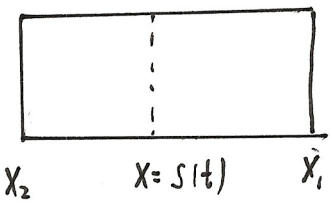
SHOCK CONDITION

SUPPOSE $\quad u_t + (Q)_x = 0$

AND THAT WE WOULD PREDICT A MULTI-VALUED SOLUTION.

$$\frac{d}{dt}\int_{x_2}^{x_1} u\,dx = Q(x_2,t) - Q(x_1,t)$$



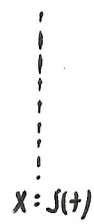
THEN $\quad \frac{d}{dt}\left[\int_{x_2}^{s} u\,dx + \int_{s}^{x_1} u\,dx\right] = Q(x_2,t) - Q(x_1,t)$

RECALL $\quad \frac{d}{dt}\int_a^t F(x,t)\,dx = F(t,t) + \int_a^t F_t(x,t)\,dx.$

$$\rightarrow \quad u(s^-,t)\frac{ds}{dt} - u(s_+,t)\frac{ds}{dt} + \int_{x_2}^{s} u_t\,dx + \int_{s}^{x_1} u_t\,dx = Q(x_2,t) - Q(x_1,t)$$

let $x_2 \to s^-$, $x_1 \to s^+$ SO THAT

$$\frac{ds}{dt} = \frac{Q(s_-,t) - Q(s_+,t)}{u(s_-,t) - u(s_+,t)}$$



NOW IF $\quad Q = Q(u)$ WE WRITE

$$\frac{ds}{dt} = \frac{Q(u_+) - Q(u_-)}{u_+ - u_-}$$

$u_- = u(s_-,t)$
$u_+ = u(s_+,t)$

<u>EX:</u> . IF $\quad Q(u) = u^2/2 \quad$ SO $\quad u_t + u u_x = 0 \quad$ THEN

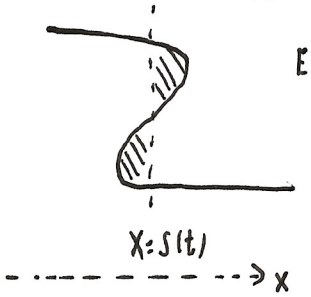
$$\begin{array}{r} y^2+y+1 \\ y-1\,\overline{)\,y^3-1} \end{array}$$

$$\frac{ds}{dt} = \frac{u_+ + u_-}{2}$$

. IF $\quad Q(u) = u^3/3 \qquad u_t + u^2 u_x = 0 \quad \rightarrow \quad \frac{ds}{dt} = \frac{1}{3}\left(u_+^2 + u_+u_- + u_-^2\right)$

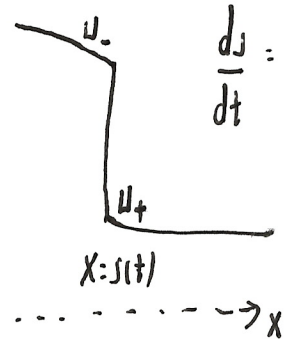
GEOMETRICAL INTERPRETATION (EQUAL AREA RULE)

$u(x,t)$ MULTI-VALUED u



EQUAL AREA RULE:
MOVING DISCONTINUITY IS INSERTED TO PRESERVE MASS.
(SHADED AREAS CANCEL)

$\longrightarrow$



$$\frac{ds}{dt} = \frac{Q(u_+) - Q(u_-)}{u_+ - u_-}$$

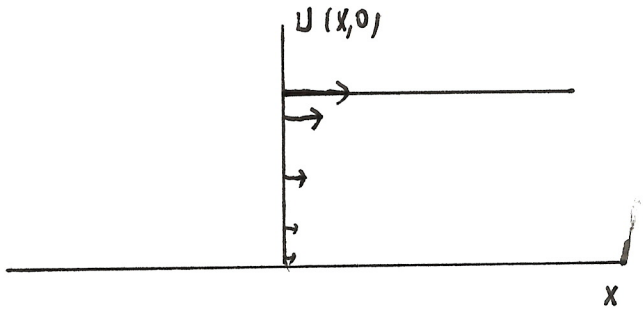
EXPANSION FAN CAUSED BY DISCONTINUOUS INITIAL DATA THAT LEADS TO A REGION IN (x,t) PLANE WITH NO CHARACTERISTICS.

EX:

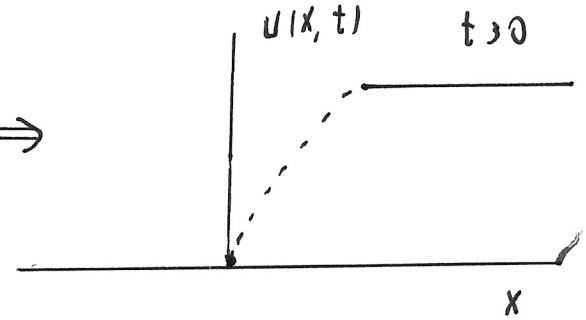
$$u_t + c(u)\,u_x = 0 \quad \text{IN } -\infty < x < \infty,\ t > 0$$

$$u(x,0) = \begin{cases} 0 & \text{IF } x \leq 0 \\ 1 & \text{IF } x > 0 \end{cases} \equiv f(\xi)$$

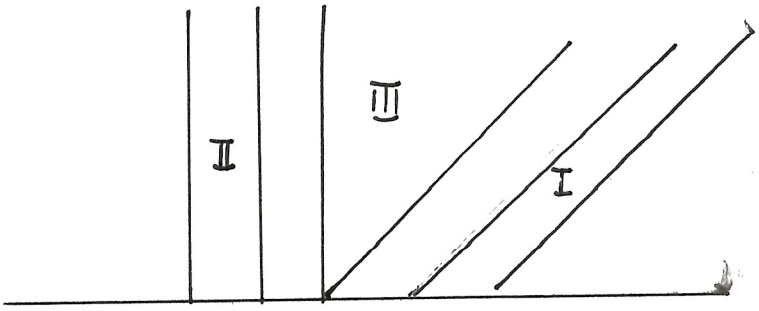
ASSUME $c(u) > 0$ IF $u > 0$ AND $c'(u) > 0$. ASSUME $c(0) = 0$



$\Longrightarrow$



INITIALLY IN x-t PLANE WE HAVE



WHAT DO WE DO IN REGION III?
there are no characteristics.
$\rightarrow$ EXPANSION FAN.

ON $dx/dt = c(u)$, $x(0) = \xi$
$du/dt = 0$, $u = f(\xi)$ AT $t = 0$

$\rightarrow$ $\begin{cases} x = c(1)t + \xi & \text{IN Region I} \\ u = 1 \end{cases}$

$\begin{cases} x = \xi & \text{IN Region II} \\ u = 0 \end{cases}$

IN THE GAP REGION WE INSERT AN <u>EXPANSION FAN</u>.

THIS SOLUTION HAS THE FORM

$$u = g(\eta) \quad \text{WITH} \quad \eta = x/t. \qquad \text{AND} \quad 0 \le x \le c(1)\,t$$

WE CALCULATE $\quad u_t = g'(\eta)\,\eta_t, \quad u_x = g'(\eta)\,\eta_x.$ NOW $\quad u_t + c(u)\,u_x = 0$

$$\rightarrow \quad g'(\eta)\left(-x/t^2\right) + c\left(g(\eta)\right) g'(\eta)\, 1/t = 0$$

$$\rightarrow \quad g'(\eta)\left[\, c\left(g(\eta)\right) - \eta \,\right] = 0 \quad \rightarrow \quad c\left(g(\eta)\right) = \eta.$$

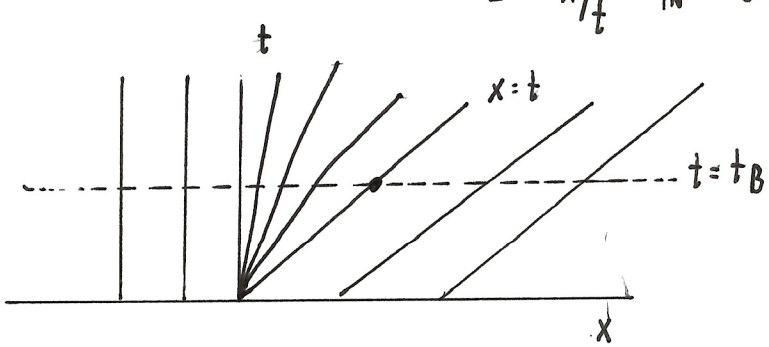
SO <u>EXPANSION FAN</u>: SOLVE $c\left(g(\eta)\right) = \eta$. WITH $0 \le \frac{x}{t} \le c(1)$

AND $\eta = x/t$ FOR $g(\eta) = \ldots$

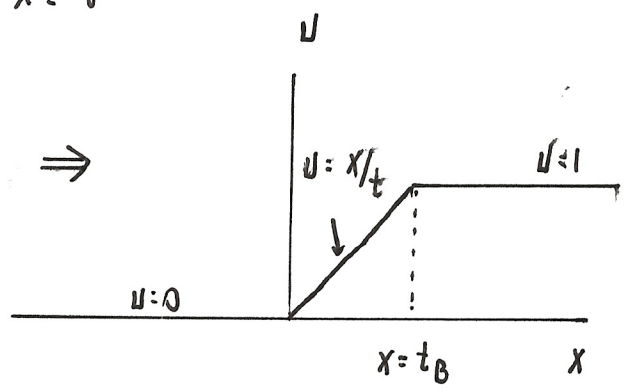
$$\longrightarrow \quad u = g(x/t) \quad \text{IN} \quad 0 \le \frac{x}{t} \le c(1).$$

- IN PARTICULAR, IF $c(u) = u \rightarrow g(\eta) = \eta$ AND $c(1) = 1$.

AND SO $\quad u = x/t \quad$ IN $\quad 0 \le x \le t$



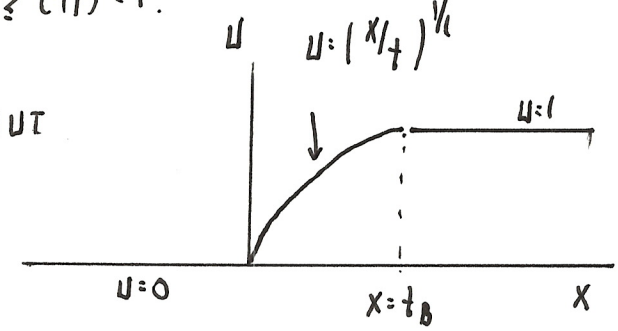
$\Rightarrow$



- IF $c(u) = u^2$ THEN SOLVE $c\left(g(\eta)\right) = \eta \rightarrow g^2 = \eta$ OR $g = \eta^{1/2}$.

THUS $\quad u = (x/t)^{1/2} \quad$ IN $\quad 0 \le \frac{x}{t} \le c(1) = 1.$

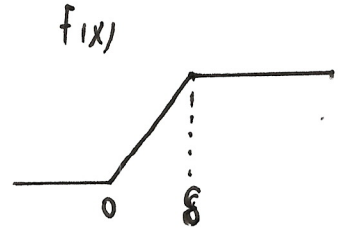
THE FAN LOOKS THE SAME BUT



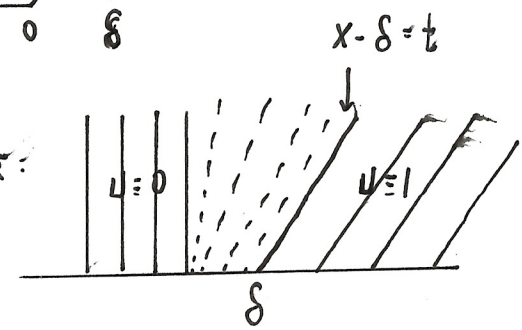
REMARK (EXPANSION FAN)

$$u_t + u\,u_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = f(x) = \begin{cases} 0, & x \le 0 \\ x/\delta, & 0 \le x \le \delta \\ 1, & x \ge \delta \end{cases}$$



FOR ANY $\delta > 0$ CHARACTERISTICS FILL THE PLANE:



<u>IN GAP</u>: ON $dx/dt = u, \quad x(0) = \xi$

$du/dt = 0, \quad u(0) = \xi/\delta$

$$\rightarrow \quad u = \xi/\delta \quad \text{AND} \quad x = \frac{\xi}{\delta}t + \xi. \quad \rightarrow \quad \xi = \frac{x}{t/\delta + 1}$$

NOW $$u = \frac{\xi}{\delta} = \frac{x}{t+\delta} \quad \text{IN} \quad 0 \le x \le t + \delta.$$

NOW LET $\delta \to 0$ $\quad u \hat{=} x/t$ IN $0 \le x \le t$

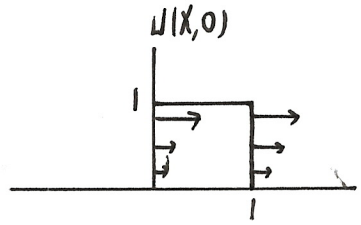
EQUIVALENT TO SOLVING $c(g) = \Lambda$.

IN LIMIT $\delta \to 0$ WHERE WE USE A SMOOTH BUT STEEP INITIAL DATA WE RECOVER THE EXPANSION FAN SOLUTION.

EXAMPLE

$$u_t + u u_x = 0 \qquad q(u) = u^2/2$$

$$u(x,0) = f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$



$c(u) = u \qquad c > 0$ if $u > 0$, $c' = 1 > 0$

- EXPANSION FAN INSERTED AT BACK AT $t=0$ CENTERED AT $x=0$
- SHOCK FIT AT $x=1$ AT $t=0$.

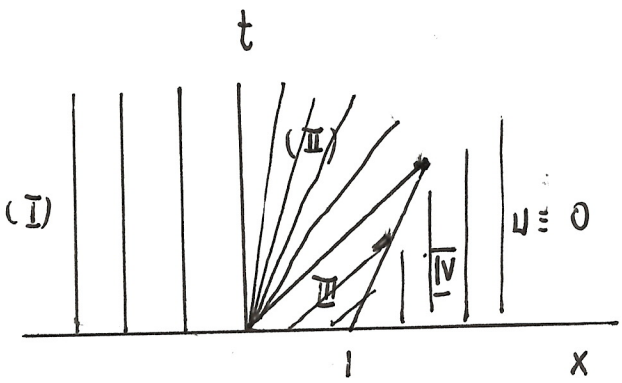
METHOD OF CHARACTERISTICS

$$\frac{dx}{dt} = u, \quad x(0) = s, \quad 0 < s < 1$$

$$\frac{du}{dt} = 0, \quad u(0) = 1$$

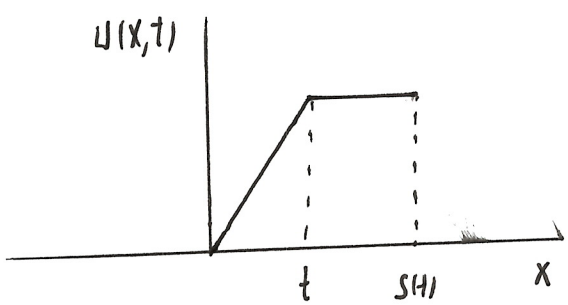
$$x = t + s, \quad 0 < s < 1$$

$$u = 1$$



FOR $t < S(t)$

$$u = \begin{cases} 0, & x < 0 \quad (\mathrm{I}) \\ x/t, & 0 \le x \le t \quad (\mathrm{II}) \quad \text{FAN} \\ 1, & t < x < S \quad (\mathrm{III}) \\ 0 & x > S \quad (\mathrm{IV}) \end{cases}$$



NOW $$\frac{ds}{dt} = \frac{u_+^2/2 - u_-^2/2}{u_+ - u_-} = \frac{1}{2}(u_+ + u_-) = \frac{1}{2}.$$

THIS GIVES $ds/dt = \frac{1}{2} t$, $s(0) = 1 \rightarrow s = t/2 + 1$.

THE FAN HITS THE SHOCK AT THE TIME WHEN WE LOSE REGION (III), i.e. WHEN $t = t/2 + 1$ WHICH GIVES $t = 2$ AND $s = 2$.

AT THIS TIME WE MUST FIT A NEW SHOCK CONNECTING REGION II (FAN) WITH REGION (IV). WE NOW HAVE $u_+ = 0$, $u_- = s_2/t$

THUS $$\frac{ds_2}{dt} = \frac{1}{2}(u_+ + u_-) = \frac{s_2}{2t}, \quad s_2(2) = 2.$$

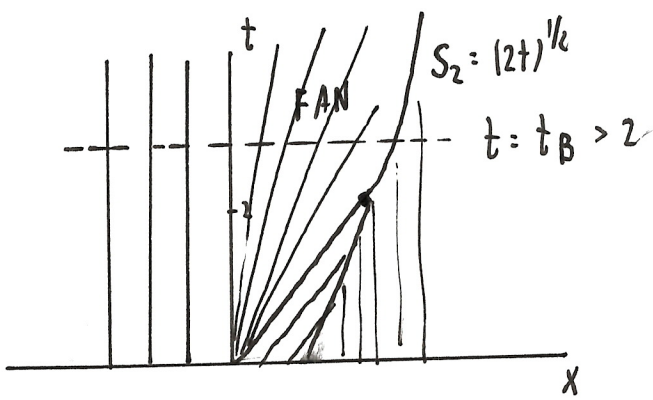
INTEGRATING $\dfrac{ds_2}{s_2} = \dfrac{dt}{2t}$, $s(2) = 2$

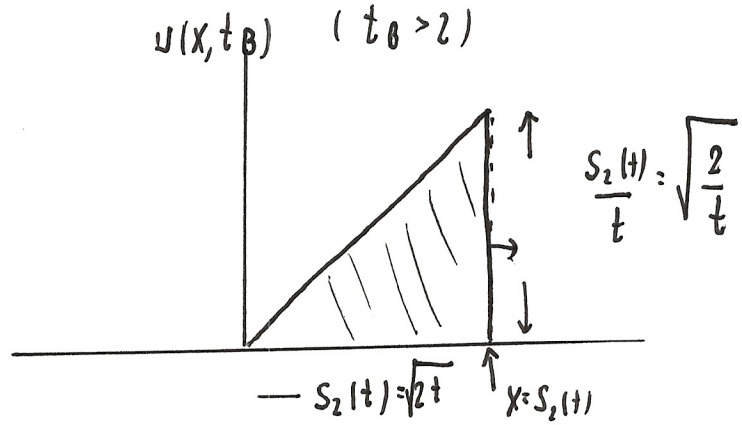
$$\log s_2 = \log(\sqrt{t}) + \text{CONSTANT}.$$

THIS GIVES $s_2 = \eta\, t^{1/2}$

NOW $s(2) = 2$ GIVES $\eta = \sqrt{2}$.

THUS $s_2 = (2t)^{1/2}$ FOR $t \geq 2$.





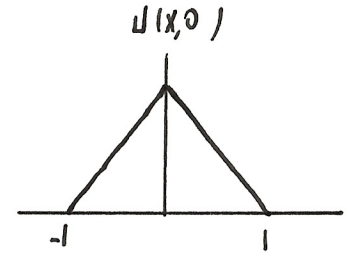
THE AREA IS

$$= \frac{1}{2}\sqrt{\frac{2}{t}}\,\sqrt{2t} = 1 \quad \forall\, t > 2.$$

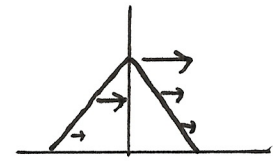
THIS IS THE SAME AS THE INITIAL AREA UNDER THE $f(x)$.

EXAMPLE $u_t + (q(u))_x = 0$ WITH $q = u^2/2$

$$u(x,0) = \begin{cases} 0 & \text{IF } x < -1 \\ 1+x & \text{IF } -1 \le x \le 0 \\ 1-x & \text{IF } 0 \le x \le 1 \\ 0 & \text{IF } x \ge 1. \end{cases}$$



$c(u) = u$, $c' > 0$.



- SKETCH CHARACTERISTICS BEFORE A SHOCK FORMS
- A SHOCK FORMS WHERE AND WHEN ?
- FIT A SHOCK AND CALCULATE SHOCK POSITION.

SOLUTION QUALITATIVELY A SHOCK WILL EVENTUALLY FORM ON THE RIGHT.

THERE ARE FOUR INITIAL REGIONS.

REGION I $u = 0$ IN $x < -1$

REGION IV $u = 0$ IN $x \ge 1$

IN **REGION II**,

$$\frac{dx}{dt} = u, \quad x = \xi, \ -1 \le \xi \le 0 \qquad \Rightarrow \quad x = (1+\xi)t + \xi, \quad u = 1+\xi$$

$$\frac{du}{dt} = 0, \quad u = 1+\xi \qquad u = 1+\xi = \frac{x-t}{t+1} + 1 = \frac{x+1}{t+1}$$

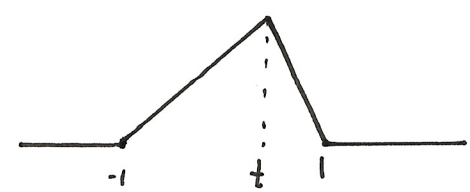
IN REGION III

$$x = (1-\xi)t + \xi, \quad 0 \le \xi \le 1 \qquad \longrightarrow \quad u = 1-\xi = 1 - \left(\frac{x-t}{1-t}\right) = \frac{1-x}{1-t}$$

$$u = 1-\xi$$

THEREFORE, FOR $0 \le t \le 1$,

$$u = \begin{cases} 0, & x \le -1 \quad (\mathrm{I}) \\ (1+x)/(1+t), & -1 \le x \le t \quad (\mathrm{II}) \\ (1-x)/(1-t), & t \le x \le 1 \quad (\mathrm{III}) \\ 0, & x \ge 1 \quad (\mathrm{IV}) \end{cases}$$



WE LOSE REGION (III) WHEN $t = 1$ AND $x = 1$. AT THIS TIME WE MUST FIT A SHOCK. $u_- = \dfrac{1+S}{1+t}$, $u_+ = 0$.

(Region II) Region IV

NOW $$\frac{ds}{dt} = \frac{\left(u_+^2/2 - u_-^2/2 \right)}{(u_+ - u_-)} = \frac{1}{2}(u_+ + u_-)$$

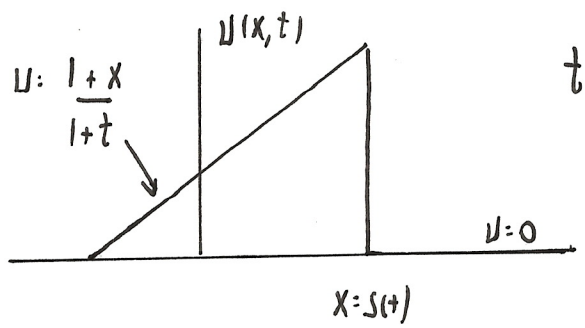
SO $$\frac{ds}{dt} = \frac{1}{2}\left(\frac{1+s}{1+t}\right), \qquad s(1) = 1.$$

$$\frac{ds}{1+s} = \frac{dt}{2(1+t)} \rightarrow \log(1+s) = \log\left(\sqrt{1+t}\right) + \text{CONSTANT}.$$

$$s = -1 + k\sqrt{1+t}.$$

NOW $s(1) = 1 \rightarrow k = \sqrt{2}.$

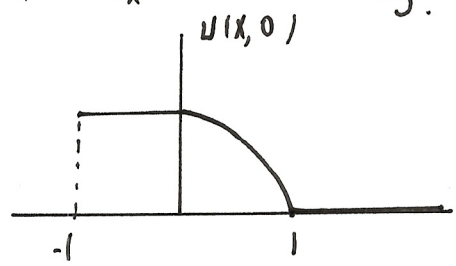
$$s = -1 + \sqrt{2(t+1)}$$



$t > 1.$

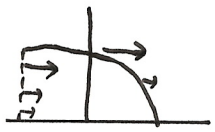
EXAMPLE $u_t + u^2 u_x = 0$ IN $-\infty < x < \infty$ $\rightarrow$ $u_t + (Q(u))_x = 0$ $\quad Q = u^3/3$.

$$u(x,0) = f(x) = \begin{cases} 0, & x < -1 \\ 1, & -1 \le x < 0 \\ \sqrt{1-x}, & 0 \le x \le 1 \\ 0, & x \ge 1 \end{cases}$$



- FIT INITIAL SHOCKS AND A FAN AS NEEDED.
- SKETCH PROFILE OF u AT $t = 1/2, 3/2$.
- FIND WHEN A NEW SHOCK IS NEEDED AND FIT THIS SHOCK. PLOT u WHEN $t = 3$.
- SKETCH CHARACTERISTICS IN (x,t) PLANE INCLUDING ALL SHOCKS.

SOLUTION $c(u) = u^2$. $c(0) = 0$, $c > 0$ IF $u > 0$, $c' > 0$. SO HIGHER u MOVE FASTER.



NEED A FAN CENTERED NEAR $x = -1$, STARTING AT $t = 0$. A SHOCK WILL EVENTUALLY FORM AT FRONT OF WAVE.

REGION I $u \equiv 0$ IN $x \le -1$

REGION II FAN REGION $c(g(\eta)) = \eta$ $\quad \eta = \dfrac{x+1}{t}$. $\quad$ IN $0 \le \dfrac{x+1}{t} \le 1$

$$\rightarrow g(\eta) = \eta^{1/2} \rightarrow u = \sqrt{\frac{x+1}{t}}$$

REGION III ON $dx/dt = u^2$, $x(0) = \xi$, $-1 \le \xi \le 0$
$du/dt = 0$, $u = 1$ AT $t = 0$ $\rightarrow$ $x = t + \xi$ IN $-1 \le \xi \le 0$, $u = 1$

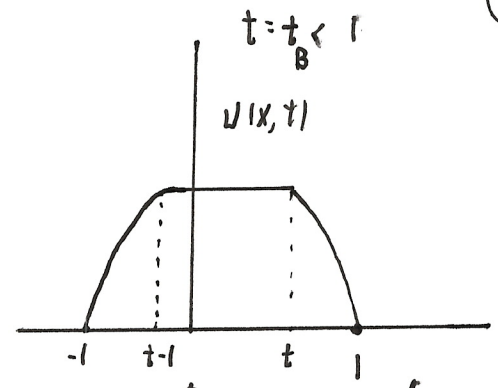
REGION IV ON $dx/dt = u^2$, $x(0) = \xi$, $0 \le \xi \le 1$
$du/dt = 0$, $u = \sqrt{1-\xi}$, $t = 0$ $\rightarrow$ $x = (1-\xi)t + \xi$, $u = \sqrt{1-\xi}$

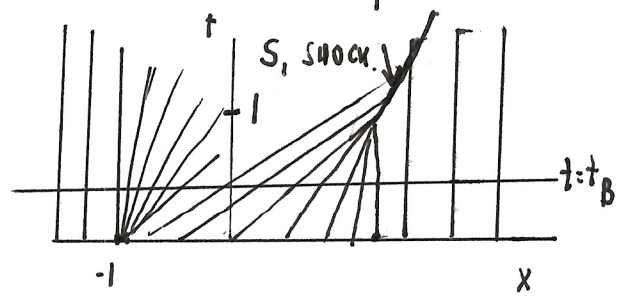
$$\rightarrow u = \sqrt{1 - \left(\frac{x-t}{1-t}\right)} \qquad t \le x \le 1$$

REGION V $u \equiv 0$.

THIS GIVES UP UNT

$$u = \begin{cases} 0, & x < -1 \quad (\mathrm{I}) \\ \sqrt{\dfrac{1+x}{t}} & -1 \le x \le t-1 \quad (\mathrm{II}) \\ 1 & t-1 \le x \le t \quad (\mathrm{III}) \\ \sqrt{\dfrac{1-x}{1-t}} & t \le x \le 1 \quad (\mathrm{IV}) \\ 0 & x \ge 1 \quad (\mathrm{V}) \end{cases}$$





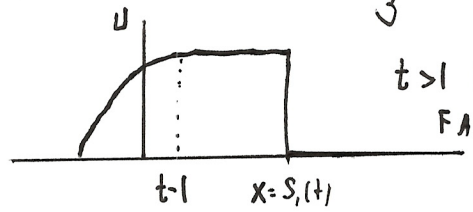
NOTICE AT $t=1$ WE LOSE REGION (IV) AND WE MUST INSERT A SHOCK THAT STARTS AT $S_1(1) = 1$ CONNECTING REGIONS (III) AND (IV).

$u_+ = 0$ (REGION (V)), $u_- = 1$ (REGION (III)).

$$\frac{ds_1}{dt} = \frac{Q(u_+) - Q(u_-)}{u_+ - u_-} \quad \text{WITH} \quad Q(u) = u^3/3.$$

SO

$$\frac{ds_1}{dt} = \frac{1}{3}\frac{(u_+^3 - u_-^3)}{(u_+ - u_-)} = \frac{1}{3}(u_+^2 + u_+u_- + u_-^2) = \frac{1}{3}$$

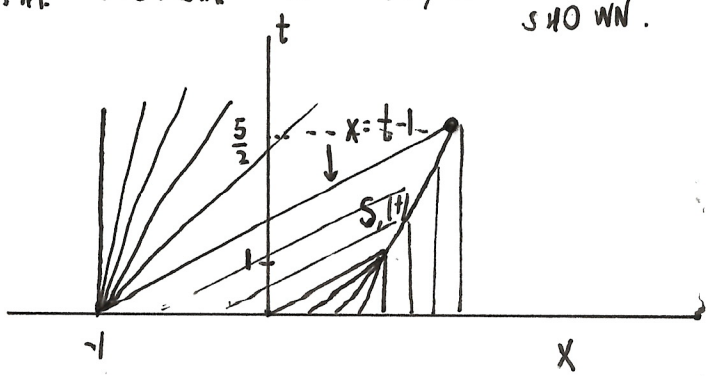


$t > 1$ BEFORE FAN HITS SHOCK.

WITH $S_1(1) = 1$ WE GET

$$S_1 = \frac{1}{3}t + \frac{2}{3}.$$

THE PICTURE IN (x, t) PLANE IS SHOWN.



NOW FAN'S RIGHT EDGE $x = t-1$ HITS $S_1(t)$ WHEN $-1+t = \frac{1}{3}t + \frac{2}{3} \rightarrow t = 5/2$

NOW $t = 5/2 \rightarrow S_1 = 3/2$.

AT THIS TIME WE LOSE REGION (III). WE MUST MAKE A NEW SHOCK THAT CONNECTS FAN TO REGION (V)

so $$\frac{dS_2}{dt} = \frac{1}{3}\left(u_+^2 + u_+ u_- + u_-^2\right)$$

$$u_+ = 0, \qquad u_- = \sqrt{\frac{1+S_2}{t}}$$

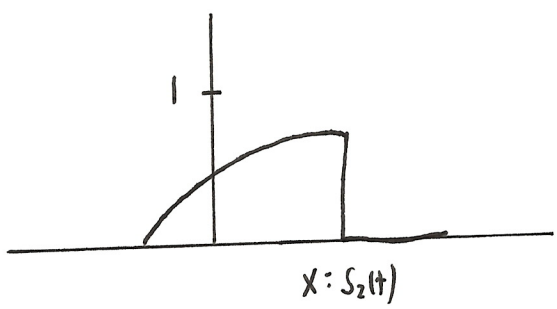
so $$\frac{dS_2}{dt} = \frac{1}{3}\left(\frac{1+S_2}{t}\right), \qquad S_2\left(5/2\right) = 3/2.$$

$$\frac{1}{1+S_2}\, dS_2 = \frac{1}{3t}\, dt \rightarrow \log(1+S_2) = \frac{1}{3}\log t + c = \log\left(t^{1/3}\right) + c.$$

so $$1 + S_2 = k\, t^{1/3}$$

$$S_2 = -1 + k\, t^{1/3}. \qquad S_2\left(5/2\right) = 3/2$$

THEN YIELDS $$S_2 = -1 + \left(\frac{25t}{4}\right)^{1/3}.$$



EXAMPLE (QUALITATIVE) LET $u_t + (g(u))_x = 0$ WITH $g = u - u^2/2$.

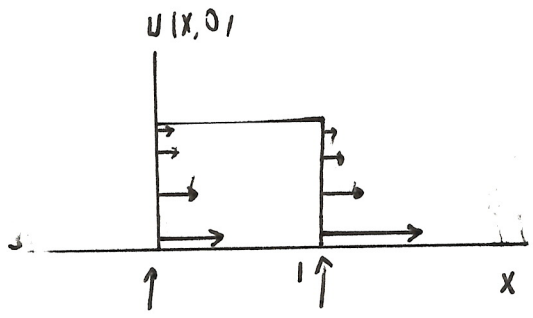
NOW SUPPOSE $u_t + (1-u)u_x = 0$, $-\infty < x < \infty$ WITH $c(u) = 1-u = g'(u)$.

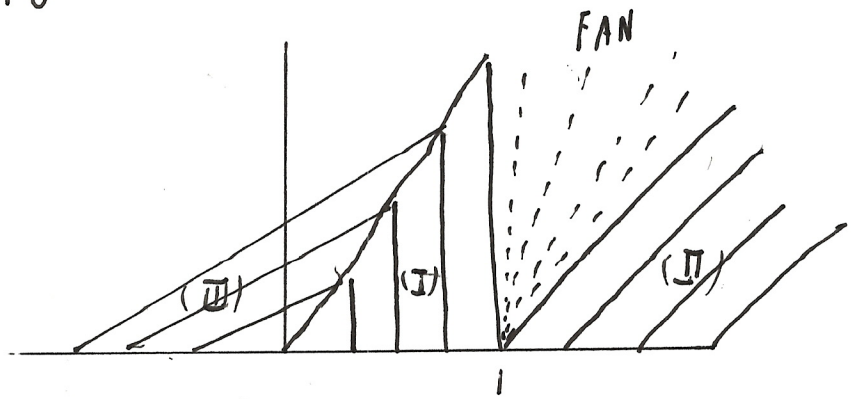
$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x < 1 \\ 0 & \text{if } x \ge 1 \end{cases}$$

WHERE WOULD YOU PUT A SHOCK? WHERE DOES THE EXPANSION FAN GO?

SOLUTION $c(u) = 1-u$ $\quad c > 0$ ON $0 < u < 1$. $\quad c(0) = 1$, $c(1) = 0$

$c' < 0$





IN REGION (I) $\dfrac{dx}{dt} = 1-u$ $\quad x(0) = \xi$ $\quad 0 < \xi < 1$

$\dfrac{du}{dt} = 0$, $\quad u = 1$ AT $t = 0$

$\longrightarrow u \equiv 1$, $\quad x = \xi$

IN REGIONS (II) AND (III) $\quad u \equiv 0$

EXPANSION FAN NOTICE IT IS CENTERED AT $x = 1$.

SO $u = g(\Lambda)$ WITH $\Lambda = \dfrac{x-1}{t} \longrightarrow c(g(\Lambda)) = \Lambda$, $0 \le \dfrac{x-1}{t} \le 1$

THUS $1 - g(\Lambda) = \Lambda \rightarrow g(\Lambda) = 1 - \Lambda$

SO $u = 1 - \dfrac{(x-1)}{t}$ IN $0 \le \dfrac{x-1}{t} \le 1$

SHOCK SPEED IF WE USE SHOCK CONDITION $\dfrac{ds}{dt} = \dfrac{g(u_+) - g(u_-)}{u_+ - u_-}$.

THEN WE OBTAIN

$$\frac{ds}{dt} = \frac{(u_+ - u_+^2/2) - (u_- - u_-^2/2)}{(u_+ - u_-)} = \frac{(u_+ - u_-) - \frac{1}{2}(u_+^2 - u_-^2)}{u_+ - u_-}.$$

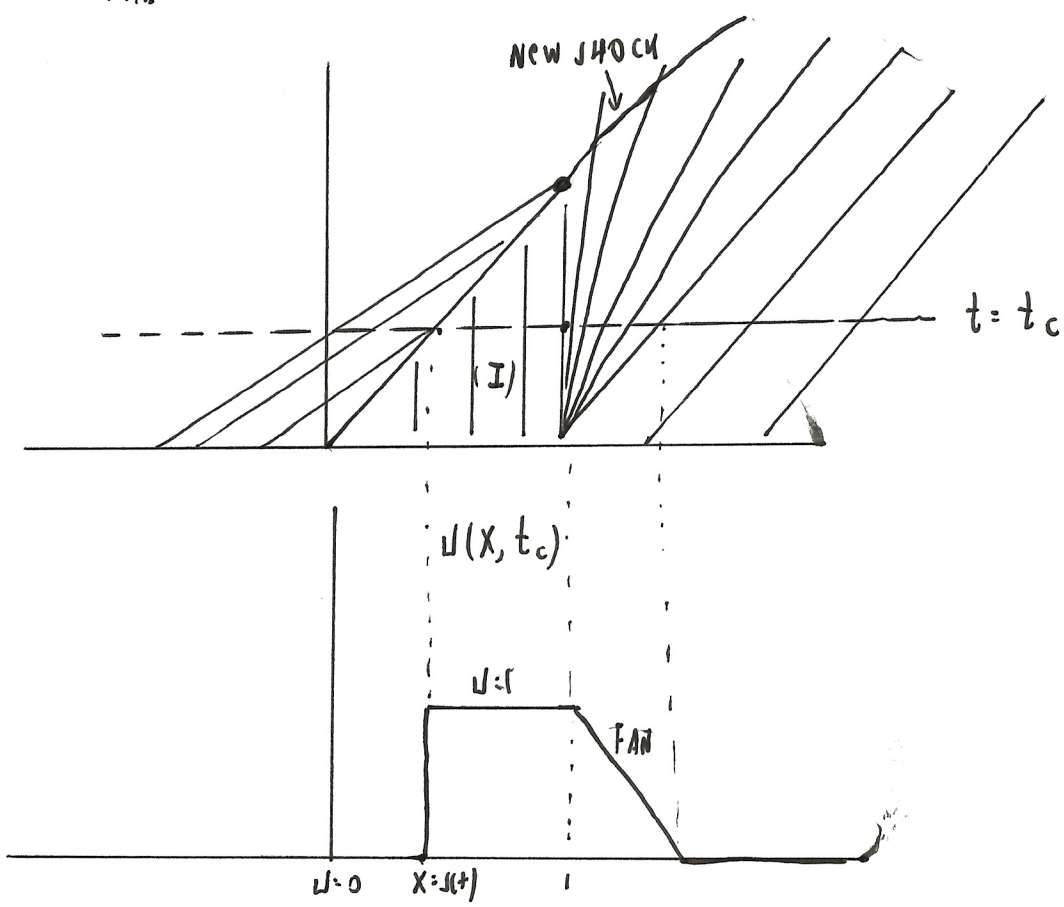
THIS GIVES $\quad \dfrac{ds}{dt} = 1 - \dfrac{1}{2}(u_+ + u_-).$

IN REGION I $\quad u_+ = 1$
REGION III $\quad u_- = 0$

$$\longrightarrow \quad \frac{ds}{dt} = \frac{1}{2}. \qquad s(0) = 0$$

$$s = t/2.$$

NOTICE AS IN PREVIOUS EXAMPLE THAT EVENTUALLY THE SHOCK WILL CATCH UP WITH THE EDGE OF THE FAN.

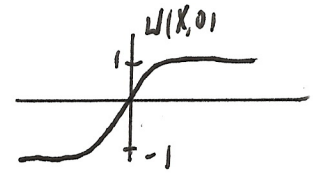


AS IN PREVIOUS EXAMPLE WE NEED TO FIT A NEW SHOCK WHEN WE LOSE REGION I, i.e. WHEN $t/2 = 1 \rightarrow t = 2$.

EXAMPLE $\quad u_t - u\,u_x = 0, \quad -\infty < x < \infty$

$u(x,0) = \tanh x$

$q = -u^2/2$



(i) SKETCH (x,t) PLANE. SHOW THAT A SHOCK FIRST FORMS AT $(t,x) = (1,0)$

(ii) SHOW THAT THE SHOCK THAT EMERGES IS STATIONARY.

SOLUTION $\quad$ ON $\dfrac{dx}{dt} = -u, \quad x(0) = \xi$

$\dfrac{du}{dt} = 0, \quad u(0) = \tanh \xi$

$c(u) = -u$

IF $u > 0 \quad c < 0$

IF $u < 0 \quad c > 0$

NOW $\quad u = \tanh \xi$

$x = -\tanh \xi \, t + \xi$

WAVE WILL STEEPEN AND FORM A SHOCK.

NOW $\quad u_x = \operatorname{sech}^2 \xi \; \xi_x$

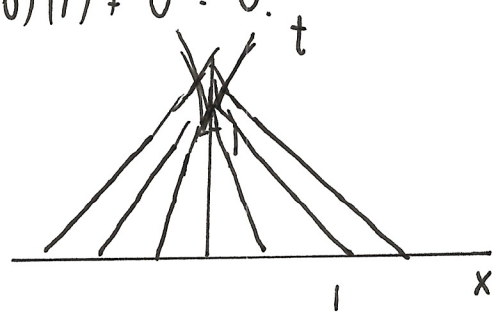
$\dfrac{d}{d\xi} \tanh \xi = \operatorname{sech}^2 \xi$

AND $\quad 1 = (-\operatorname{sech}^2 \xi \, t + 1)\,\xi_x$

SO $\quad u_x = \dfrac{\operatorname{sech}^2 \xi}{1 - t \operatorname{sech}^2 \xi}$

$t_B = \dfrac{1}{\max\limits_{\xi} |\operatorname{sech}^2 \xi|} = 1$ WHICH OCCURS AT $\xi = 0$.

SO $\quad t_B = 1, \; \xi_B = 0 \longrightarrow x_B = -\tanh(0)(1) + 0 = 0.$

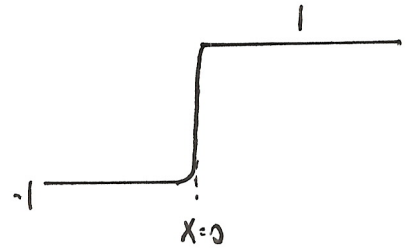


NOW SHOCK SPEED IS

$$\frac{ds}{dt} = -\frac{1}{2}\left(u^+ + u^-\right) = 0$$

$u^+ = -u^-$ (U IS ODD)

SO THE SHOCK THAT FORMS IS STATIONARY

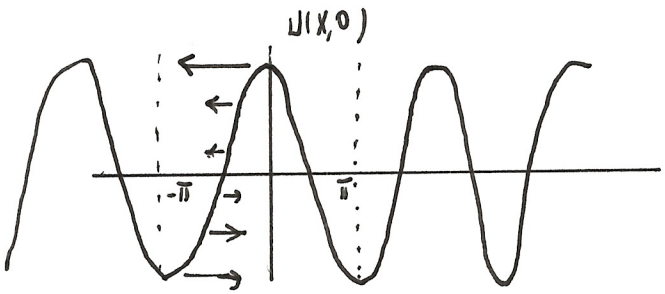


EXAMPLE $\quad u_t - u\,u_x = 0, \quad -\infty < x < \infty$

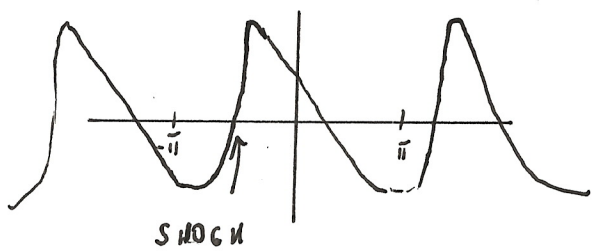
$u(x,0) = \cos x.$ $\qquad g(u) = -u^2/2$ FOR FITTING SHOCK

- SKETCH HOW INITIAL WAVE STEEPENS
- FIND BREAKING TIME AND WHERE IT OCCURS
- SHOW THAT THE SHOCK THAT EMERGES IS STATIONARY.

SOLUTION $\quad c(u) = -u \begin{cases} >0 & \text{if } u<0 \\ <0 & \text{if } u>0 \end{cases} \qquad c(0)=0, \quad c' < 0.$



IN $-\pi < x < \pi$ THE WAVE WILL STEEPEN AS SHOWN. EXTEND THIS PICTURE PERIODICALLY



SHOCK WILL FORM.

CHARACTERISTICS: ON $\quad dx/dt = -u, \quad x(0) = \xi \quad \rightarrow \quad x = -\cos\xi\, t + \xi$

$du/dt = 0, \quad u(0) = \cos\xi \qquad u = \cos\xi$

$u_x = -\sin\xi\, \xi_x$

$\xi_x + \sin\xi\, \xi_x\, t = 1 \qquad \rightarrow \quad u = \cos(x + ut)$

NOW $\quad u_x = -\dfrac{\sin\xi}{1 + t\sin\xi}.$

$t_B = \dfrac{1}{\max\limits_{-\pi<\xi<0} |\sin\xi|} = 1$ WHEN $\xi = -\pi/2$ $\quad$ BY PERIODICITY $\rightarrow \xi = -\pi/2 + (2n-1)\pi/2$

THIS YIELDS THAT $x_B = \xi.$

BREAKING TIME AT $t_B = 1$ THAT OCCURS WHEN $\xi_B = (2n-1)\pi/2$

WHICH CORRESPONDS TO $x_B = -\cos(\xi)t + \xi = 0\cdot 1 + \xi_B.$

INFINITE # POINTS WHERE WAVE BREAKS.

SINCE $\dfrac{ds}{dt} = -\dfrac{1}{2}\dfrac{(u_+^2 - u_-^2)}{(u_+ - u_-)} = -\dfrac{1}{2}(u_+ + u_-)$ WE HAVE BY SYMMETRY THAT $u_+ = -u_-$ SO $ds/dt = 0$

THE SHOCK IS STATIONARY.