# MODIFIED BESSEL FUNCTIONS



THE <u>MODIFIED</u> BESSEL EQUATION OF ORDER $\nu \geq 0$ IS FOR $y(x)$:

$$(*) \quad x^2 y'' + x y' - [x^2 + \nu^2] y = 0 \quad \text{FOR } x > 0.$$

[RECALL THAT BESSEL'S EQUATION WITH $J_\nu(x)$ AND $Y_\nu(x)$ AS SOLUTIONS HAS A DIFFERENT SIGN, I.E. $x^2 y'' + x y' + [x^2 - \nu^2] y = 0$].

- NOW AS $x \to 0$ WE CLAIM THAT BOTH SOLUTIONS TO $(*)$ ARE

$$y \cong \text{SPAN}\{x^\nu, x^{-\nu}\} \text{ FOR } \nu > 0 \, ; \quad y \cong \text{SPAN}\{x^0, \ln x\} \text{ FOR } \nu = 0.$$

IN OTHER WORDS THE DIFFERENT SIGN BETWEEN BESSEL'S EQUATION AND THE MODIFIED BESSEL EQUATION HAS <u>NO</u> <u>IMPACT</u> TO A FIRST APPROXIMATION.

- HOWEVER, TO FIND BEHAVIOR AS $x \to +\infty$ WE PUT

$$y(x) = f(x)\, \psi(x)$$

INTO THE ODE $(*)$ AND CHOOSE $f(x)$ TO ELIMINATE FIRST DERIVATIVE TERM. THE CALCULATION IS VIRTUALLY THE SAME AS FOR BESSEL'S EQUATION AND WE FIND THAT

$$f(x) = x^{-1/2} \quad \text{AND} \quad \psi'' - \left[1 + \frac{(\nu^2 - 1/4)}{x^2}\right]\psi = 0.$$

THIS SHOWS THAT AS $x \to +\infty$, $\psi \cong \text{SPAN}\{e^x, e^{-x}\}$.

SO THAT $$y(x) \cong \text{SPAN}\left\{\frac{e^x}{\sqrt{x}}, \frac{e^{-x}}{\sqrt{x}}\right\} \quad \text{AS } x \to \infty;$$

THIS MEANS THAT ONE SOLUTION IS BOUNDED AS $x \to +\infty$ WHILE THE OTHER IS UNBOUNDED.

IN THIS WAY, THE GENERAL SOLUTION TO $(*)$ HAS THE FORM

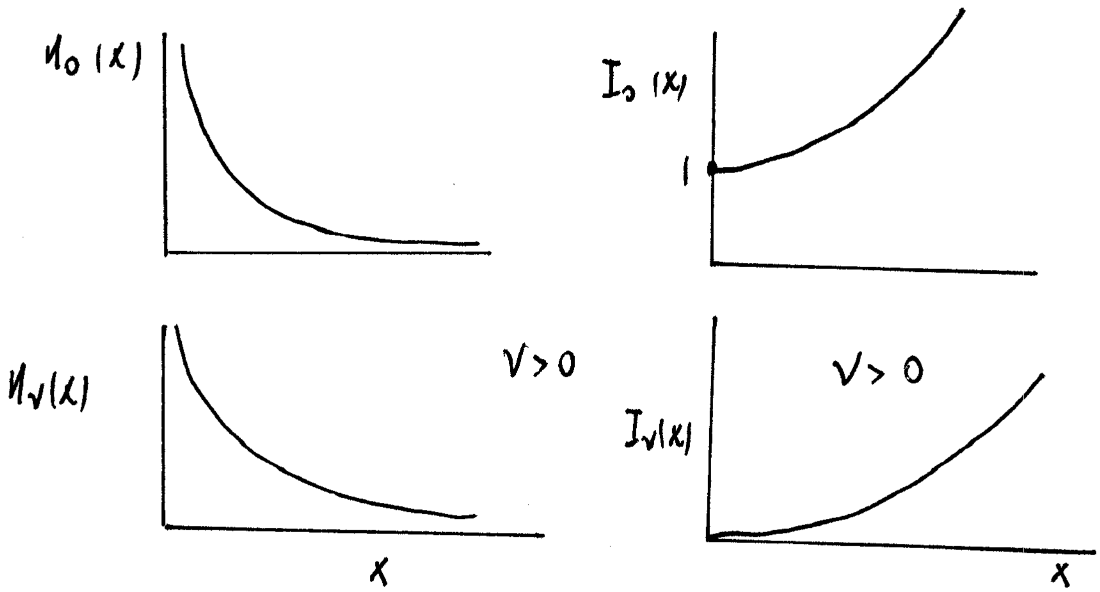
$$y(x) = C_1 I_\nu(x) + C_2 K_\nu(x)$$

WHERE $I_\nu(x)$ AND $K_\nu(x)$ ARE "MODIFIED BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND", RESPECTIVELY, "OF ORDER $\nu$".

THEY SATISFY:

- $I_\nu(0) = 0$ FOR $\nu > 0$ ; $I_0(0) = 1$ ; $I_\nu(x) \to +\infty$ AS $x \to \infty$ FOR $\nu \geq 0$
- $K_\nu(x) \to 0$ AS $x \to \infty$ ; $|K_\nu(x)| \to +\infty$ AS $x \to \infty$.

PLOTS FROM WOLFRAM ARE:




**EXAMPLE 1** IN TERMS OF THE APPROPRIATE MODIFIED BESSEL FUNCTION FIND THE SOLUTION TO:

(i) $u'' + \frac{1}{r} u' - 4u = 0$ IN $0 \leq r < a$ ; $u(a) = 1$; $u, u'$ BOUNDED AS $r \to 0$

(ii) $u'' + \frac{1}{r} u' - 9u = 0$ IN $r \geq a$ ; $u(a) = 1$; $u$ BOUNDED AS $r \to \infty$.

**SOLUTION (i)** WRITE $r^2 u'' + r u' - 4r^2 u = 0$. BY SCALE INVARIANCE PUT $x = 2r$ SO $y(x) = u[x/2]$ SATISFIES

$$x^2 y'' + x y' - x^2 y = 0.$$

THE SOLUTION THAT IS BOUNDED AS $x \to 0$ IS $y = I_0(x)$.

SO WE PUT $u = C\, I_0(2r)$. SET $u(a) = 1 = C\, I_0(2a)$ SO

$C = 1/I_0(2a)$ AND $u(r) = \dfrac{I_0(2r)}{I_0(2a)}$.

SOLUTION (ii) WRITE $r^2 u'' + r u' - 9r^2 u = 0$. PUT $x = 3r$ AND $y(x) = u(x/3)$.

SO THAT $x^2 y'' + x y' - x^2 y = 0$. THE BOUNDED SOLUTION IS $y = K_0(x)$, WHICH IS BOUNDED AS $x \to \infty$. SO

$$u = C\, K_0(3r).$$

IMPOSING $u(a) = 1$ GIVES $C = 1/K_0(3a)$

$$\longrightarrow \quad u = \frac{K_0(3r)}{K_0(3a)}.$$

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MODIFIED BESSEL FUNCTIONS ARE NOT ARISING FROM STURM LIOUVILLE PROBLEMS. THEY ARISE NATURALLY IN TRYING TO SOLVE $\nabla^2 u - \beta^2 u = 0$ FOR $\beta > 0$ INSIDE OR OUTSIDE DISKS IN 2-D.

EXAMPLE 2 FIND THE SOLUTION TO

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} - \beta^2 u = 0 \quad \text{IN} \quad r \geq a,\ 0 \leq \varphi \leq 2\pi, \ \text{WITH } \beta > 0,$$

FOR $u(r, \varphi)$ WITH

$$u(a, \varphi) = 2\cos(3\varphi) \quad ; \quad u,\ u_\varphi,\ 2\pi \text{ periodic}$$

$$u \longrightarrow 0 \text{ AS } r \to \infty.$$

SOLUTION WE PUT $u = F(r)\cos(3\varphi)$ INTO THE PDE TO GET:

$$F'' + \frac{1}{r} F' - \frac{9}{r^2} F - \beta^2 F = 0 \ \rightarrow \ r^2 F'' + r F' - [9 + \beta^2 r^2] F = 0.$$

THE SOLUTION THAT IS BOUNDED AS $r \to \infty$ IS $F(r) = C K_3(\beta r)$ FOR SOME $C$. WE NEED $F(a) = 2$ SO $F(r) = 2 K_3(\beta r)/K_3(\beta a)$

$$\longrightarrow \quad u(r, \varphi) = 2\, \frac{K_3(\beta r)}{K_3(\beta a)} \cos(3\varphi).$$

EXAMPLE 3  FIND THE SOLUTION $u(r,\varphi)$ TO

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} - \beta^2 u = 0 \quad \text{IN} \quad r \geq a, \; 0 \leq \varphi \leq 2\pi$$

FOR $\beta > 0$ WITH

$$u(a,\varphi) = 2\sin^2(\varphi) \; ; \quad u \text{ BOUNDED AS } r \to \infty$$

$$u, u_\varphi \;\; 2\pi \text{ PERIODIC}$$

SOLUTION  WE WRITE $2\sin^2(\varphi) = 2[1 - \cos(2\varphi)]/2 = 1 - \cos(2\varphi)$.

SO WE PUT $u(r,\varphi) = f_0(r) + f_2(r)\cos(2\varphi)$

AND OBTAIN

$$\left.\begin{array}{l} f_0'' + \frac{1}{r} f_0' - \beta^2 f_0 = 0, \quad \text{IN } 0 \leq r \leq a \\ f_0(a) = 1, \quad f_0 \text{ BOUNDED AS } r \to 0 \end{array}\right\} \rightarrow f_0(r) = \frac{K_0(\beta r)}{K_0(\beta a)} \quad \text{(BOUNDED AS } r \to \infty)$$

$$\left.\begin{array}{l} f_2'' + \frac{1}{r} f_2' - \left[\frac{4}{r^2} + \beta^2\right] f_2 = 0 \quad \text{IN } 0 \leq r \leq a \\ f_2(a) = -1, \quad f_2 \text{ BOUNDED AS } r \to 0 \end{array}\right\} \rightarrow f_2(r) = -\frac{K_2(\beta r)}{K_2(\beta a)}$$

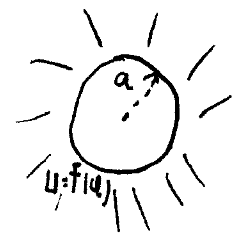
SO $$u(r,\varphi) = \frac{K_0(\beta r)}{K_0(\beta a)} - \frac{K_2(\beta r)}{K_2(\beta a)} \cos(2\varphi)$$

FINALLY, WE GIVE AN EXAMPLE ON HOW TO SOLVE PROBLEMS WITH MODIFIED BESSEL FUNCTIONS THROUGH EIGENFUNCTION EXPANSIONS.

# EIGENFUNCTION EXPANSIONS

**EXAMPLE 4** FIND AN EIGENFUNCTION EXPANSION SOLUTION FOR THE SOLUTION $u(r,\varphi)$ TO

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} - B^2 u = 0 \quad \text{IN } r \geq a,\ 0 \leq \varphi < 2\pi \\ u(a,\varphi) = f(\varphi), \quad u \text{ BOUNDED AS } r \to \infty \\ u,\ u_\varphi \ 2\pi \text{ PERIODIC} \end{cases}$$




HERE $B > 0$ AND $f(\varphi)$ IS $2\pi$-PERIODIC IN $\varphi$.

**SOLUTION** SEPARATE VARIABLES $u(r,\varphi) = \Phi(r)\psi(\varphi)$:

$$[\Phi'' + 1/r\,\Phi' - B^2\Phi]\psi + \frac{1}{r^2}\Phi\psi'' = 0 \to \frac{r^2[\Phi'' + \frac{1}{r}\Phi' - B^2\Phi]}{\Phi} = -\frac{\psi''}{\psi} = \sigma.$$

SO $\psi'' + \sigma\psi = 0$, $0 < \varphi < 2\pi$, $\psi, \psi'$ $2\pi$ PERIODIC $\rightarrow$ $\psi = \text{SPAN}\{\cos n\varphi, \sin n\varphi\}$, $\sigma = n^2$, $n = 0, 1, 2, \ldots$

THIS GIVES: $r^2\Phi'' + r\Phi' - [n^2 + r^2B^2]\Phi = 0$. THE SOLUTION THAT IS BOUNDED AS $r \to \infty$ IS $K_n(Br)$. SO BY SUPERPOSITION

$$u(r,\varphi) = c_0 K_0(Br) + \sum_{n=1}^{\infty}[c_n \cos n\varphi + b_n \sin(n\varphi)]K_n(Br).$$

NOW FROM $r = a$ WE WILL FIND THE COEFFICIENTS $b_n, c_n$.

$$f(\varphi) = c_0 K_0(Ba) + \sum_{n=1}^{\infty}[c_n \cos(n\varphi) + b_n \sin(n\varphi)]K_n(Ba).$$

BY ORTHOGONALITY:

$$c_0 = \frac{1}{2\pi K_0(Ba)}\int_0^{2\pi} f(\varphi)\,d\varphi$$

↑ FOURIER SERIES!!

$$c_n = \frac{1}{\pi K_n(Ba)}\int_0^{2\pi} f(\varphi)\cos(n\varphi)\,d\varphi$$

$$b_n = \frac{1}{\pi K_n(Ba)}\int_0^{2\pi} f(\varphi)\sin(n\varphi)$$

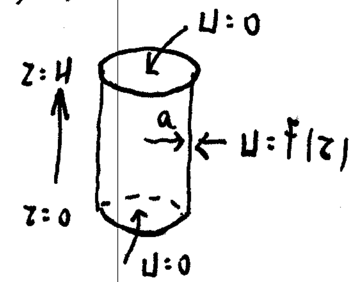
**EXAMPLE 5** AXIALLY-SYMMETRIC STEADY-STATE DIFFUSION IN A CYLINDER

FIND AN EIGENFUNCTION EXPANSION SOLUTION $u(r,z)$ TO

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0 \quad \text{IN } 0 \le r \le a,\ 0 \le z \le H$$

$$u(a,z) = f(z)\ ; \quad u, u_r \text{ BOUNDED AS } r \to 0$$

$$u(r,0) = u(r,H) = 0$$




**SOLUTION** SEPARATING VARIABLES WE WILL USE EIGENFUNCTIONS IN Z-DIRECTION SINCE WE HAVE HOMOGENEOUS BC IN Z-DIRECTION.

PUT $u = \Phi(r) Z(z) \longrightarrow \dfrac{\Phi'' + \frac{1}{r}\Phi'}{\Phi} = -\dfrac{Z''}{Z} = \lambda. \rightarrow \begin{array}{l} Z'' + \lambda Z = 0 \\ \Phi'' + \frac{1}{r}\Phi' - \lambda \Phi = 0 \end{array}$

SO $\left.\begin{array}{l} Z'' + \lambda Z = 0, \quad 0 < z < H \\ Z(0) = 0, \quad Z(H) = 0 \end{array}\right\} \longrightarrow Z_n(z) = \sin\left(\dfrac{n\pi z}{H}\right), \quad \lambda_n = \dfrac{n^2\pi^2}{H^2}, \quad n = 1, 2, \ldots$

NOW WE HAVE THAT

$$\Phi'' + \frac{1}{r}\Phi' - \lambda \Phi = 0 \quad \text{IN } 0 < r < a$$

$$\rightarrow \quad r^2 \Phi'' + r\Phi' - \frac{n^2\pi^2}{H^2} r^2 \Phi = 0.$$

WE WANT A SOLUTION BOUNDED AS $r \to 0$. WE CANNOT SATISFY BC ON $r = a$ YET!

SO $\Phi_n(r) = I_0\left(\dfrac{n\pi r}{H}\right)$ WHERE $I_0(z)$ IS MODIFIED BESSEL FUNCTION OF FIRST KIND OF ORDER 0.

NOW BY SUPERPOSITION : $(*)\ \left\{ u(r,z) = \displaystyle\sum_{n=1}^{\infty} A_n I_0\left(\frac{n\pi r}{H}\right) \sin\left(\frac{n\pi z}{H}\right). \right.$ TO

FIND COEFFICIENTS, SATISFY $u(a,z) = f(z) = \displaystyle\sum_{n=1}^{\infty} A_n I_0\left(\frac{n\pi a}{H}\right) \sin\left(\frac{n\pi z}{H}\right)$,

WHICH IS A FOURIER-SINE SERIES. SO,

$$A_n I_0\left(\frac{n\pi a}{H}\right) = \frac{2}{H}\int_0^H f(z)\sin\left(\frac{n\pi z}{H}\right)dz \rightarrow A_n = \frac{2}{H I_0\left(\frac{n\pi a}{H}\right)}\int_0^H f(z)\sin\left(\frac{n\pi z}{H}\right)dz.$$

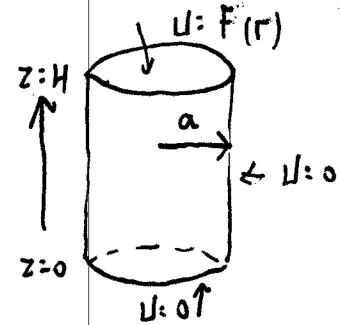
THE SOLUTION $u(r,z)$ IS GIVEN BY $(*)$.

**EXAMPLE 6** NOW CONSIDER THE SAME PROBLEM BUT WHERE INHOMOGENEOUS BOUNDARY TERM OCCURS ON TOP OF CYLINDER.

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0, \quad \text{IN } 0 \le r \le a, \; 0 \le z \le H$$

$$u(a, z) = 0; \quad u, u_r \text{ BOUNDED AS } r \to 0$$

$$u(r, 0) = 0, \quad u(r, H) = F(r).$$




SOLUTION KEY POINT: HOMOGENEOUS BC OCCUR IN RADIAL DIRECTION NOW. SO MAKE RADIAL DIRECTION BE SL PROBLEM $\rightarrow$ GET $J_0$ NOT $I_0$.

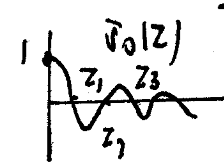
PUT $u = \Phi(r) Z(z) \rightarrow \dfrac{\Phi'' + \frac{1}{r}\Phi'}{\Phi} = -\dfrac{Z''}{Z} = -\lambda$ (DIFFERENT SIGN) $\rightarrow$

$$\begin{cases} \Phi'' + \frac{1}{r}\Phi' + \lambda \Phi = 0 \quad \text{(EIG PROB)} \\ Z'' - \lambda Z = 0 \\ \text{IMPOSE } Z(0) = 0 \text{ ONLY!} \end{cases}$$

SO WE HAVE

$$\begin{cases} \Phi'' + \frac{1}{r}\Phi' + \lambda \Phi = 0, \quad 0 \le r \le a \\ \Phi(a) = 0; \quad \Phi, \Phi' \text{ BOUNDED AS } r \to 0 \end{cases} \longrightarrow \Phi = J_0(\sqrt{\lambda}\, r)$$

$$J_0(\sqrt{\lambda}\, a) = 0 \rightarrow \lambda_n = z_n^2 / a^2, \quad n = 1, 2, \ldots$$




IN SL FORM: $(r\Phi')' + \lambda r \Phi = 0$ (weight $w = r$.)

NOW $Z'' - \lambda Z = 0$ WITH $Z(0) = 0 \longrightarrow Z(z) = \sinh[\sqrt{\lambda}\, z] = \sinh\left(\dfrac{z_n z}{a}\right)$

BY SUPERPOSITION $\quad u(r, z) = \sum_{n=1}^{\infty} A_n J_0(\sqrt{\lambda_n}\, r) \sinh[\sqrt{\lambda_n}\, z].$

IMPOSING $u(r, H) = F(r) \longrightarrow F(r) = \sum_{n=1}^{\infty} A_n \sinh[\sqrt{\lambda_n}\, H] \, J_0(\sqrt{\lambda_n}\, r)$

BY ORTHOGONALITY OF $\Phi_n$ WITH WEIGHT $w = r$ WE GET THAT $A_n$ IS

$$A_n \sinh[\sqrt{\lambda_n}\, H] = \int_0^a r F(r) J_0(\sqrt{\lambda_n}\, r)\, dr \Big/ \int_0^a r [J_0(\sqrt{\lambda_n}\, r)]^2 \, dr, \quad n = 1, 2, \ldots$$

THIS DETERMINES $A_n$ AND RECALL $\displaystyle\int_0^a r [J_0(\sqrt{\lambda_n}\, r)]^2 \, dr = \frac{a^2}{2} [J_0'(\sqrt{\lambda_n}\, a)]^2.$