LAPLACE TRANSFORMS

THIS IS VERY USEFUL FOR AN INITIAL VALUE PROBLEM WITH INITIAL CONDITIONS GIVEN AT $t=0$. LET $F(t)$ BE PIECEWISE CONTINUOUS WITH

$$|f(t)| \leq K e^{\alpha t} \quad \forall t \quad (\text{possibly } \alpha > 0).$$

THEN DEFINE
$$\hat{F}(s) = \mathcal{L}(F(t)) = \int_0^\infty e^{-st} f(t)\, dt.$$

WE HAVE THAT $\hat{F}(s)$ IS <u>ANALYTIC</u> FOR $Re(s) > \alpha$. WE RECALL SOME BASIC PROPERTIES:

(i) $\mathcal{L}(F'') = s^2 \mathcal{L}(F) - F'(0) - sF(0)$, $\quad \mathcal{L}(F') = s\mathcal{L}(F) - F(0)$

(ii) $\mathcal{L}(F(t)e^{at}) = F(s-a)$ WITH $F(s) = \mathcal{L}(F(t))$.

(iii) $\mathcal{L}(t\, f(t)) = -F'(s)$ WITH $F(s) = \mathcal{L}(f(t))$

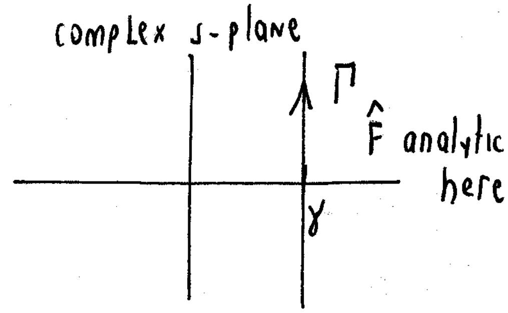
(iv) $\mathcal{L}(H_\lambda(t)) = \dfrac{e^{-s\lambda}}{s}$ WITH $H_\lambda(t) = \begin{cases} 0, & 0 \leq t < \lambda \\ 1, & t \geq \lambda \end{cases}$ AND $\lambda > 0$.

(V) <u>CONVOLUTION</u> $\mathcal{L}[g * h] = \hat{G}(s)\hat{H}(s)$ $\quad \hat{G}(s) = \mathcal{L}(g(t))$, $\hat{H}(s) = \mathcal{L}(h(t))$

AND $g * h = \int_0^t g(t-\tau)\, h(\tau)\, d\tau = \int_0^t g(\tau) h(t-\tau)\, d\tau.$

(Vi) <u>INVERSION FORMULA</u>

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \hat{F}(s) e^{st}\, ds$$



CALCULATING INVERSE LAPLACE TRANSFORMS REQUIRES COMPLEX VARIABLE THEORY ⟶ M301

(SEE APPENDIX A)

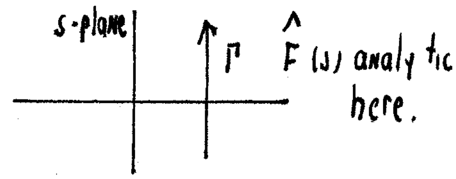
APPENDIX A (CULTURAL)

WE WANT TO DERIVE THE LAPLACE TRANSFORM PAIR

$$\hat{F}(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt.$$

WITH $|f(t)| \le K e^{\alpha t}$ $\forall t$, THEN $\hat{F}(s)$ IS ANALYTIC IN $\mathrm{RE}(s) > \alpha$. THEN,

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \hat{F}(s) e^{st}\,ds \quad \text{WITH } \gamma > \alpha.$$



DERIVATION LET $\mathcal{U}(t) = \begin{cases} f(t) & t \ge 0 \\ 0 & t < 0 \end{cases}$ IN FOURIER TRANSFORM.

THEN SINCE $f(t)$ COULD GROW AT ∞, i.e. $|f(t)| \le K e^{\alpha t}$ WE LET

$$\mathcal{U}_\gamma(t) \equiv \begin{cases} f(t) e^{-\gamma t}, & t \ge 0 \\ 0, & t < 0 \end{cases} \quad \text{FOR ANY } \gamma > \alpha, \text{ SO } \int_0^\infty |f(t)| e^{-\gamma t}\,dt < \infty$$

THEN THE FOURIER TRANSFORM OF $\mathcal{U}_\gamma(t)$ IS $\hat{\mathcal{U}}_\gamma(K) = \int_0^\infty f(t) e^{-\gamma t} e^{-iKt}\,dt$.

THEN THE INVERSE TRANSFORM IS

$$\mathcal{U}_\gamma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\mathcal{U}}_\gamma(K) e^{iKt}\,dK = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^\infty f(t') e^{-\gamma t'} e^{-iKt'}\,dt'\right) e^{iKt}\,dK.$$

NOW LET $s = \gamma + iK$ BE A NEW VARIABLE TO REPLACE K. THEN $dK = ds/i$ AND

$$\mathcal{U}_\gamma(t) = \frac{1}{2\pi}\int_{-i\infty+\gamma}^{i\infty+\gamma}\left(\int_0^\infty f(t') e^{-st'}\,dt'\right) e^{(s-\gamma)t}\,\frac{ds}{i} = \frac{e^{-\gamma t}}{2\pi i}\int_{-\infty}^{\infty}\left(\int_0^\infty f(t') e^{-st'}\,dt'\right) e^{st}\,ds.$$

THUS IF WE RECALL $\mathcal{U}_\gamma(t) = f(t) e^{-\gamma t}$ FOR $t > 0$, THEN CANCELLING $e^{-\gamma t}$ GIVES

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} \hat{F}(s)\, e^{st}\,ds \quad \text{WITH } \hat{F}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

AND γ CHOSEN WITH $\gamma > \alpha$; RECALL $\hat{F}(s)$ ANALYTIC FOR $\mathrm{RE}(s) > \alpha$.

BRIEF TABLE OF LAPLACE TRANSFORMS

(i) $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$ FOR $n = 0, 1, 2, \ldots$ (ii) $\mathcal{L}(\cos \omega t) = \dfrac{s}{s^2+\omega^2}$

(iii) $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$ (iv) $\mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2}$

(v) $\mathcal{L}[t^k] = \dfrac{\Gamma(k+1)}{s^{k+1}}$ FOR k REAL AND $k > -1$. $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$

(vi) $\mathcal{L}\left[\int_0^t F(\tau)\, d\tau\right] = \dfrac{1}{s} \mathcal{F}(s)$, $\mathcal{F}(s) = \mathcal{L}[F(t)]$. " GAMMA FUNCTION. $\Gamma(1/2) = \sqrt{\pi}$

COMMON LAPLACE TRANSFORMS FOR DIFFUSION EQUATIONS ARE

(vi) $\mathcal{L}\left[\operatorname{ERFC}\left(\lambda / 2\sqrt{t}\right)\right] = \dfrac{e^{-\lambda\sqrt{s}}}{s}$ FOR $\lambda \geq 0$

WHERE $\operatorname{ERFC}(z) \equiv \dfrac{2}{\sqrt{\pi}} \int_z^\infty e^{-\zeta^2}\, d\zeta$. $\operatorname{ERFC}(\infty) = 0$, $\operatorname{ERFC}(0) = 1$

" COMPLEMENTARY ERROR FUNCTION"

RECALL: $\operatorname{ERF}(z) \equiv \dfrac{2}{\sqrt{\pi}} \int_0^z e^{-\zeta^2}\, d\zeta$ " ERROR FUNCTION", $\operatorname{ERF}(0) = 0$, $\operatorname{ERF}(\infty) = 1$

AND $\operatorname{ERF}(z) + \operatorname{ERFC}(z) = 1 \quad \forall z \geq 0$.

(vii) $\mathcal{L}\left[\dfrac{\lambda}{2\sqrt{\pi}\, t^{3/2}} e^{-\lambda^2/4t}\right] = e^{-\lambda\sqrt{s}}$, FOR $\lambda > 0$.

(viii) $\mathcal{L}\left[\dfrac{1}{\sqrt{t}} e^{-\lambda^2/4t}\right] = \left(\dfrac{\pi}{s}\right)^{1/2} e^{-\lambda\sqrt{s}}$ FOR $\lambda > 0$

(ix) $\mathcal{L}\left[-e^{ak} e^{a^2 t} \operatorname{ERFC}\left(a\sqrt{t} + \dfrac{k}{2\sqrt{t}}\right) + \operatorname{ERFC}\left(\dfrac{k}{2\sqrt{t}}\right)\right] = \dfrac{a e^{-k\sqrt{s}}}{s(a+\sqrt{s})}$

FOR $k \geq 0$.

(x) $\mathcal{L}\left[\dfrac{1}{a} e^{a^2 t} \operatorname{ERF}(a\sqrt{t})\right] = \dfrac{1}{\sqrt{s}(s-a^2)}$.

PROBLEM 1 CONSIDER THE DIFFUSION PROBLEM FOR $u(x,t)$:

$$u_t = D u_{xx}, \quad 0 < x < \infty, \quad t > 0$$

$$u(0,t) = h(t); \quad u(x,0) = 0; \quad u \to 0 \text{ AS } x \to \infty \text{ FOR } t \text{ FIXED.}$$

HERE $D > 0$ IS A CONSTANT.

(i) FOR ARBITRARY $h(t)$ FIND $u(x,t)$ IN TERMS OF AN INTEGRAL.

(ii) IF $h(t) = 1$ FIND A COMPACT FORMULA FOR $u(x,t)$ AND CALCULATE THE FLUX, WITH $u_x = u_x(t,\tau)$.

$$\mathcal{F}(t) \equiv \int_0^t \left[-D u_x \right]\Big|_{x=0} d\tau$$

SOLUTION

(i) LET $\hat{u}(x,s) = \mathcal{L}[u(x,t)]$. THEN $\mathcal{L}[u_t] = D\mathcal{L}[u_{xx}]$

THIS GIVES $s\hat{u} - u(x,0) = D\hat{u}_{xx}$ WITH $\hat{u}(0,s) = \hat{H}(s) = \mathcal{L}[h(t)]$.

WE MUST SOLVE:

$$(+)\quad \begin{cases} \hat{u}_{xx} - \dfrac{s}{D}\hat{u} = 0, & 0 < x < \infty \\ \hat{u}(0,s) = \hat{H}(s), & \hat{u} \text{ BOUNDED AS } x \to \infty \text{ FOR } s > 0. \end{cases}$$

SINCE $\hat{u} = \text{SPAN}\{e^{-\sqrt{s/D}\,x}, e^{\sqrt{s/D}\,x}\}$ WE CHOOSE DECAYING EXPONENTIAL TO GET $\hat{u} = \hat{H}(s)\, e^{-\sqrt{s/D}\,x}$.

NOW INVERT THE TRANSFORM USING CONVOLUTION PROPERTY.

SINCE $\mathcal{L}^{-1}\left[e^{-\lambda\sqrt{s}}\right] = \dfrac{\lambda}{2\sqrt{\pi}\,t^{3/2}} e^{-\lambda^2/4t}$, WE SET $\lambda = x/\sqrt{D}$

SO THAT $\mathcal{L}^{-1}\left[e^{-x\sqrt{s/D}}\right] = \dfrac{x}{2D\sqrt{\pi}\,t^{3/2}} e^{-x^2/4Dt}$, $\mathcal{L}^{-1}[\hat{H}(s)] = h(t)$.

SO BY CONVOLUTION

$$u(x,t) = \int_0^t \frac{h(t-\tau)}{2\sqrt{\pi D}} \frac{x}{\tau^{3/2}} e^{-x^2/4D\tau}\, d\tau. \quad (*)$$

(ii) IF $h(t) \equiv 1$ WE CAN USE (*).

$$u(x,t) = \left(\int_0^t \frac{1}{\tau^{3/2}} e^{-x^2/4D\tau} \, d\tau \right) \frac{x}{2\sqrt{\pi D}}.$$

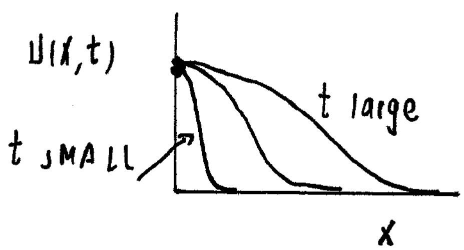
SET $\zeta \equiv x/2\sqrt{D\tau}$ SO $\sqrt{\tau} = \dfrac{x}{2\zeta\sqrt{D}}$. $\rightarrow \frac{1}{2}\tau^{-1/2} d\tau = -\dfrac{x}{2\zeta^2\sqrt{D}} d\zeta.$

SO $$\tau^{-3/2} d\tau = \frac{-x/\zeta^2\sqrt{D}}{\tau} d\zeta = -\frac{x}{\zeta^2\sqrt{D}} \frac{4\zeta^2 D}{x^2} d\zeta = -\frac{4\sqrt{D}}{x} d\zeta.$$

THIS GIVES, $$u(x,t) = \left(\frac{x}{2\sqrt{\pi D}}\right)\left(-\frac{4\sqrt{D}}{x}\right) \int_\infty^{x/2\sqrt{Dt}} e^{-\zeta^2} d\zeta = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{Dt}}^{\infty} e^{-\zeta^2} d\zeta.$$

WE IDENTIFY, $$u(x,t) = \text{ERFC}\left(x/2\sqrt{Dt} \right)$$

WHERE ERFC(Z) IS COMPLEMENTARY ERROR FUNCTION.



__REMARK__ AN EASIER WAY TO SOLVE THIS IS TO RETURN TO THE BVP (+) ON PREVIOUS PAGE. IF $h(t) = 1$, $\mathcal{L}(h(t)) = \frac{1}{s}$ SO $\hat{u}$ SATISFIES

$$\hat{u}_{xx} - \frac{s}{D}\hat{u} = 0, \quad 0 < x < \infty$$

$$\hat{u}(0,s) = \frac{1}{s}$$

$$\rightarrow \hat{u}(x,s) = \frac{1}{s} e^{-\sqrt{s/D}\,x} \rightarrow \text{USING } \mathcal{L}^{-1}\left[\frac{e^{-\lambda\sqrt{s}}}{s}\right] = \text{ERFC}\left(\frac{\lambda}{2\sqrt{t}}\right)$$

WITH $\lambda = x/\sqrt{D} \geq 0$ WE GET $u(x,t) = \text{ERFC}\left(x/2\sqrt{Dt} \right)$.

(iii) NOW CALCULATE THE FLUX INTO $X=0$, WHEN $h(t)=1$

$$\mathcal{F}(t) \equiv \int_0^t [-D\, U_x]\Big|_{x=0} \, d\tau.$$

LET $g(\tau) \equiv -D\, U_x(0,\tau)$.

WE HAVE
$$\mathcal{L}[\mathcal{F}(t)] = \frac{\mathcal{L}[g(t)]}{S} = \frac{-D\, \hat{U}_x(0,S)}{S}.$$

NOW SINCE $\hat{U}(X,S) = \frac{1}{S} e^{-\sqrt{S/D}\, X}$ SO $\hat{U}_x(0,S) = -\frac{1}{\sqrt{D}\, S^{1/2}}$.

THUS
$$\mathcal{L}[\mathcal{F}(t)] = \frac{-D\left[\frac{-1}{\sqrt{D}\, S^{1/2}}\right]}{S} = \frac{\sqrt{D}}{S^{3/2}}.$$

NOW USING
$$\mathcal{L}^{-1}\left[\frac{\Gamma(K+1)}{S^{K+1}}\right] = t^K \quad \text{FOR } K = 1/2$$

WE GET
$$\mathcal{L}^{-1}\left[\frac{1}{S^{3/2}}\right] = \frac{t^{1/2}}{\Gamma(3/2)}$$

THUS
$$\mathcal{F}(t) = \mathcal{L}^{-1}\left[\frac{\sqrt{D}}{S^{3/2}}\right] = \frac{\sqrt{Dt}}{\Gamma(3/2)} \qquad \text{CLAIM } \Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\sqrt{\pi}.$$

THE INTEGRAL OF TOTAL FLUX DIVERGES AS $t \to \infty$ LIKE $\approx C\sqrt{Dt}$ WHERE C IS A CONSTANT. IN FACT $\mathcal{F}(t) = 2\sqrt{Dt}/\sqrt{\pi}$.

<u>REMARK</u> $\Gamma(z+1) = \int_0^\infty t^z e^{-t}\, dt$. USE IBP WITH $u = t^z$, $dv = -e^{-t}\, dt$.

SO
$$\Gamma(z+1) = \underbrace{-t^z e^{-t}\Big|_0^\infty}_{(=0 \text{ IF } z>0)} + z\int_0^\infty t^{z-1} e^{-t}\, dt = z\,\Gamma(z) \quad \text{IF } z > 0.$$

SO $\Gamma(z+1) = z\,\Gamma(z)$. THUS $\Gamma(3/2) = \frac{1}{2}\,\Gamma(1/2)$.

$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t}\, dt$. LET $t = \zeta^2 \to dt = 2\zeta\, d\zeta$ SO $\Gamma(1/2) = 2\int_0^\infty e^{-\zeta^2}\, d\zeta$

$$\to \Gamma(1/2) = \sqrt{\pi}\left(\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\zeta^2}\, d\zeta\right) = \sqrt{\pi}\ \text{ERF}(\infty) = \sqrt{\pi}.$$

EXAMPLE DIFFUSION AND WAVE PROPAGATION OUTSIDE A SPHERE.

SOLVE EACH OF THE FOLLOWING:

(I) $$U_t = D\left[U_{rr} + \frac{2}{r} U_r \right] \quad \text{IN} \quad r \geq a,\ t>0$$

$$U(a,t) = 1, \quad U \text{ BOUNDED AS } r \to \infty$$

$$U(r,0) = 0.$$

(II) $$U_{tt} = c^2 \left[U_{rr} + \frac{2}{r} U_r \right] \quad \text{IN} \quad r \geq a,\ t>0$$

$$U(a,t) = 1, \quad U \text{ BOUNDED AS } r \to \infty$$

$$U(r,0) = 0, \quad U_t(r,0) = 0.$$

SOLUTION I WE DEFINE $\hat{U}(r,s) = \int_0^\infty e^{-st} U(r,t)\, dt$. TAKING THE LT WE GET

$$s\hat{U} - U(r,0) = D\left[\hat{U}_{rr} + \frac{2}{r} \hat{U}_r \right]$$

$$\hat{U}(a,s) = 1/s, \quad \hat{U} \text{ BOUNDED AS } r \to \infty.$$

THIS GIVES

$$(*) \begin{cases} \hat{U}_{rr} + \dfrac{2}{r} \hat{U}_r - \dfrac{s}{D} \hat{U} = 0, \quad r \geq a,\ t>0. \\ \hat{U}(a,s) = \dfrac{1}{s} \quad \hat{U} \text{ BOUNDED AS } r \to \infty. \end{cases}$$

PUT $\hat{U} = \frac{f(r)}{r} \rightarrow \quad f'' - \frac{s}{D} f = 0 \rightarrow f = Ae^{-\sqrt{\frac{s}{D}}\, r}.$

SO $\hat{U} = A\frac{e^{-\sqrt{s/D}\, r}}{r}$. WHEN $r = a$ WE HAVE $\hat{U} = \frac{1}{s} = \frac{Ae^{-\sqrt{\frac{s}{D}}\, a}}{a}$

THIS GIVES $$\hat{U} = \left(\frac{a}{r}\right) \frac{e^{-\sqrt{s/D}\,(r-a)}}{s}.$$

NOW RECALLING $\mathcal{L}^{-1}\left[\frac{e^{-\lambda\sqrt{s}}}{s} \right] = \text{ERFC}\left(\lambda/2\sqrt{t} \right)$ FOR $\lambda \geq 0$

WE SET $\lambda = (r-a)/\sqrt{D}$ SO THAT

$$U(r,t) = \left(\frac{a}{r}\right) \text{ERFC}\left(\frac{(r-a)}{2\sqrt{Dt}} \right) \quad \text{FOR } r \geq a,\ t>0$$

SOLUTION II WE PROCEED IN EXACTLY THE SAME WAY. RECALL

$$\mathcal{L}[u_{tt}] = s^2 \mathcal{L}[u] - u_t(x,0) - s\,u(x,0) \quad \text{WITH} \quad u(x,0) = u_t(x,0) = 0$$

WE GET THAT (*) NOW BECOMES

$$\begin{cases} \hat{U}_{rr} + \dfrac{2}{r}\hat{U}_r - \dfrac{s^2}{c^2}\hat{U} = 0 & \text{IN } r \geq a,\ t > 0 \\ \hat{U}(a,s) = 1/s, \quad \hat{U} \text{ BOUNDED AS } r \to \infty. \end{cases}$$

THE SOLUTION IS

$$\hat{U}(r,s) = \left(\frac{a}{r}\right) \frac{e^{-s/c\,(r-a)}}{s} \quad \text{IN } r \geq a.$$

NOW RECALL

$$\mathcal{L}^{-1}\left[e^{-s\lambda}/s\right] = H_\lambda(t) = \begin{cases} 0, & \text{IF } 0 \leq t < \lambda \\ 1, & \text{IF } t \geq \lambda \end{cases} \quad \text{WITH } \lambda \geq 0.$$

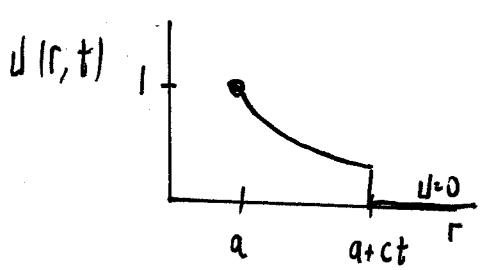
SET $\lambda = (r-a)/c$ SO THAT

$$u(r,t) = \left(\frac{a}{r}\right) H_{\frac{(r-a)}{c}}(t) = \left(\frac{a}{r}\right) \begin{cases} 0 & \text{IF } 0 \leq t < \frac{r-a}{c} \\ 1 & \text{IF } t \geq \frac{r-a}{c}, \quad r \geq a \end{cases}$$

THIS IS EQUIVALENT TO WRITING

$$u(r,t) = \left(\frac{a}{r}\right) \begin{cases} 0 & \text{IF } r > a + ct \\ 1 & \text{IF } a \leq r \leq a + ct. \end{cases}$$

WE CONCLUDE THAT $u = 0$ "AHEAD" OF THE "WAVE" TRAVELLING AT SPEED C.



→ FINITE-SPEED OF PROPAGATION OF "SIGNALS".

EXAMPLE (HEAT KERNEL IN 1-D)

FIND THE SOLUTION $u(x,t)$ TO

$$u_t = D u_{xx}, \quad -\infty < x < \infty, \ t > 0, \quad D \text{ CONSTANT}$$
$$u(x,0) = \delta(x); \quad u \to 0 \text{ AS } |x| \to \infty \text{ FOR EACH } t > 0.$$

SOLUTION WE TAKE LAPLACE TRANSFORM $\hat{u}(x,s) = \int_0^\infty e^{-st} u(x,t)\, dt$ TO GET $s\hat{u} - u(x,0) = D\hat{u}_{xx}$. THUS $\hat{u}(x,s)$ SOLVES

$$(1) \quad \begin{cases} D\hat{u}_{xx} - s\hat{u} = -\delta(x), & -\infty < x < \infty \\ \hat{u} \to 0 \text{ AS } |x| \to \infty. \end{cases}$$

NOW FOR $x \neq 0$, $\hat{u} = \text{SPAN}\{ e^{-\sqrt{s/D}\,x}, e^{\sqrt{s/D}\,x} \}$. WE WANT DECAYING EXPONENTIALS FOR $x \to \infty$ AND $x \to -\infty$. THE "CONTINUITY" CONDITIONS ACROSS $x=0$ ARE $\hat{u}$ IS CONTINUOUS AND $D\hat{u}_x\big|_{x=0^+} - D\hat{u}_x\big|_{x=0^-} = -1$. TO SEE THIS SECOND CONDITION WE LET $\varepsilon > 0$ AND INTEGRATE

$$\int_{-\varepsilon}^{\varepsilon} (D\hat{u}_{xx} - s\hat{u})\, dx = -\int_{-\varepsilon}^{\varepsilon} \delta(x)\, dx = -1.$$

→ LET $\varepsilon \to 0^+$ AND USE $\hat{u}$ CONTINUOUS TO GET $D\hat{u}_x\big|_{0^+} - D\hat{u}_x\big|_{0^-} = -1$.

SO USING DECAY AS $|x| \to \infty$ AND ENFORCING CONTINUITY OF $\hat{u}$ AT $x=0$ GIVES

$$\hat{u}(x,s) = \begin{cases} A e^{\sqrt{s/D}\,x} & \text{IF } x < 0 \\ A e^{-\sqrt{s/D}\,x} & \text{IF } x > 0. \end{cases}$$

USING JUMP CONDITION $D\hat{u}_x\big|_{0^+} - D\hat{u}_x\big|_{0^-} = -1 \to \left(-2\sqrt{\frac{s}{D}}\, A\right) D = -1 \to A = \frac{1}{2\sqrt{D}\sqrt{s}}$.

$$\text{SO} \quad \hat{u}(x,s) = \frac{1}{2\sqrt{D}\sqrt{s}}\, e^{-\sqrt{s/D}\,|x|}.$$

RECALLING $\mathcal{L}^{-1}[s^{-1/2} e^{-\Delta\sqrt{s}}] = \frac{1}{\sqrt{\pi t}}\, e^{-\Delta^2/4t}$ WE SET $\Delta = |x|/\sqrt{D}$ TO GET

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}}\, e^{-x^2/4Dt}$$

"GAUSSIAN PROFILE → SOURCE SOLUTION OF HEAT EQUATION".

IN FACT, IT FOLLOWS THAT $\int_{-\infty}^{\infty} u(x,t)\, dx = 1 \quad \forall\, t \geq 0$.

REMARK IF $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ THEN

large s behavior of $F(s) \longleftrightarrow$ small t behavior of $f(t)$.

WE NOW USE THIS QUALITATIVE FACT IN AN EXAMPLE.

PROBLEM 3 CONSIDER

$$u_t = D\,u_{xx}, \quad 0 < x < 1,\ t > 0,\ D > 0 \text{ CONSTANT}$$

$$u_x(0,t) = 0, \quad u(1,t) = 1, \quad u(x,0) = 0$$

A STRAIGHTFORWARD SEPARATION OF VARIABLES SOLUTION GIVES

$$(*) \quad u(x,t) = 1 + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1/2} \cos\left[\left(k+\tfrac{1}{2}\right)\pi x\right] e^{-D(k+1/2)^2 \pi^2 t}$$

NOTICE THAT IF t IS LARGE ONLY A FEW TERMS IN THIS INFINITE SERIES ARE NEEDED TO APPROXIMATE $u(x,t)$ RATHER ACCURATELY, HOWEVER, IF t IS SMALL WE NEED <u>MANY</u> TERMS IN THE INFINITE SUM IN $(*)$. CAN WE GET ANOTHER REPRESENTATION FOR $u(x,t)$ THAT <u>CONVERGES QUICKLY</u> WHEN t IS SMALL? YES, USE LAPLACE TRANSFORMS.

DEFINE $\quad U(x,s) = \mathcal{L}(u(x,t))$.

THEN $\quad U_{xx} - \frac{s}{D} U = 0, \quad 0 < x < 1$

$$U_x = 0 \text{ ON } x = 0, \quad U = \frac{1}{s} \text{ ON } x = 1$$

THE SOLUTION IS $\quad U = A \cosh\left[\sqrt{\frac{s}{D}}\, x\right] \qquad A = \frac{1}{s \cosh\left(\sqrt{\frac{s}{D}}\right)}$

THEN $$\bar{U}(x,s) = \frac{\cosh\left(\sqrt{\frac{s}{D}}\,x\right)}{s\cosh\left(\sqrt{\frac{s}{D}}\right)}$$

NOW INVERT THE TRANSFORM. USE $\cosh y = (e^{y} + e^{-y})/2$.

SO $$\bar{U}(x,s) = \frac{e^{\sqrt{s/D}\,x} + e^{-\sqrt{s/D}\,x}}{s\left[e^{\sqrt{s/D}} + e^{-\sqrt{s/D}}\right]} = \frac{1}{s} e^{-\sqrt{s/D}\,(1-x)} \left(\frac{1 + e^{-2\sqrt{s/D}\,x}}{1 + e^{-2\sqrt{s/D}}}\right)$$

NOW FOR $|z| < 1$ $$\frac{1}{1+z} = 1 - z + z^2 - z^3 \ldots = \sum_{n=0}^{\infty} (-1)^n z^n.$$

LET $z = e^{-2\sqrt{s/D}}$ SO THAT $$\frac{1}{1 + e^{-2\sqrt{s/D}}} = \sum_{n=0}^{\infty} (-1)^n e^{-2n\sqrt{s/D}}$$

THEN $$\bar{U}(x,s) = \frac{e^{-\sqrt{s/D}\,(1-x)}}{s} \left(1 + e^{-2\sqrt{s/D}\,x}\right) \sum_{n=0}^{\infty} (-1)^n e^{-2n\sqrt{s/D}}$$

WE WRITE THIS AS
$$\bar{U}(x,s) = \frac{1}{s}\left(\sum_{n=0}^{\infty} (-1)^n e^{-\sqrt{\frac{s}{D}}\,(2n+1-x)} + \sum_{n=0}^{\infty} (-1)^n e^{-\sqrt{\frac{s}{D}}\,(1+2n+x)}\right)$$

NOW INVERT EACH TERM USING $\mathcal{L}^{-1}\left[\frac{e^{-A\sqrt{s}}}{s}\right] = \mathrm{ERFC}\left(A/2\sqrt{t}\right)$.

THIS GIVES
$$(+) \qquad u(x,t) = \sum_{n=0}^{\infty} (-1)^n \left[\mathrm{ERFC}\left(\frac{2n+1-x}{2\sqrt{Dt}}\right) + \mathrm{ERFC}\left(\frac{2n+1+x}{2\sqrt{Dt}}\right)\right]$$

THIS IS AN EXACT SOLUTION AND THUS SINCE THE SOLUTION IS UNIQUE, (+) AND (*) (EIGENFUNCTION SOLUTION) MUST BE EQUIVALENT. SINCE $\mathrm{ERFC}(y) \to 0$ AS $y \to \infty$ THEN ONLY A FEW TERMS IN (+) ARE NEEDED TO CALCULATE $u(x,t)$ ACCURATELY WHEN t IS SMALL.

NOTICE THAT THE LEADING TERM IN (+) IS TO TAKE $n=0$ AND THEN TAKE THE ERFC WHOSE ARGUMENT COULD VANISH AT $X=1$.

HENCE, $u(x,t) \approx \operatorname{erfc}\left(\frac{1-x}{2\sqrt{Dt}}\right)$ FOR t SMALL.

THIS APPROXIMATION CAN BE OBTAINED VERY QUICKLY FROM

$$\bar{U}(x,s) = \frac{\cosh\left(\sqrt{\frac{s}{D}}\,x\right)}{s\cosh\left(\sqrt{\frac{s}{D}}\right)}.$$

LET $s \to \infty$ (t SMALL) TO OBTAIN

$$\bar{U}(x,s) \approx \frac{1}{s}\,\frac{e^{\sqrt{s/D}\,x}}{e^{1\sqrt{s/D}}} = \frac{1}{s}\,e^{-\sqrt{s/D}\,(1-x)}$$

LET $\lambda = \frac{(1-x)}{\sqrt{D}} \geq 0$ ON $0<x<1$. THEN WITH $\mathcal{L}^{-1}\left[\frac{e^{-\lambda\sqrt{s}}}{s}\right] = \operatorname{erfc}\left(\frac{\lambda}{2\sqrt{t}}\right)$

WE OBTAIN $u(x,t) \approx \operatorname{erfc}\left(\frac{1-x}{2\sqrt{Dt}}\right)$ FOR t SMALL.