{ds} = -x, \quad x(0) = \tau$$

$$\frac{dy}{ds} = -q, \quad y(0) = 0$$

$$\frac{du}{ds} = -px - q^2, \quad u(0) = \tau^2 - \frac{1}{6}\tau^4$$

$$\frac{dp}{ds} = -2x, \quad p(0) = 2\tau - \frac{2}{3}\tau^3$$

$$\frac{dq}{ds} = -q, \quad q(0) = \tau^2.$$

IT IS PERHAPS EASIER TO NOTICE THAT

$$\frac{dq}{dx} = \frac{q}{x}, \quad q = \tau^2 \text{ when } x = \tau \longrightarrow q = \tau x.$$

$$\frac{dy}{dx} = \frac{q}{x} = \tau, \quad y = 0 \text{ when } x = \tau \longrightarrow y = \tau x - \tau^2$$

$$\frac{dp}{dx} = 2, \quad p = 2\tau - \frac{2}{3}\tau^3 \text{ when } x = \tau \rightarrow p = 2x - \frac{2}{3}\tau^3.$$

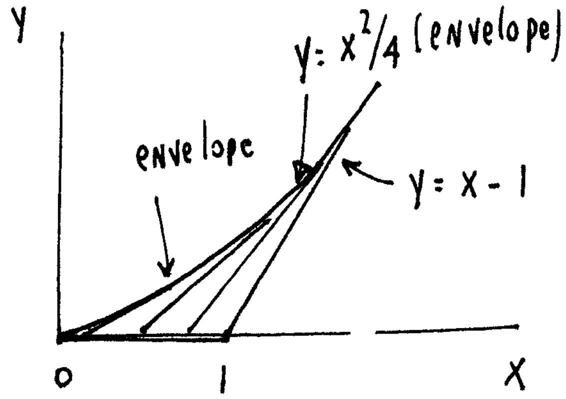
NOW FINALLY $u = px + q^2/_2 - x^2$.

THIS GIVES $u = \left(2x - \frac{2}{3}\tau^3\right)x + \frac{\tau^2 x^2}{2} - x^2.$

OR $u = x^2 + \frac{\tau^2 x^2}{2} - \frac{2}{3}x\tau^3$

WITH $y = \tau x - \tau^2$ IS IMPLICIT FORM

FOR THE SOLUTION. $0 \leq \tau \leq 1$



characteristics are

$$y = \tau x - \tau^2.$$

LET'S CALCULATE THE ENVELOPE OF THE CHARACTERISTICS.

WE WRITE $\quad G(y, x, \tau) = 0 \qquad G = y - \tau x + \tau^2$

$$G_\tau(y, x, \tau) = 0$$

SO $\quad G_\tau = 0 \longrightarrow -x + 2\tau = 0 \longrightarrow \tau = x/2$

HENCE $\quad y - \dfrac{x^2}{2} + \dfrac{x^2}{4} = 0 \longrightarrow y = x^2/4$

IS ENVELOPE

EIKONAL EQUATION

WE BEGIN WITH THE WAVE EQUATION $\phi = \phi(X, Y, t)$

$$\phi_{tt} = c^2 (\phi_{XX} + \phi_{YY}).$$

LET $\phi = e^{-i\omega t}\psi(X,Y)$ TO GET $\psi_{XX} + \psi_{YY} + K^2\psi = 0$

WHERE $K = \omega/c$. NOW IF WE NON-DIMENSIONALIZE BY SETTING $x = X/L$, $y = Y/L$

WE OBTAIN

$$\psi_{xx} + \psi_{yy} + \kappa^2 \psi = 0 \qquad \kappa = L^2 K.$$

ASSUME THAT $\kappa \gg 1$ (HIGH SPATIAL FREQUENCY $\to$ LOW WAVELENGTH).

THEN LET $\psi = A(x,y)\, e^{i\kappa u(x,y)}$ $\qquad u$ = phase of wave.

A = amplitude

WE CALCULATE $\psi_x = i\kappa u_x A e^{i\kappa u} + A_x e^{i\kappa u}$

$$\psi_{xx} = -\kappa^2 u_x^2 A e^{i\kappa u} + i\kappa u_{xx} A e^{i\kappa u} + 2i\kappa u_x A_x e^{i\kappa u} + A_{xx} e^{i\kappa u}.$$

SUBSTITUTING INTO $\psi_{xx} + \psi_{yy} + \kappa^2\psi = 0$ WE OBTAIN

$$\underline{-\kappa^2 A (u_x^2 + u_y^2)} + i\kappa\left[(u_{xx} + u_{yy})A + 2\nabla u \cdot \nabla A\right] + (A_{xx} + A_{yy}) + \underline{\kappa^2 A}$$

THE TWO LARGEST TERMS PROPORTIONAL TO κ^2 BALANCE AND SO

$$-(u_x^2 + u_y^2) = -1 \longrightarrow u_x^2 + u_y^2 = 1$$

EIKONAL EQUATION.

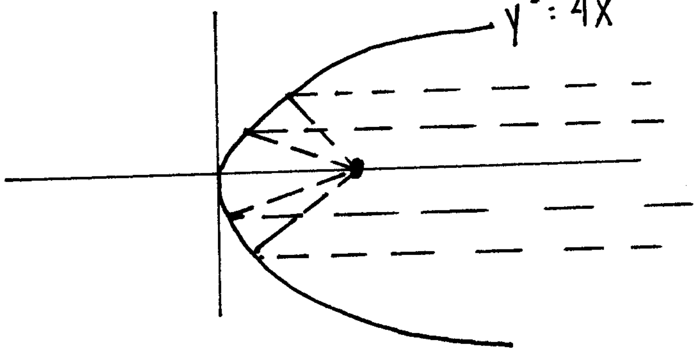
SPECIAL SOLUTIONS ARE $u = -x$, $A = 1 \longrightarrow \psi = e^{-i\kappa x}$

OR $\phi = e^{-i\omega t} e^{-i\kappa x} = e^{-i\kappa(x+ct)}$ $\qquad c = \omega/\kappa$

THIS IS A WAVE PROPAGATING TO THE LEFT.

IF $u = x$, $A = 1$ THEN $\phi = e^{i\kappa(x-ct)}$ A WAVE MOVING TO RIGHT.

EXAMPLE PARABOLIC REFLECTOR.



$p^2 + q^2 = 1$ $\quad p = u_x$

$q = u_y$

$x = \tau^2, \; y = 2\tau \;, -\infty < \tau < \infty$

parametrizes the reflector

WE WANT TO SHOW THAT RAYS OF LIGHT (i.e. THE CHARACTERISTICS) ALL MEET AT THE FOCAL POINT OF THE PARABOLA.

WE ASSUME $u = -x$ (i.e. $\phi = e^{-ik(x+ct)}$ A WAVE MOVING TO THE LEFT IS INCIDENT ON THE PARABOLA. WE WANT TO CALCULATE REFLECTED FIELD. WE WANT $u = -x$ ON BOUNDARY OF PARABOLA

NOW THE CHARPIT SYSTEM IS

$$\frac{dx}{ds} = F_p = 2p \;, \qquad x(0) = \tau^2 = x_0(\tau)$$

$$\frac{dy}{ds} = F_q = 2q \;, \qquad y(0) = 2\tau = y_0(\tau)$$

$$\frac{du}{ds} = pF_p + qF_q = 2 \;, \qquad u(0) = -\tau^2 = u_0(\tau)$$

$$\frac{dp}{ds} = -F_x - pF_u = 0 \;, \qquad p(0) = p_0$$

$$\frac{dq}{ds} = -F_y - qF_u = 0 \;, \qquad q(0) = q_0$$

TO CALCULATE (p_0, q_0) WE NOTICE THAT ON T

$$\frac{du_0}{d\tau} = \frac{\partial u_0}{\partial x}\frac{dx_0}{d\tau} + \frac{\partial u_0}{\partial y}\frac{dy_0}{d\tau} \longrightarrow -2\tau = 2p_0\tau + 2q_0$$

THUS $\quad p_0^2 + q_0^2 = 1,\quad$ AND $\quad q_0 = -\tau(1+p_0)$

WE SOLVE $\quad p_0^2 + (1+p_0)^2\tau^2 = 1$

HENCE $\quad p_0 = \dfrac{-2\tau^2 \pm \sqrt{4\tau^4 - 4(\tau^2-1)(\tau^2+1)}}{2\tau^2+2} = \dfrac{-2\tau^2 \pm 2}{2\tau^2+2}.$

WE WANT + SIGN SO THAT $p_0 \neq -1$ (i.e. $u \neq -x$)

HENCE $\quad p_0 = \dfrac{-2\tau^2+2}{2\tau^2+2} \qquad q_0 = -\tau(1+p_0) = -\tau\left(1 + \dfrac{(2-2\tau^2)}{(2+2\tau^2)}\right) = \dfrac{-2\tau}{1+\tau^2}.$

THEN SOLVING CHARPIT'S SYSTEM WE OBTAIN

$$p = \frac{1-\tau^2}{1+\tau^2}, \quad q = -\frac{2\tau}{1+\tau^2}, \quad u = 2s - \tau^2$$

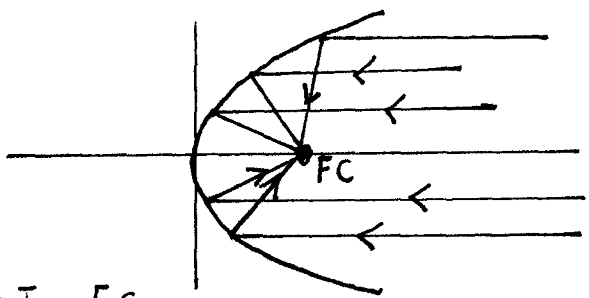
AND $\quad x = 2\left(\dfrac{1-\tau^2}{1+\tau^2}\right)s + \tau^2, \qquad y = -\dfrac{4\tau}{(1+\tau^2)}s + 2\tau.$

WE THEN ELIMINATE S TO OBTAIN: $\quad x - \tau^2 = 2\dfrac{(1-\tau^2)}{(1+\tau^2)}\dfrac{(y-2\tau)}{(-4\tau)}(1+\tau^2).$

CLEANING THIS UP ONE OBTAINS THE FAMILY OF CURVES

$$x - \tau^2 = (\tau^2-1)\left(\frac{y}{2\tau} - 1\right) \longrightarrow 2\tau(x-\tau^2) = (\tau^2-1)(y-2\tau)$$

THIS IS A FAMILY OF STRAIGHT LINES FOR EACH τ IN $-\infty < \tau < \infty$



NOTICE THAT FOR ANY τ

$x = 1$ WHEN $y = 0$ $\longleftarrow$ FOCAL POINT FC.

EXAMPLE 2 (CAUSTIC IN A LIQUID SURFACE UNDER AN OBLIQUE LIGHT SOURCE)

please see http://www.ballandclaw.com/Caustic/index.html
↑capital

FOR A Java applet of the phenomenon. $F = p^2 + q^2 - 1$

WE WANT TO SOLVE $p^2 + q^2 = 1$ WITH $u = -x$ ON

THE PORTION OF THE CIRCLE $x^2 + y^2 = 1$ WITH $x_0 = \cos\tau$, $y_0 = \sin\tau$

AND $\pi/2 < \tau < 3\pi/2$.

THE CHARPIT SYSTEM IS

$$\frac{dx}{ds} = 2p \quad , \qquad x(0) = \cos\tau$$

$$\frac{dy}{ds} = 2q \quad , \qquad y(0) = \sin\tau$$

$$\frac{du}{ds} = 2 \quad , \qquad u(0) = -\cos\tau$$

$$\frac{dp}{ds} = 0 \quad , \qquad p(0) = p_0(\tau)$$

$$\frac{dq}{ds} = 0 \quad , \qquad q(0) = q_0(\tau)$$

TO DETERMINE p_0, q_0 WE USE $p_0^2 + q_0^2 = 1$

Together WITH $du_0/d\tau = p_0 x_0' + q_0 y_0' \rightarrow \sin\tau = -\sin\tau\, p_0 + \cos\tau\, q$

BY INSPECTION ONE SOLUTION IS $p_0 = -1$, $q_0 = 0$. THIS

CORRESPONDS TO $u = -x$ INCIDENT WAVE. WE, HOWEVER,

WANT THE REFLECTED WAVE.

THE OTHER SOLUTION IS $q_0 = \sin(2\tau)$, $p_0 = \cos(2\tau)$

SINCE $\quad -\sin\tau \cos(2\tau) + \sin(2\tau)\cos\tau = \sin(2\tau - \tau) = \sin\tau$.

THEN WE OBTAIN
$$p = p_0 = \cos(2\tau)$$
$$q = q_0 = \sin(2\tau)$$

AND
$$\frac{dx}{ds} = 2\cos(2\tau), \qquad x = 2\cos(2\tau)s + \cos\tau$$
$$\frac{dy}{ds} = 2\sin(2\tau), \qquad y = 2\sin(2\tau)s + \sin\tau$$

NOW WE ELIMINATE s: $\quad (x - \cos\tau) = 2\cos(2\tau)\dfrac{(y - \sin\tau)}{2\sin(2\tau)}$.

THIS GIVES
$$(x - \cos\tau)\sin 2\tau = (y - \sin\tau)\cos 2\tau$$

OR EQUIVALENTLY $\sin 2\tau\, x - \cos(\tau)\sin 2\tau = y\cos 2\tau - \sin\tau\cos 2\tau$.

THIS GIVES:
$$x\sin(2\tau) - y\cos(2\tau) = \sin\tau, \qquad \pi/2 < \tau < 3\pi/2$$

THESE ARE THE CHARACTERISTICS: STRAIGHT LINES!

NOW WHAT IS THE ENVELOPE?

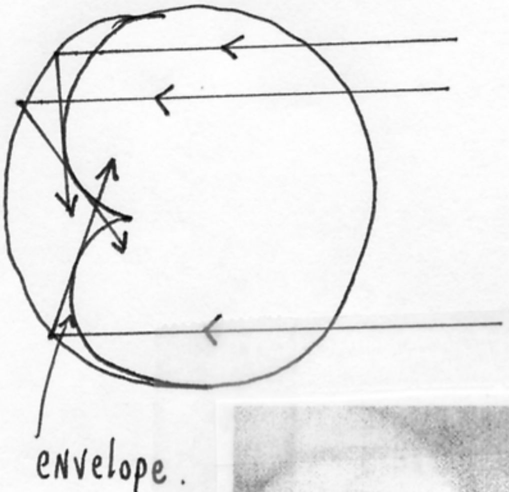
$$F = x\sin(2\tau) - y\cos(2\tau) - \sin\tau = 0$$
$$F_\tau = 2x\cos(2\tau) + 2y\sin(2\tau) - \cos\tau = 0$$
$$\begin{pmatrix} \sin 2\tau & -\cos 2\tau \\ \cos 2\tau & \sin 2\tau \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin\tau \\ \frac{1}{2}\cos\tau \end{pmatrix}$$

RECALLING: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc}$

THEN $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \sin 2\tau & \cos 2\tau \\ -\cos 2\tau & \sin 2\tau \end{pmatrix} \begin{pmatrix} \sin \tau \\ \frac{1}{2}\cos\tau \end{pmatrix}$

parametric EQUATION OF envelope $\left\{ \begin{aligned} X &= \sin\tau \sin 2\tau + \frac{1}{2}\cos\tau \cos 2\tau \\ Y &= -\sin\tau \cos 2\tau + \frac{1}{2}\cos\tau \sin 2\tau \end{aligned} \right.$ $\quad \pi/2 < \tau < 3\pi/2$

IF YOU PLOT THIS CURVE THE PICTURE IS AS FOLLOWS



THE ENVELOPE IS A NEPHROID OR (KIDNEY)