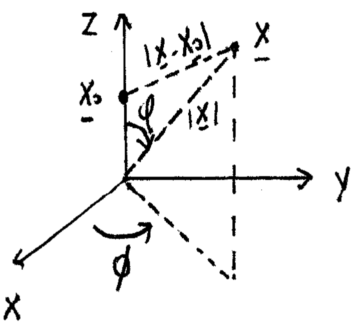
APPENDIX A

DERIVATION OF GENERATING FUNCTION

LET $\underline{X}_0$ BE SOME FIXED POINT IN $\mathbb{R}^3$. THEN $1/|\underline{X}-\underline{X}_0|$ IS A HARMONIC FUNCTION IN $\mathbb{R}^3$ SO THAT $\Delta\left(\frac{1}{|\underline{X}-\underline{X}_0|}\right) = 0$. THIS FOLLOWS SINCE IF WE LET $p = |\underline{X}-\underline{X}_0|$ THEN $\Delta(1/p) \equiv (1/p)'' + \frac{2}{p}(1/p)' = 0$.

NOW WITHOUT LOSS OF GENERALITY LET $\underline{X}_0$ BE ALIGNED WITH POSITIVE Z-AXIS. WE TAKE $\underline{X}_0$ TO BE A UNIT VECTOR. WE GET THE PICTURE SHOWN WITH $r = |\underline{X}|$:



$$
\begin{aligned}
\text{NOW } |\underline{X}-\underline{X}_0|^2 &= (\underline{X}-\underline{X}_0)^T \cdot (\underline{X}-\underline{X}_0) \\
&= \underline{X}^T\underline{X} - 2\,\underline{X}^T\underline{X}_0 + \underline{X}_0^T\underline{X}_0 \\
&= r^2 - 2r\cos\varphi + 1
\end{aligned}
$$

(NOTE: $\underline{X}^T\underline{X} = |\underline{X}|^2 = r^2$, $\underline{X}^T\underline{X}_0 = \underline{X}\cdot\underline{X}_0 = |\underline{X}||\underline{X}_0|\cos\varphi = r\cos\varphi$, $\underline{X}_0^T\underline{X}_0 = |\underline{X}_0|^2 = 1$

THUS $\frac{1}{|\underline{X}-\underline{X}_0|} = \frac{1}{(r^2-2r\cos\varphi+1)^{1/2}}$ IS A SOLUTION TO LAPLACE'S EQUATION IN 3-D WITH AZIMUTHAL SYMMETRY. (INDEPENDENT OF ANGLE ϕ).

HENCE FOR $r \neq 1$, $\varphi \neq 0$, WE MUST HAVE FOR SOME A_n THAT

$$\frac{1}{(r^2-2r\cos\varphi+1)^{1/2}} = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\varphi).$$

FOR SOME CONSTANTS A_n (INDEPENDENT OF φ AND r). TO FIND A_n SET $\varphi = 0$ AND TAKE $r<1$. THEN $(r^2-2r\cos\varphi+1)^{-1/2} = (r^2-2r+1)^{-1/2} = (1-r)^{-1}$.

HENCE $\frac{1}{1-r} = \sum_{n=0}^{\infty} A_n P_n(1) r^n$. BUT $P_n(1) = 1 \rightarrow \frac{1}{1-r} = \sum_{n=0}^{\infty} A_n r^n \rightarrow A_n = 1, \forall n$

THUS WE HAVE

$$\frac{1}{(r^2-2r\cos\varphi+1)^{1/2}} = \sum_{n=0}^{\infty} r^n P_n(\cos\varphi).$$

NOW LET $x=\cos\varphi$, $t=r$ WITH $0<t<1$ WE GET THE <u>GENERATING FUNCTION</u> FOR LEGENDRE POLYNOMIALS GIVEN BY

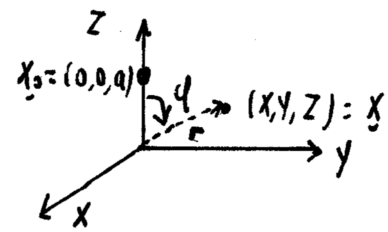
$$(*) \qquad \frac{1}{(t^2-2tx+1)^{1/2}} = \sum_{n=0}^{\infty} t^n P_n(x). \qquad \square$$

<u>EXAMPLE</u> SHOW THAT THE ELECTROSTATIC POTENTIAL FOR A POINT CHARGE AT $\underline{x}_0=(0,0,a)$ ON THE Z-AXIS CAN BE EXPANDED AS

$$\frac{1}{|\underline{x}-\underline{x}_0|} = \frac{1}{a}\sum_{n=0}^{\infty} P_n(\cos\varphi)\left(\frac{r}{a}\right)^n \quad \text{WITH } r=|\underline{x}| \text{ AND } 0<r<a.$$

TO DERIVE THIS WE REFER TO THE PICTURE BELOW AND CALCULATE

$$|\underline{x}-\underline{x}_0|^2 = x^2+y^2+(z-a)^2 = x^2+y^2+z^2-2az+a^2$$



BUT $z=r\cos\varphi$ WITH φ LATTITUDE AND $x^2+y^2+z^2=r^2$. THEREFORE

$$|\underline{x}-\underline{x}_0|^2 = r^2+a^2-2ar\cos\varphi.$$

SO

$$\frac{1}{|\underline{x}-\underline{x}_0|} = \frac{1}{\sqrt{r^2+a^2-2ar\cos\varphi}} = \frac{1}{a\sqrt{\frac{r^2}{a^2}-\frac{2r}{a}\cos\varphi+1}}.$$

NOW LET $t=r/a<1$ AND LABEL $x=\cos\varphi$ (THIS IS NOT THE COORDINATE x !!)

THEN $\dfrac{1}{|\underline{x}-\underline{x}_0|} = \dfrac{1}{a\sqrt{t^2-2tx+1}}$. BY USING THE GENERATING FUNCTION IN $(*)$ WE CONCLUDE THAT

$$\frac{1}{|\underline{x}-\underline{x}_0|} = \frac{1}{a}\sum_{n=0}^{\infty} P_n(\cos\varphi)\left(\frac{r}{a}\right)^n \quad \text{WHICH CONVERGES IF } 0<r<a.$$

THE GENERATING FUNCTION IN (*) IS USEFUL FOR DERIVING MANY RESULTS FOR LEGENDRE POLYNOMIALS.

<u>EXAMPLE</u> USE THE GENERATING FUNCTION TO PROVE THAT

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1} \quad \text{FOR } n=0,1,2..$$

<u>PROOF</u> WE BEGIN WITH

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n \quad \text{FOR } 0<t<1.$$

SQUARE BOTH SIDES

$$\frac{1}{1-2xt+t^2} = \Big(\sum_{m=0}^{\infty} P_m(x)\,t^m\Big)\Big(\sum_{n=0}^{\infty} P_n(x)\,t^n\Big) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)\,P_n(x)\,t^{m+n}.$$

NOW INTEGRATE WRT x AND USE ORTHOGONALITY $\int_{-1}^{1} P_m P_n\,dx = 0$ IF $m \neq n$.

THIS GIVES

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} t^{m+n}\int_{-1}^{1} P_m P_n\,dx = \sum_{n=0}^{\infty}\Big(\int_{-1}^{1} P_n^2\,dx\Big)t^{2n}. \qquad (+)$$

NOW CALCULATE THE LEFT SIDE:

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \frac{1}{-2t}\int_{-1}^{1} \frac{dx}{\left[x-\frac{(1+t^2)}{2t}\right]} = -\frac{1}{2t}\log\left[\,\left|x-\frac{(1+t^2)}{2t}\right|\,\right]\Big|_{-1}^{1}$$

$$= -\frac{1}{2t}\left[\log\left|1-\frac{(1+t^2)}{2t}\right| - \log\left|-1+\frac{(1+t^2)}{2t}\right|\right] = -\frac{1}{2t}\left[\log\left|\frac{(1-t)^2}{2t}\right| - \log\left|\frac{(1+t)^2}{2t}\right|\right]$$

THIS GIVES

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = -\frac{1}{t}\left[\log(1-t) - \log(1+t)\right] = \frac{1}{t}\log\left(\frac{1+t}{1-t}\right) \quad \text{FOR } 0<t<1.$$

PUTTING THIS INTO (+) GIVES

$$\frac{1}{t}\log\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty}\Big(\int_{-1}^{1} P_n^2(x)\,dx\Big)\,t^{2n}. \qquad (**)$$

THE FINAL STEP IS TO EXPAND THE LEFT SIDE OF (**) IN A TAYLOR SERIES FOR $t \ll 1$ AND EQUATE THE TAYLOR COEFFICIENTS TO THE RIGHT HAND SIDE. WE CALCULATE

$$\frac{1}{1-t} = 1 + t + t^2 + \dots \quad \text{FOR } |t|<1$$

$$\rightarrow \quad \log(1-t) = -t - t^2/2 - t^3/3 \dots \quad \text{FOR } |t|<1$$

REPLACING $t \mapsto -t$: $\log(1+t) = t - t^2/2 + t^3/3 - \dots$

SO $$\frac{1}{t}\left[\log(1+t) - \log(1-t)\right] = \frac{1}{t}\left(2t + \frac{2t^3}{3} + \frac{2t^5}{5} \dots\right) = 2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1} .$$

IN THIS WAY WE GET

$$\sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n} = \sum_{n=0}^{\infty} \left(\int_{-1}^{1} (P_n(x))^2 dx\right) t^{2n} .$$

EQUATING POWERS OF t^{2n} GIVES

$$\int_{-1}^{1} (P_n(x))^2 dx = \frac{2}{2n+1} , \quad n = 0, 1, 2, \dots$$

FINALLY, IF WE PUT $x = \cos\varphi$ THEN $dx = -\sin\varphi \, d\varphi$ SO

$$\int_0^{\pi} (P_n(\cos\varphi))^2 \sin\varphi \, d\varphi = \frac{2}{2n+1} \quad \text{FOR } n = 0, 1, 2 \dots$$

EXAMPLE USE THE GENERATING FUNCTION TO DERIVE THE RECURSION RELATION

$$(n+1)\,P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x) \quad \text{FOR} \quad n \geq 1 \text{ integer.}$$

PROOF WE BEGIN WITH

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,t^n \quad \text{FOR} \quad 0<t<1.$$

DIFFERENTIATE WRT t:

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} n\,P_n(x)\,t^{n-1}$$

MULTIPLY BY $(1-2xt+t^2)$:

$$(x-t)\,(1-2xt+t^2)^{-3/2} = (1-2xt+t^2)\sum_{n=1}^{\infty} n\,P_n(x)\,t^{n-1}.$$

PUT IN GENERATING FUNCTION ON LHS AND SIMPLY THE RHS:

$$(x-t)\sum_{n=0}^{\infty} P_n(x)\,t^n = \sum_{n=1}^{\infty} n\,P_n(x)\,t^{n-1} - 2x\sum_{n=1}^{\infty} n\,P_n(x)\,t^n + \sum_{n=1}^{\infty} n\,P_n(x)\,t^{n+1}$$

NOW SHIFT INDICES ON RHS SO THAT WE GET t^n:

$$(x-t)\sum_{n=0}^{\infty} P_n(x)\,t^n = \sum_{n=0}^{\infty}(n+1)\,P_{n+1}\,t^n - \sum_{n=1}^{\infty} 2x\,n\,P_n\,t^n + \sum_{n=2}^{\infty}(n-1)\,P_{n-1}\,t^n$$

SIMILARLY FOR THE LHS WE GET TAKING OUT $n=0,1$ TERMS FROM RHS:

$$\sum_{n=0}^{\infty} x\,P_n\,t^n - \sum_{n=1}^{\infty} P_{n-1}\,t^n = P_1\,t^0 + t^1\left[\,2P_2 - 2x\,P_1\,\right]$$
$$+ \sum_{n=2}^{\infty} t^n\left[\,(n+1)\,P_{n+1} - 2xn\,P_n + (n-1)\,P_{n-1}\right]t^n.$$

SIMILARLY FOR THE LEFT SIDE WE WRITE FACTORING OUT $n=0,1$ TERMS

$$x\,P_0\,t^0 + t\left[x\,P_1 - P_0\right] + \sum_{n=2}^{\infty}\left(x\,P_n - P_{n-1}\right)t^n = P_1\,t^0 + t^1\left[\,2P_2 - 2x\,P_1\right]$$
$$+ \sum_{n=2}^{\infty} t^n\left[\,(n+1)\,P_{n+1} - 2xn\,P_n + (n-1)\,P_{n-1}\right]t^n$$

EQUATING COEFFICIENTS GIVES

$$P_1 = x P_0 \quad \text{WITH} \quad P_0 = 1 \rightarrow P_1 = x.$$

$$2P_2 - 2xP_1 = xP_1 - P_0 \quad \rightarrow \quad 2P_2 = 3xP_1 - P_0$$

AND FINALLY $\quad x P_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x).$
$n \geq 2$

WE SIMPLIFY THIS TO GET

$$(n+1)P_{n+1}(x) = x(2n+1)P_n(x) - n P_{n-1}(x) \quad \text{FOR} \quad n \geq 2.$$

OBSERVE THAT FOR $n=1$ WE GET $2P_2 = 3xP_1 - P_0$.

THIS MEANS THAT WITH $P_0(x) = 1$, $P_1(x) = x$ WE HAVE THE RECURRENCE RELATION

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - n P_{n-1}(x) \quad \text{FOR} \quad n \geq 1$$

THAT CAN YIELD ALL THE LEGENDRE POLYNOMIALS.

WE CAN ALSO DERIVE OTHER IDENTITIES INVOLVING LEGENDRE POLYNOMIALS USING THE GENERATING FUNCTION. TWO OF WHICH ARE:

(i) $P_n(-1) = (-1)^n.$ FOR $n \geq 0$.

(ii) $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$ FOR $n \geq 1$.

<u>PROOF OF (i)</u> PUT $x=-1$ IN GENERATING FUNCTION TO GET

$$[(1+t)^2]^{-1/2} = \frac{1}{1+t} = \sum_{n=0}^{\infty} P_n(-1)t^n. \quad \text{BUT} \quad \frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n \quad \text{FOR } 0<t<1.$$

EQUATING POWERS OF t^n GIVES $P_n(-1) = (-1)^n$.

PROOF OF (ii)

WE BEGIN BY DIFFERENTIATING THE GENERATING FUNCTION WRT X:

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

WHICH GIVES
$$t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n'(x)\, t^n$$

MULTIPLY BY $(1-2xt+t^2)$ AND USE GENERATING FUNCTION:

$$t(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^{n+1} = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)\, t^n.$$

SHIFTING INDICES AND WRITING IN TERMS OF t^n:

$$\sum_{n=1}^{\infty} P_{n-1}\, t^n = \sum_{n=0}^{\infty} P_n'\, t^n - 2x\sum_{n=1}^{\infty} P_{n-1}'\, t^n + \sum_{n=2}^{\infty} P_{n-2}'\, t^n$$

EXTRACT $n=0$ AND $n=1$ TERMS:

$$t P_0 + \sum_{n=2}^{\infty} P_{n-1}\, t^n = P_0' + P_1' t - 2x P_0' t + \sum_{n=2}^{\infty} (P_n' - 2x P_{n-1}' + P_{n-2}')\, t^n.$$

SINCE $P_0 = 1 \rightarrow P_0' = 0$, AND $P_1 = x \rightarrow P_1' = 1$. SO WE ONLY NEED TO COMPARE t^n TERMS FOR $n \geq 2$. THIS GIVES

$$P_{n-1} = P_n' - 2x P_{n-1}' + P_{n-2}' \quad \text{FOR} \quad n \geq 2$$

REPLACING n BY $n+1$ GIVES AFTER RE-ARRANGING:

$$P_{n+1}' + P_{n-1}' = P_n + 2x P_n' \quad \text{FOR} \quad n = 1, 2 \ldots \qquad (1)$$

RECALL OUR PREVIOUS RECURSION RELATION

$$(n+1) P_{n+1} = (2n+1)\, x P_n - n P_{n-1} \quad \text{FOR} \quad n = 1, 2, \ldots$$

NOW DIFFERENTIATE WRT X TO GET

$$(n+1) P_{n+1}' = (2n+1)\, x P_n' + (2n+1) P_n - n P_{n-1}' \quad \text{FOR} \quad n = 1, 2 \ldots \qquad (2)$$

WE SIMPLIFY BY ELIMINATING P_n' BETWEEN (1) AND (2): TO DO SO, MULTIPLY (2) BY A FACTOR OF 2

$$2(n+1)P_{n+1}' = (2n+1)\left[2xP_n'\right] + 2(2n+1)P_n - 2nP_{n-1}'$$

$$2(n+1)P_{n+1}' = (2n+1)\left[P_{n+1}' + P_{n-1}' - P_n\right] + 2(2n+1)P_n - 2nP_{n-1}'$$

THIS SIMPLIFIES TO

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x) \quad \text{FOR} \quad n \geq 1 \;\square$$

FINALLY, THERE IS A SINGLE FORMULA, KNOWN AS <u>RODRIGUEZ FORMULA</u> THAT CAN BE USED TO CALCULATE <u>ALL</u> THE LEGENDRE POLYNOMIALS $P_n(x)$ THAT SATISFY $P_n(1) = 1$.

<u>RESULT</u> FOR ALL $n = 1, 2, \ldots$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]. \quad \text{(RODRIGUEZ)}.$$

<u>PROOF</u> DEFINE $y = \dfrac{d^n}{dx^n}\left[(x^2-1)^n\right]$. WE FIRST WANT TO SHOW THAT $y(x)$ SATISFIES LEGENDRE'S EQUATION

$$(*)\; \Big\}\; (1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

TO DO SO, WE DEFINE $h(x) \equiv (1-x^2)^n$. WE CALCULATE

$$h'(x) = -2nx(1-x^2)^{n-1}$$

SO THAT $$(1-x^2)h' + 2nx(1-x^2)^n = 0$$

WHICH IMPLIES $$(1-x^2)h' + 2nxh = 0. \;\Big\}\; (*)$$

OUR GOAL NOW IS TO DIFFERENTIATE $(*)$ A TOTAL OF $n+1$ TIMES.

NOW LET A, B BE ANY TWO FUNCTIONS OF X. BY REPEATED APPLICATION OF CHAIN RULE

$$(AB)'' = A''B + 2A'B' + AB''$$

$$(AB)''' = A'''B + 3A''B' + 3A'B'' + AB'''$$

$$(AB)^{(m)} = A^{(m)}B + mA^{(m-1)}B' + \binom{m}{2}A^{(m-2)}B'' + \dots + mAB^{(m-1)} + AB^{(m)}.$$

(BINOMIAL COEFFICIENTS)

NOW DIFFERENTIATE (+) $n+1$ TIMES

$$[(1-x^2)h']^{(n+1)} + 2n[xh]^{(n+1)} = 0$$

SET $A = h'$, $B = (1-x^2)$, $m = n+1$ IN FIRST TERM

$A = h$, $B = X$, $m = n+1$ IN SECOND TERM.

SO

$$(1-x^2)h^{(n+2)} + (n+1)h^{(n+1)}(-2x) + \binom{n+1}{2}h^{(n)}(-2)$$
$$+ 2n[x\,h^{(n+1)} + (n+1)h^{(n)}] = 0.$$

THUS

$$(1-x^2)\,h^{(n+2)} + h^{(n+1)}[-2x(n+1)+2nx] + h^{(n)}\left[2n(n+1) - 2\binom{n+1}{2}\right] = 0.$$

BUT $\binom{n+1}{2} = \frac{(n+1)!}{(n-1)!\,2!} = \frac{n(n+1)}{2}$.

THIS GIVES $2n(n+1) - 2\frac{n(n+1)}{2} = n(n+1)$. WE CONCLUDE THAT

$h^{(n)}(x) = \frac{d^n}{dx^n}[(1-x^2)^n]$ SATISFIES $(1-x^2)(h^{(n)})'' - 2x(h^{(n)})' + n(n+1)h^{(n)} = 0.$

THUS $y = (-1)^n h_0(x)$ SATISFIES (*) ON PREVIOUS PAGE (LEGENDRE'S DIFFERENTIAL EQUATION).

SO $y = \frac{d^n}{dx^n}[(x^2-1)^n]$ IS A SOLUTION.

FINALLY, SINCE THE EQUATION IS HOMOGENEOUS WE HAVE FOR ANY $A_n > 0$ THAT

$$y = A_n \frac{d^n}{dx^n}\left[(x^2-1)^n\right] = A_n \frac{d^n}{dx^n}\left[(x-1)^n(x+1)^n\right]$$

IS A SOLUTION TO LEGENDRE'S EQUATION. WE MUST ENFORCE $y(1)=1$ BY DIFFERENTIATING n TIMES, WE HAVE THAT THE ONLY NONZERO TERM IS

$$y(1) = A_n\left[\frac{d^n}{dx^n}(x-1)^n\right](x+1)^n\Big|_{x=1} = n!\,2^n A_n,$$

SO WE MUST SET $A_n = \dfrac{1}{n!\,2^n}$.

THIS GIVES $y(x) = P_n(x) = \dfrac{1}{n!\,2^n}\dfrac{d^n}{dx^n}\left[(x^2-1)^n\right]$. $\square$

FINALLY WE WILL USE THE GENERATING FUNCTION TO CALCULATE POINT VALUES OF THE LEGENDRE POLYNOMIALS.

<u>CLAIM</u> $P_{2m}(0) = \dfrac{(-1)^m (2m)!}{2^{2m}(m!)^2}$, $m = 0, 1, 2 \ldots$

<u>PROOF</u> SINCE $P_n(x)$ IS EVEN IF $n = 0,2,4,\ldots \rightarrow P_{2m}'(0) = 0$.

WE WOULD LIKE TO FIND $P_{2m}(0)$. WE SET $x=0$ IN GENERATING FUNCTION

$$\frac{1}{\sqrt{t^2-2xt+1}} = \sum_{n=0}^{\infty} P_n(x)\,t^n$$

TO GET

$$\frac{1}{\sqrt{1+t^2}} = \sum_{n=0}^{\infty} P_n(0)\,t^n.$$

WE MUST FIND MACLAURIN SERIES OF

$$f(t) = \frac{1}{\sqrt{1+t}} \quad \text{IN} \quad 0 \le t < 1$$

WE CALCULATE

$$f(0) = 1$$

$$f'(0) = -\frac{1}{2}(1+t)^{-3/2}\Big|_{t=0} = -\frac{1}{2}$$

$$f''(0) = \frac{1}{2}\left(\frac{3}{2}\right)(1+t)^{-5/2}\Big|_{t=0} = \frac{1\cdot 3}{2\cdot 2}$$

$$f'''(0) = -\frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)(1+t)^{-7/2}\Big|_{t=0} = -\frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}$$

CONTINUING ON WE GET

$$f^{(n)}(0) = (-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n}. \quad \text{FOR } n = 1, 2, \ldots; \quad f^{(0)}(0) = 1.$$

BUT NOW

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{[1\cdot 3 \cdots (2n-1)]}{2^n \cdot n!} t^n \quad \text{IN} \quad 0 \le t < 1.$$

REPLACING t BY t^2,

$$\frac{1}{\sqrt{1+t^2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{[1\cdot 3 \cdots (2n-1)]}{2^n\, n!} t^{2n} = \sum_{n=0}^{\infty} P_n(0)\, t^n.$$

REWRITING WE HAVE

$$1 + \sum_{n=1}^{\infty} (-1)^n \frac{[1\cdot 3 \cdots (2n-1)]\, t^{2n}}{2^n\, n!} = P_0(0) + \sum_{n=1}^{\infty} P_n(0)\, t^n.$$

$P_0(0) = 1$ AND THIS RESULT SAYS $P_n(0) = 0$ FOR $n = 1, 3, \ldots$ AS EXPECTED.

SO REPLACING $n \longmapsto 2n$ IN LAST SUM

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1\cdot 3 \cdots (2n-1)}{2^n\, n!}\right) t^{2n} = \sum_{n=1}^{\infty} P_{2n}(0)\, t^{2n}.$$

WE CONCLUDE THAT

$$P_{2n}(0) = (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2^n\, n!}, \quad n = 1,2,3\ldots$$

FINALLY OBSERVE $\quad 1\cdot 3\cdots(2n-1) = 1\cdot\frac{2}{2}\cdot 3\cdot\frac{4}{4}\cdots\frac{(2n-2)}{2n-2}(2n-1)\frac{(2n)}{2n}$

$$= \frac{(2n)!}{2n(2n-2)\cdots(4)(2)} = \frac{(2n)!}{2^n\cdot n!}.$$

THUS YIELDS

$$P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}, \quad n = 1,2,\ldots$$

AND $P_0(0) = 1.$ $\square$

<u>CLAIM</u> FOR $m = 1,2,3\ldots$ WE HAVE

$$P_{2m-1}'(0) = -(2m)\,P_{2m}(0) = \frac{(-1)^{m+1}\, 2m\,(2m)!}{2^{2m}(m!)^2}.$$

<u>PROOF</u> DIFFERENTIATE THE GENERATING FUNCTION WRT X AND SET $X=0$

$$\frac{1}{\sqrt{t^2-2xt+1}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

YIELDS

$$\frac{t}{(\sqrt{t^2-2xt+1})^3}\bigg|_{x=0} = \frac{t}{(\sqrt{1+t^2})^3} = \sum_{n=0}^{\infty} P_n'(0)\, t^n.$$

SO

$$\frac{t}{(1+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(0)\, t^n. = P_0'(0) + \sum_{n=1}^{\infty} P_n'(0)\, t^n$$

$$= \sum_{n=0}^{\infty} P_n'(0)\, t^n \quad \text{SINCE } P_0'(0) = 0.$$

HOWEVER PREVIOUSLY WE SHOWED THAT

$$\frac{1}{\sqrt{1+t^2}} = P_0(0) + \sum_{n=1}^{\infty} P_n(0)\, t^n.$$

DIFFERENTIATING WRT t GIVES

$$\frac{-t}{(1+t^2)^{3/2}} = \sum_{n=1}^{\infty} n\, P_n(0)\, t^{n-1} = \sum_{n=0}^{\infty} (n+1) P_{n+1}(0)\, t^n$$

SO WE CONCLUDE THAT

$$P_n'(0) = -(n+1)\, P_{n+1}(0).$$

LET $n = 2m-1$ TO GIVE

$$P_{2m-1}'(0) = -(2m)\, P_{2m}(0)$$

AND THE RESULT FOLLOWS. $\square$