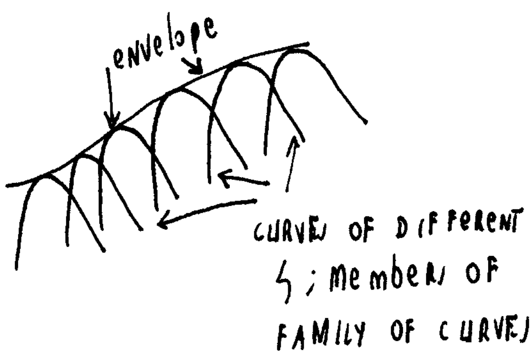
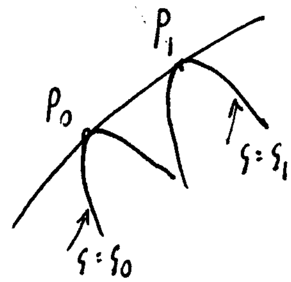
ENVELOPES OF FAMILIES OF CURVES

CONSIDER A FAMILY OF CURVES

$$G(x, t, \varsigma) = 0$$

THE ENVELOPE OF THE FAMILY OF CURVES (IF IT EXISTS) MUST CONTAINS POINTS ON THE FAMILY OF CURVES, AND THE FAMILY OF CURVES MUST INTERSECT TANGENTIALLY WITH THE ENVELOPE:





SO ON THE POINT P_0, $\varsigma = \varsigma_0$ AND AT THE POINT P_1, $\varsigma = \varsigma_1$.

AT THE POINT OF CONTACT WITH THE FAMILY OF CURVES $G(x, t, \varsigma) = 0$ WITH $\varsigma = \varsigma(x, t)$.

THUS $\quad G(x, t, \varsigma(x, t)) = 0$

NOW LET (x, t) AND $(x + dx, t + dt)$ BE TWO NEIGHBORING POINTS ON THE ENVELOPE CORRESPONDING TO ς AND $\varsigma + d\varsigma$ RESPECTIVELY.

THUS $\quad dG = G_x\,dx + G_t\,dt + G_\varsigma\,d\varsigma = 0 \quad$ (SINCE $G(x, t, \varsigma(x, t)) = 0$)

HOWEVER $G_x\,dx + G_t\,dt = (G_x, G_t) \cdot (dx, dt) = 0$ BY THE FACT THAT THE FAMILY OF CURVES INTERSECT TANGENTIALLY. THIS IMPLIES $G_\varsigma = 0$.

THUS THE ENVELOPE OF A FAMILY OF CURVES (IF IT EXISTS) IS THE SOLUTION OF $\quad G(x, t, \varsigma) = 0, \quad G_\varsigma(x, t, \varsigma) = 0.$

EXAMPLE A CONCENTRATED SOUND PULSE EMITTED AT $t=0$ IN A QUIESCENT MEDIUM PROPAGATES THROUGH SPACE AS AN EVER EXPANDING SPHERE WHOSE RADIUS IS ct AT ANY TIME t. ASSUME THAT THE SOURCE IS MOVING ALONG X-AXIS AT A SPEED $U>0$. THEN THE FAMILY OF WAVEFRONTS SATISFY

$$(x+Ut)^2+y^2+z^2=c^2t^2$$

REMARK (i) IF WE CONSIDER THE 2-D VERSION OF THIS AND LET $t=\varsigma$ WE CAN WRITE $G(x,t,\varsigma)=0$ $\quad G(x,t,\varsigma)\equiv(x+\varsigma t)^2+y^2-c^2\varsigma$ WHICH IS PRECISELY OF THE FORM OF THE THEORY.

WE WRITE $(x+Ut)^2+y^2+z^2-c^2t^2\equiv G(x,y,z,t)=0$.

THE EQUATION OF THE ENVELOPE IF IT EXISTS IS TO SOLVE $G=0$ TOGETHER WITH $G_t=0$.

$$G=0 \longrightarrow (x+Ut)^2+y^2+z^2=c^2t^2$$

$$G_t=0 \longrightarrow 2(x+Ut)U-2c^2t=0 \longrightarrow U(x+Ut)=c^2t.$$

NOW WE COMBINE TO GET $\quad x^2+2Utx+U^2t^2+y^2+z^2=U(x+Ut)t$

THIS GIVES: $\quad x^2+Uxt+y^2+z^2=0$

WE THEN SOLVE FOR t IN $U(x+Ut)=c^2t$ TO GET $\quad t=Ux/(U^2-c^2)$

THIS GIVES $\quad x^2+Ux\left(\dfrac{Ux}{U^2-c^2}\right)+y^2+z^2=0.$

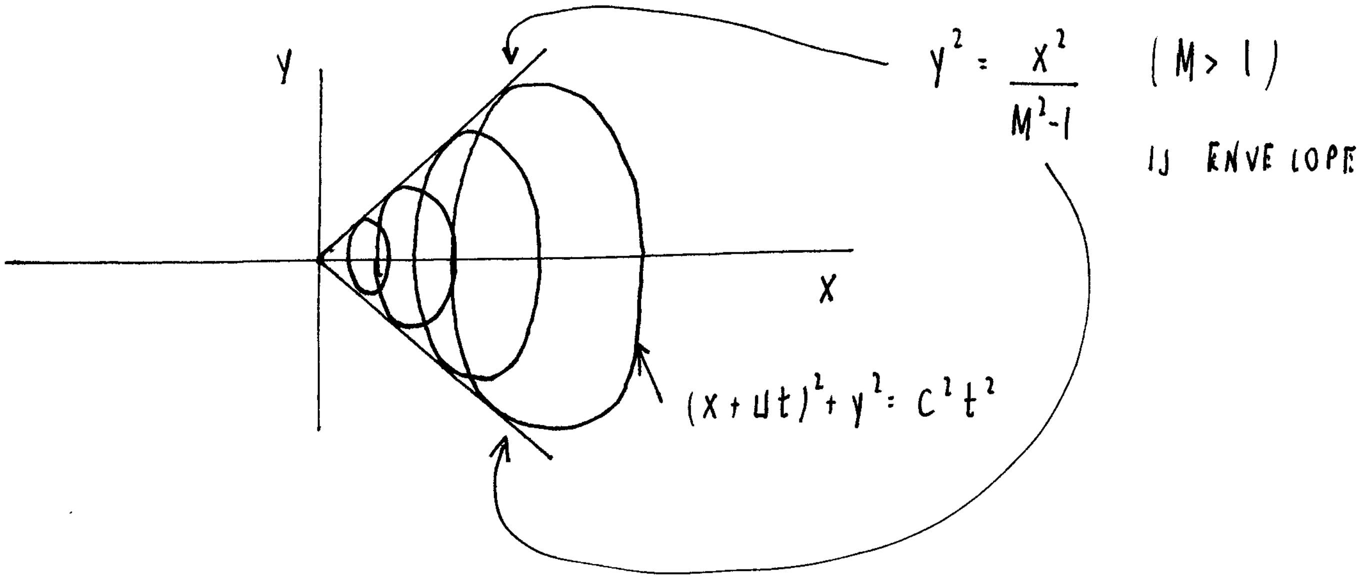
WE SOLVE TO GET

$$y^2+z^2=\frac{x^2}{M^2-1} \qquad M=\frac{U}{c} \text{ MACH NUMBER}$$

$M<1$ SUBSONIC

$M>1$ SUPERSONIC

IF $M > 1$ THEN THE ENVELOPE OF THE FAMILY OF CURVES IS SIMPLY A CONE (CALLED A MACH CONE) WITH SEMI-ANGLE $\varphi = \sin^{-1}(1/M)$. THE PICTURE IN THE 2-D CASE IS:



NOTICE THAT WHEN $M < 1$ SUBSONIC, THEN $y^2 + z^2 = \dfrac{x^2}{M^2-1} < 0$.

HENCE THERE IS NO ENVELOPE IN THE SUBSONIC CASE.