SAMPLE PROBLEMS

PROBLEM 1 CONSIDER THE FOLLOWING EIGENVALUE PROBLEM FOR $\phi(x)$:

$$\phi'' - 2x\phi' + \lambda\phi = 0, \quad 0 < x < 1; \quad \phi(0) = 0, \quad \phi'(1) = -\phi(1)$$

PROVE THAT ANY EIGENVALUE λ MUST BE REAL AND POSITIVE.

ALSO, WHAT IS THE ORTHOGONALITY RELATION FOR EIGENFUNCTIONS?

PROBLEM 2 CONSIDER AXIALLY SYMMETRIC DIFFUSION FOR $u(r, z, t)$ IN A FINITE CYLINDER OF RADIUS $a > 0$ AND HEIGHT $H > 0$ WITH INSULATING BOUNDARY CONDITIONS MODELED BY

$$u_t = u_{rr} + \frac{1}{r} u_r + u_{zz}, \quad 0 \le r \le a, \quad 0 \le z \le H, \quad t \ge 0$$

WITH $\quad u_z = 0$ ON $z = 0, H$; $\quad u_r = 0$ ON $r = a$, $\quad u, u_r$ BOUNDED AS $r \to 0$

$$u(r, z, 0) = f(r, z)$$

- FIND AN EIGENFUNCTION EXPANSION REPRESENTATION FOR THE SOLUTION $u(r, z, t)$.
- CALCULATE $\displaystyle\lim_{t \to \infty} u(r, z, t)$.

PROBLEM 3 (SHORT ANSWER QUESTIONS)

(i) SOLVE $\quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \quad$ IN $\quad 0 \le r \le a, \quad 0 \le \varphi \le 2\pi$

$$u(a, \varphi) = 2\cos^2\varphi + 1; \quad u, u_\varphi \text{ ARE } 2\pi \text{ PERIODIC}$$

$$u, u_r \text{ BOUNDED AS } r \to 0.$$

(HINT: $\cos^2\varphi = \dfrac{1 + \cos 2\varphi}{2}$)

(ii) CONSIDER THE FORCED WAVE EQUATION

$$u_{tt} + a u_t = c^2 u_{xx} + \sin(\omega t), \quad 0 < x < L,\ t > 0$$

$$u(0,t) = 0, \quad u(L,t) = 0; \quad u(x,0) = u_t(x,0) = 0.$$

HERE $c > 0$ WHILE $a > 0$ IS SMALL. FOR WHAT FREQUENCIES ω WILL WE OBTAIN A VERY LARGE RESPONSE FOR u WHEN $a > 0$ IS SMALL? (HINT: WHAT ARE THE RESONANT FREQUENCIES IF $a = 0$?)

(iii) CONSIDER RADIALLY SYMMETRIC DIFFUSION IN THE UNIT SPHERE WHERE $u(r,t)$ SATISFIES

$$u_t = D\left(u_{rr} + \frac{2}{r} u_r \right) \quad \text{IN } 0 \leq r \leq 1,\ t > 0$$

$$u(1,t) = 0; \quad u, u_r \text{ BOUNDED AS } r \to 0; \quad u(r,0) = f(r).$$

HERE $D > 0$ IS A CONSTANT.

SHOW THAT FOR LARGE TIME, i.e. $t \gg 1$, THAT

$$u(r,t) \cong A e^{-\sigma t} \frac{\sin(\pi r)}{r}, \quad \text{FOR SOME } A, \sigma$$

THAT YOU ARE TO FIND.

(iv) FIND THE SOLUTION TO

$$u_{rr} + \frac{1}{r} u_r - 4u = 0 \quad \text{IN } r \geq a$$

$$u(a) = 1, \quad u \to 0 \text{ AS } r \to \infty.$$

PROBLEM 4 CONSIDER LAPLACE'S EQUATION FOR $u = u(r, \varphi)$ IN THE WEDGE $0 \le r \le R$, $0 \le \varphi \le \alpha$ SATISFYING

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \quad \text{IN} \quad 0 \le r \le R, \; 0 \le \varphi \le \alpha$$

$$u(r, 0) = 0, \quad u(r, \alpha) = 0 \quad \text{FOR} \quad 0 \le r \le R$$

$$u(R, \varphi) = 1 \quad \text{FOR} \quad 0 \le \varphi \le \alpha.$$

HERE $0 < \alpha < 2\pi$, AND $R > 0$.

(i) FIND AN INFINITE SERIES REPRESENTATION FOR $u(r, \varphi)$.

(HINT: YOU CAN USE $\int_0^\alpha \sin\left(\frac{n\pi\varphi}{\alpha}\right) d\varphi = \frac{\alpha}{n\pi}\left[1 - (-1)^n\right]$).

(ii) SUM THE INFINITE SERIES IN (i).

(iii) AS $r \to 0$ SHOW THAT $u \approx A r^{\beta} \sin(\beta\varphi)$ FOR SOME A AND $\beta > 0$ TO BE FOUND. FOR WHAT VALUES OF THE WEDGE ANGLES α IS u_r UNBOUNDED AS $r \to 0$?

PROBLEM 5 CONSIDER RADIALLY SYMMETRIC HEAT EQUATION FOR THE TEMPERATURE $u(r,t)$ IN A SPHERE OF RADIUS $a>0$ WITH AN INSULATING BOUNDARY AND AN INTERNAL SOURCE $G(r)$:

$$u_t = D\left(u_{rr} + \frac{2}{r} u_r \right) + G(r), \qquad 0 \le r \le a, \quad t > 0$$

$$u_r = 0 \quad \text{ON} \quad r = a \,; \qquad u,\ u_r \ \text{BOUNDED AS} \ r \to 0$$

$$u(r,0) = F(r)$$

HERE $D>0$ IS A CONSTANT. HERE $G(r)$ IS A SMOOTH FUNCTION.

(i) DETERMINE THE CONDITION ON $G(r)$ THAT IS NEEDED IN ORDER FOR A STEADY-STATE SOLUTION $U(r)$ TO EXIST.

(ii) FIND AN EIGENFUNCTION EXPANSION REPRESENTATION FOR THE SOLUTION $u(r,t)$ FOR ANY $G(r)$

(iii) FIND AN APPROXIMATION TO THE SOLUTION THAT IS VALID WHEN t IS LARGE. DOES THIS CONFIRM YOUR RESULT IN (i)?

PROBLEM 6 LET $u(x,t)$ SOLVE WITH $D>0$, $k>0$ CONSTANTS

$$u_t = D u_{xx} - k u \quad \text{IN} \quad x>0,\ t>0$$

$$u(0,t) = t \ \text{ FOR } t \ge 0\,; \quad u(x,0) = 0, \quad u \ \text{BOUNDED AS} \ x \to \infty,\ t>0.$$

(i) FIND THE SOLUTION $u(x,t)$ IN TERMS OF A CONVOLUTION INTEGRAL

(ii) CALCULATE THE FLUX $J(t) \equiv -D u_x(0,t)$.

(RECALL: $\mathcal{L}^{-1}\left[e^{-\lambda\sqrt{s}} \right] = \left(\lambda / 2\sqrt{\pi}\, t^{3/2} \right) e^{-\lambda^2/4t}$ FOR $\lambda > 0$.)

PROBLEM (*) $\phi'' - 2x\phi' + \lambda\phi = 0$, $0<x<1$; $\phi(0)=0$, $\phi'(1) = -\phi(1)$.

WE PUT IN SL FORM: MULTIPLY BY p:

$$p\phi'' - \underline{2xp}\,\phi' + \lambda p\phi = 0.$$

WANT $$(p\phi')' + \lambda p\phi = 0$$

SO $$p\phi'' + \underline{p'}\phi' + \lambda p\phi = 0$$

WE ENFORCE $p' = -2xp$ SO $(\log p)' = -2x \to \log p = -x^2 \to p = e^{-x^2}$.

THUS (*) IS EQUIVALENT TO

$$\left.\begin{aligned} &[e^{-x^2}\phi']' + \lambda e^{-x^2}\phi = 0 \\ \text{WITH}\quad &\phi(0)=0,\quad \phi'(1) = -\phi(1). \end{aligned}\right\}\ (+)$$

TO PROVE EIGENVALUES ARE REAL DEFINE $\mathcal{L}\phi \equiv -[e^{-x^2}\phi']'$ SO

$$\mathcal{L}\phi = \lambda w\phi \quad \text{WITH} \quad w = e^{-x^2}.$$

WE USE LAGRANGE'S IDENTITY WITH ϕ AND $\bar\phi$, WHERE $\mathcal{L}\bar\phi = \bar\lambda w\bar\phi$, SO THAT

$$\int_0^1 \phi\,\mathcal{L}\bar\phi\,dx = \int_0^1 \bar\phi\,\mathcal{L}\phi\,dx$$

$$\downarrow$$

$$\int_0^1 \phi\,\bar\lambda w\bar\phi\,dx = \int_0^1 \bar\phi\,\lambda w\phi\,dx \to (\bar\lambda - \lambda)\int_0^1 |\phi|^2 w\,dx = 0$$

WITH $|\phi|^2 = \phi\bar\phi$. THUS $\lambda = \bar\lambda$ REAL. THE ORTHOGONALITY RELATION IS $\int_0^1 e^{-x^2}\phi_j\phi_k\,dx = 0$ FOR $j \neq k$.

TO PROVE POSITIVITY MULTIPLY (+) BY ϕ AND INTEGRATE:

$$\int_0^1 \phi[e^{-x^2}\phi']'\,dx + \lambda\int_0^1 e^{-x^2}\phi^2\,dx = 0.$$

USE IBP TO GET $$\phi e^{-x^2}\phi'\Big|_0^1 - \int_0^1 \phi'^2 e^{-x^2}dx + \lambda\int_0^1 e^{-x^2}\phi^2\,dx$$

USING $\phi(0)=0$ AND $\phi'(1) = -\phi(1)$ WE GET

$$-[\phi(1)]^2 e^{-1} - \int_0^1 \phi'^2 e^{-x^2}dx = -\lambda\int_0^1 e^{-x^2}\phi^2\,dx.$$

THUS $\lambda\int_0^1 e^{-x^2}\phi^2\,dx = [\phi(1)]^2 e^{-1} + \int_0^1 \phi'^2 e^{-x^2}dx$. WE CONCLUDE THAT $\lambda \geq 0$.

HOWEVER $\lambda \neq 0$ SINCE IF WE SET $\lambda = 0$ WE GET ONLY TRIVIAL SOLUTION $\phi \equiv 0$.

THUS $\lambda > 0$.

PROBLEM 2 $\quad u_t = u_{rr} + \frac{1}{r} u_r + u_{zz}, \quad 0 < r < a, \quad 0 < z < H, \quad t > 0$

$u_r = 0$ ON $r = a$, u BOUNDED AS $r \to 0$

$u_z = 0$ ON $z = 0, H$

$u(r, z, 0) = f(r, z)$.

NOW WE SEPARATE VARIABLES TO OBTAIN

$$\frac{T'}{T} = \frac{R'' + 1/r\, R'}{R} + \frac{Z''}{Z} = -\lambda.$$

THEN $\quad Z'' + \mu Z = 0 \qquad Z_m(z) = \cos\left(\frac{m\pi z}{H}\right), \quad \mu_m = \frac{m^2\pi^2}{H^2}, \quad m = 0, 1, 2,$

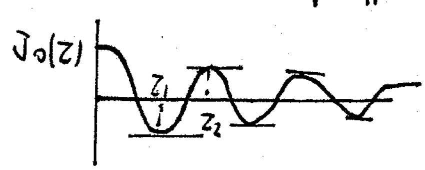
$Z'(0) = Z'(H) = 0$

THEN WE HAVE $\quad R'' + \frac{1}{r} R' + (\lambda - \mu) R = 0.$

$R'(a) = 0 \qquad R(0)$ NICE

let $\sigma = \lambda - \mu$. THEN $\quad R_n(r) = J_0(\sqrt{\sigma_n}\, r)$ $\quad$ ALSO NOTE $\sigma = 0$, $R_0 = 1$ IS EIGENPA

WITH $\quad J_0'(\sqrt{\sigma_n}\, a) = 0 \qquad \sqrt{\sigma_n}\, a = z_n$



SO $\quad \sigma_n = z_n^2 / a^2$.

THEN $\quad \lambda_{mn} = \sigma_n + \mu_m = \sigma_n + m^2\pi^2/H^2$. $\quad$ DEFINE $\sigma_0 = 0$ SINCE $J_0(0) = 1$ will get FIRST eigenpair FOR $R(r)$.

THEN $\quad u(r, z, t) = A_{00} + \sum_{\substack{m=0 \\ (m,n) \neq (0,0)}}^{\infty} \sum_{n=0}^{\infty} A_{mn} J_0(\sqrt{\sigma_n}\, r) \cos\left(\frac{m\pi z}{H}\right) e^{-\lambda_{mn} t}$

NOW FOR $t > 0$ $\quad A_{00} = \dfrac{2\pi \int_0^a \int_0^H r f(r, z)\, dr\, dz}{\pi a^2 H}$.

SO FOR $t \gg 1$, $u(r, z, t) \approx A_{00}$.

THEN $\quad A_{mn} = \dfrac{2 \int_0^a \int_0^H r J_0(\sqrt{\sigma_n}\, r) \cos(m\pi z/H)\, dr\, dz}{H \int_0^a r J_0^2(\sqrt{\sigma_n}\, r)\, dr}$.

PROBLEM 3

(i) THE SOLUTION HAS THE FORM

$$u(r,\varphi) = \sum_{n=0}^{\infty} A_n r^n \cos(n\varphi)$$

WITH $u(a,\varphi) = 2\cos^2\varphi + 1 = \dfrac{2\left[1+\cos(2\varphi)\right]}{2} + 1 = 2 + \cos(2\varphi)$.

SO ALL WE NEED IS

$$u = A_0 + A_2 r^2 \cos(2\varphi)$$

WITH $A_0 = 2$ AND $A_2 = \dfrac{1}{a^2} \longrightarrow u = 2 + \left(r/a\right)^2 \cos(2\varphi)$

(ii) EIGENFUNCTIONS ARE $\Phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

SO WE LOOK FOR $u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(n\pi x/L\right)$

OBSERVE THAT $1 = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{L}\right) \longrightarrow F_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx$

$$F_n = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

SUBSTITUTING WE GET

$$B_n'' + a B_n' + \omega_n^2 B_n = F_n \sin(\omega t)$$

WITH $\omega_n = cn\pi/L$, $n = 1,2\ldots$

SO IF a IS SMALL AND IF $F_n \neq 0$, i.e. n IS ODD, THE FREQUENCIES ω THAT GIVE LARGEST RESPONSE IS

$$\omega = \omega_n \quad n \text{ ODD.}$$

(iv) $u_{rr} + \frac{1}{r} u_r - 4u = 0$ $\quad u = \text{SPAN}\{I_0(2r), K_0(2r)\}$. SINCE $r > a$ WE WANT DECAY AND SO $u = A K_0(2r)$. BUT $u(a) = 1 = A_0 K_0(2a)$

SO $\quad u = K_0(2r)/K_0(2a)$

(iii) BY SEPARATING VARIABLES

$$u(r,t) = \Phi(r)T(t) \longrightarrow \frac{T'}{DT} = \frac{\Phi'' + \frac{2}{r}\Phi'}{\Phi} = -\lambda$$

weight FUNCTION.

SO $$\Phi'' + \frac{2}{r}\Phi' + \lambda\Phi = 0 \longrightarrow \begin{cases} (r^2\Phi')' + \lambda r^2 \Phi = 0 \\ \Phi(1) = 0, \quad \Phi, \Phi' \text{ BOUNDED AS } r \to 0 \end{cases}$$

THE SOLUTION IS $\Phi(r) = \dfrac{\sin(n\pi r)}{r}$, $\lambda_n = n^2\pi^2$.

SO $T_n = e^{-D\lambda_n t}$

AND $$u(r,t) = \sum_{n=1}^{\infty} c_n e^{-D\lambda_n t} \frac{\sin(n\pi r)}{r} \quad \text{WITH} \quad \lambda_n = n^2\pi^2$$

SINCE $u(r,0) = f(r) = \sum_{n=1}^{\infty} c_n \sin(n\pi r)/r$

$$\longrightarrow \quad c_n = \frac{2}{1}\int_0^1 f(r) \frac{\sin(n\pi r)}{r} \cdot r^2 \, dr$$

THIS GIVES $$c_n = 2\int_0^1 r f(r) \sin(n\pi r)\, dr.$$

FOR LARGE t WE

$$u(r,t) \cong c_1 e^{-D\lambda_1 t} \sin(\pi r)/r$$

WITH $$c_1 = 2\int_0^1 r f(r) \sin(\pi r)\, dr$$

$$D\lambda_1 = +\pi^2 D.$$

PROBLEM 4

(i) $u(r,\varphi) = \sum_{n=1}^{\infty} A_n r^{n\pi/\alpha} \sin(n\pi\varphi/\alpha)$ BY SEPARATION OF VARIABLES

$$A_n R^{n\pi/\alpha} = \frac{2}{\alpha}\int_0^{\alpha} \sin\left(\frac{n\pi\varphi}{\alpha}\right) d\varphi = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$A_n = \begin{cases} \frac{4}{n\pi} R^{-n\pi/\alpha} & n \text{ IS ODD} \\ 0 & n \text{ EVEN} \end{cases}$$

THUS $$u(r,\varphi) = \frac{4}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{R}\right)^{(2m-1)\pi/\alpha} \frac{\sin[(2m-1)\pi\varphi/\alpha]}{2m-1}.$$

(ii) LET $z = \left(\frac{R}{r}\right)^{-\pi/\alpha} e^{i\pi\varphi/\alpha}$. THEN,

$$u(r,\varphi) = \frac{4}{\pi}\,\mathrm{IM}\left(\sum_{m=1}^{\infty}\frac{z^{2m-1}}{2m-1}\right) = \frac{4}{\pi}\,\mathrm{IM}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right] = \frac{2}{\pi}\,\mathrm{IM}\left[\log\left(\frac{1+z}{1-z}\right)\right]$$

SO $$u(r,\varphi) = \frac{2}{\pi}\tan^{-1}(V_I/V_R) \qquad \frac{1+z}{1-z} = V_R + iV_I.$$

MULTIPLY BY CONJUGATE $$V_R + iV_I = \frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1-|z|^2+2i\,\mathrm{IM}(z)}{|1-z|^2}.$$

THUS, $$\frac{V_I}{V_R} = \frac{2\,\mathrm{IM}(z)}{1-|z|^2} = \frac{2(r/R)^{\pi/\alpha}\sin(\pi\varphi/\alpha)}{1-(r/R)^{2\pi/\alpha}}.$$

SO $$u(r,\varphi) = \frac{2}{\pi}\tan^{-1}\left(\frac{2(r/R)^{\pi/\alpha}\sin(\pi\varphi/\alpha)}{1-(r/R)^{2\pi/\alpha}}\right).$$

(ii) NOW AS $r \to 0$ $\tan^{-1}(z) \sim z$ SO THAT

$$u(r,\varphi) \approx \frac{4}{\pi}\left(\frac{r}{R}\right)^{\pi/\alpha}\sin(\pi\varphi/\alpha)$$

$$u_r = \frac{4}{\alpha R}\left(\frac{r}{R}\right)^{\pi/\alpha - 1}\sin(\pi\varphi/\alpha)$$

u_r IS BOUNDED IF $0 < \alpha < \pi$.

AND UNBOUNDED IF $\pi < \alpha < 2\pi$

PROBLEM 5 $\quad u_t = D\left(u_{rr} + \frac{2}{r} u_r\right) + G(r) \quad$ IN $0 \le r \le a,\ t \ge 0$

$u_r(a,t) = 0; \quad u, u_r$ BOUNDED AS $r \to 0$

$u(r,0) = f(r).$

(i) STEADY-STATE SATISFIES $\quad \bar{u}_{rr} + \frac{2}{r} \bar{u}_r = -\frac{G(r)}{D}, \quad \bar{u}'(a) = 0, \quad \bar{u}, \bar{u}'$ BOUNDED AS $r \to 0.$

WE MULTIPLY BY r^2 TO GET $\quad (r^2 \bar{u}')' = -\frac{r^2}{D} G(r)$

INTEGRATING ONCE GIVES: $\quad r^2 \bar{u}' = -\frac{1}{D}\int_0^r \lambda^2 G(\lambda)\, d\lambda \quad$ WHICH IS BOUNDED AS $r \to 0$. IT IS INCOMPATIBLE WITH $\bar{u}'(a) = 0$ UNLESS $\int_0^a \lambda^2 G(\lambda)\, d\lambda = 0.$

THUS NO steady-state when $\int_0^a \lambda^2 G(\lambda)\, d\lambda \neq 0.$

(ii) NOW SET $G = 0$. THE EIGENFUNCTION PROBLEM IS

$$\Phi'' + \frac{2}{r}\Phi' + \lambda \Phi = 0 \quad \text{IN } 0 \le r \le a \qquad \text{WEIGHT IS } w = r^2 \text{ SINCE}$$

$$\Phi'(a) = 0; \quad \Phi, \Phi' \text{ BOUNDED AS } r \to 0. \qquad (r^2 \Phi')' + \lambda r^2 \Phi = 0.$$

NOW $\Phi_0 = 1$ AND $\Phi_n(r) = \sin(\sqrt{\lambda}\, r)/r$. WITH $\Phi'(a) = 0 \to \sqrt{\lambda}\frac{\cos(\sqrt{\lambda} a)}{a} - \frac{\sin(\sqrt{\lambda} a)}{a^2} = 0$

THUS $\quad \sqrt{\lambda}\, a = \tan(\sqrt{\lambda}\, a) \to \sqrt{\lambda_n}\, a = z_n \quad$ WITH $\tan(z_n) = z_n, \quad n = 1, 2, 3, \ldots$

SO $\quad \Phi_0(r) = 1, \quad \Phi_n(r) = \frac{\sin(z_n r/a)}{r}, \quad n = 1, 2, \ldots$

NOW EXPAND $\quad u(r,t) = \sum_{n=0}^{\infty} b_n(t)\, \Phi_n(r)$. PUT INTO PDE

$$\sum_{n=0}^{\infty} b_n' \Phi_n = \sum_{n=0}^{\infty} D b_n \left(\Phi_n'' + \frac{2}{r}\Phi_n'\right) + G(r) = -\sum_{n=0}^{\infty} D \lambda_n \Phi_n b_n + G(r).$$

EXPAND $\quad G(r) = \sum_{n=0}^{\infty} g_n \Phi_n(r) \quad$ SO THAT $\quad g_n = \frac{\int_0^a r^2 G(r) \Phi_n\, dr}{\int_0^a r^2 \Phi_n^2\, dr}. \qquad (1)$

THEN $\quad \int_0^a r^2 \Phi_n^2\, dr = \int_0^a \sin^2(\sqrt{\lambda_n}\, r)\, dr = \frac{1}{2}\int_0^a \left(1 - \cos(2\sqrt{\lambda_n}\, r)\right) dr = \frac{1}{2}\left[a - \frac{1}{2\sqrt{\lambda_n}} \sin(2\sqrt{\lambda_n}\, a)\right]$

$$= \frac{1}{2}\left[1 - \frac{1}{2\sqrt{\lambda_n}}\, 2 \sin(\sqrt{\lambda_n}\, a) \cos(\sqrt{\lambda_n}\, a)\right] = \frac{1}{2}\left[1 - \frac{1}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n}\, a) \frac{1}{\sqrt{\lambda_n}\, a} \sin(\sqrt{\lambda_n}\, a)\right]$$

(2) $\quad \int_0^a r^2 \Phi_n^2\, dr = \frac{1}{2}\left[1 - \frac{1}{\lambda_n a} \sin^2(\sqrt{\lambda_n}\, a)\right], \quad n = 1, 2, \ldots, \quad$ IF $n = 0$, $\int_0^a r^2 \Phi_0^2\, dr = a^3/3.$

NOW WE HAVE $\sum_{n=0}^{\infty} (b_n' + D b_n \lambda_n)\Phi_n = \sum_{n=0}^{\infty} g_n \Phi_n$

WE CONCLUDE THAT

$$b_n' + D\lambda_n b_n = g_n \quad \text{FOR } n = 0,1,2,\ldots \qquad (3)$$

WITH $u(r,0) = f(r) = \sum_{n=0}^{\infty} b_n(0)\Phi_n(r) \rightarrow b_n(0) = \dfrac{\int_0^a r^2 f(r)\Phi_n\,dr}{\int_0^a r^2 \Phi_n^2\,dr}$. (4).

NOW WE HAVE $u(r,t) = \sum_{n=0}^{\infty} b_n(t)\Phi_n(r)$.

DECOMPOSING, WE GET $u(r,t) = b_0(t)\Phi_0(r) + \sum_{n=1}^{\infty} b_n(t)\Phi_n(r)$

$\underline{n=0 \text{ MODE}}$ SINCE $\Phi_0 = 1$ AND $\lambda_0 = 0$ WE GET $g_0 = \dfrac{3}{a^3}\int_0^a r^2 G(r)\,dr$,

AND $b_0(0) = \dfrac{3}{a^3}\int_0^a r^2 f(r)\,dr$. SO $b_0' = g_0$ AND $b_0(t) = g_0 t + b_0$.

WE CONCLUDE THAT $b_0(t) = \dfrac{3}{a^3}\left[t\int_0^a r^2 G(r)\,dr + \int_0^a r^2 f(r)\,dr\right]$ (5).

$\underline{n=1,2,3,\ldots}$ WE CALCULATE

$$b_n(t) = \frac{g_n}{D\lambda_n} + \left(b_n(0) - \frac{g_n}{D\lambda_n}\right)e^{-D\lambda_n t}$$

SO THAT

$$u(r,t) = \frac{3}{a^3}\left[t\int_0^a r^2 G(r)\,dr + \int_0^a r^2 f(r)\,dr\right] + \sum_{n=1}^{\infty}\left[\frac{g_n}{D\lambda_n} + \left(b_n(0) - \frac{g_n}{D\lambda_n}\right)e^{-D\lambda_n t}\right]\Phi_n(r).$$

(iii) FOR $t \to \infty$ WE HAVE

$$u(r,t) \sim \frac{3t}{a^3}\int_0^a r^2 G(r)\,dr .$$

THUS IF $\int_0^a r^2 G(r)\,dr \neq 0$, i.e. NO steady-state, then the temperature RISES linearly IN t.

IF $G(r) > 0$, ON $0 \le r \le a$

SOLUTION 6

TAKE LT IN TIME: $\hat{U}(x,s) = \mathcal{L}[\,u(x,t)\,]$. THIS GIVES,

$$D\hat{U}_{xx} - k\hat{U} = s\hat{U} - u(x,0)$$

OR $$\hat{U}_{xx} - \frac{(k+s)}{D}\hat{U} = 0 \quad \text{IN} \quad x>0$$

$$\hat{U}(0,s) = \mathcal{L}[\,t\,] = \frac{1}{s^2}, \quad \hat{U} \text{ BOUNDED AS } x \to \infty.$$

(i) THE SOLUTION IS

$$\hat{U}(x,s) = A\, e^{-\sqrt{\frac{k+s}{D}}\,x} \qquad \text{WITH} \qquad A = \frac{1}{s^2}.$$

NOW $$\hat{U}(x,s) = \hat{F}(s)\;\hat{G}(s+k)$$

WHERE $$\hat{F}(s) = \frac{1}{s^2} \qquad \hat{G}(s) = e^{-\sqrt{\frac{s}{D}}\,x}. \qquad \lambda = x/\sqrt{D}$$

WE HAVE $$f(t) = \mathcal{L}^{-1}[\,\hat{F}(s)\,] = t$$

$$g(t) = \mathcal{L}^{-1}[\,\hat{G}(s)\,] = \frac{x}{2\sqrt{\pi D}\;t^{3/2}}\, e^{-x^2/4Dt}$$

SO $$\mathcal{L}^{-1}[\,\hat{G}(s+k)\,] = \frac{x}{2\sqrt{\pi D}\;t^{3/2}}\, e^{-kt}\, e^{-x^2/4Dt}$$ BY

SHIFT PROPERTY. FINALLY, USE CONVOLUTION

$$u(x,t) = \int_0^t f(t-\tau)\,\frac{x}{2\sqrt{\pi D}\;\tau^{3/2}}\, e^{-k\tau}\, e^{-x^2/4D\tau}\, d\tau$$

BUT $f(t-\tau) = t-\tau$ SO FOR $x>0$,

$$u(x,t) = \frac{x}{2\sqrt{\pi D}} \int_0^{\tau} \frac{(t-\tau)}{\tau^{3/2}}\, e^{-k\tau}\, e^{-x^2/4D\tau}\, d\tau.$$

(ii) NOW CALCULATE

$$-D\hat{U}_x(0,S) = \frac{-D}{S^2}\sqrt{\frac{K+S}{D}}$$

THUS $\hat{J}(s) = \mathcal{L}[J(t)] = \sqrt{D}\,\frac{\sqrt{K+S}}{S^2}$.

WE NEED TO INVERT $\mathcal{L}^{-1}\left[\frac{\sqrt{K+S}}{S^2}\right]$.

MULTIPLY TOP AND BOTTOM BY $\sqrt{K+S}$:

$$\frac{\sqrt{K+S}}{S^2} = \frac{(K+S)}{S^2\sqrt{K+S}} = \frac{K}{S^2\sqrt{K+S}} + \frac{1}{S\sqrt{S+K}}.$$

NOW RECALL $\mathcal{L}^{-1}\left[\frac{1}{S}\right] = 1$, $\mathcal{L}^{-1}\left[\frac{1}{\sqrt{S}}\right] = \frac{1}{\sqrt{\pi t}}$.

$$\mathcal{L}^{-1}\left[\frac{1}{S^2}\right] = t$$

SO BY SHIFT FORMULA AND CONVOLUTION

$$\mathcal{L}^{-1}\left[\frac{\sqrt{K+S}}{S^2}\right] = K\int_0^t (t-\tau)\frac{e^{-K\tau}}{\sqrt{\pi\tau}}\,d\tau + \int_0^t \frac{e^{-K\tau}}{\sqrt{\pi\tau}}\,d\tau.$$

THIS MEANS THAT

$$J(t) = K\sqrt{D}\int_0^t \frac{(t-\tau)}{\sqrt{\pi\tau}}e^{-K\tau}\,d\tau + \int_0^t \frac{e^{-K\tau}}{\sqrt{\pi\tau}}\,d\tau.$$