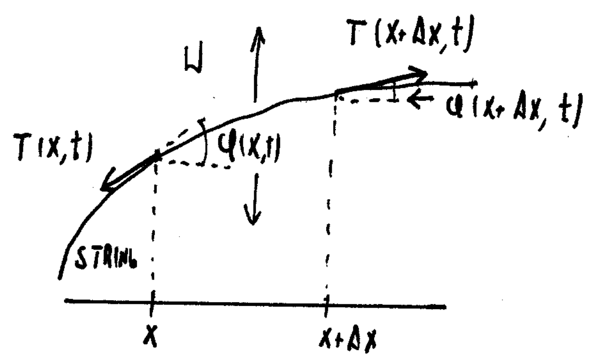
WAVE EQUATION

WE FIRST DERIVE THE 1-D WAVE EQUATION FOR THE VIBRATIONS OF A STRING.



TAKE A SMALL SEGMENT OF A STRING WITH MASS DENSITY $\rho_0(x)$ (UNITS MASS/length). THE MASS IN THE SEGMENT IS $\rho_0(x)\,\Delta x$. ASSUME WE CAN NEGLECT HORIZONTAL DISPLACEMENTS AND THAT THE MOTION IS ENTIRELY VERTICAL AS SHOWN.

USING NEWTON'S LAW IN THE VERTICAL DIRECTION AND IF Q IS THE VERTICAL COMPONENT OF THE BODY FORCE PER UNIT MASS WE GET

$$\rho_0\,\Delta x\; u_{tt} = T(x+\Delta x,t)\sin(\varphi(x+\Delta x,t)) - T(x,t)\sin(\varphi(x,t)) + \rho_0\,\Delta x\,Q.$$

DIVIDING BY Δx AND LETTING $\Delta x \to 0$ GIVES

$$\rho_0\,u_{tt} = \frac{\partial}{\partial x}\left[T\sin\varphi\right] + \rho_0\,Q.$$

NOW $\partial u/\partial x = \tan\varphi$. ASSUMING SMALL DEFLECTIONS SO THAT $\tan\varphi \cong \sin\varphi$ WE GET

$$u_{tt} = \frac{1}{\rho_0}\left(T u_x\right)_x + Q.$$

FINALLY, IF THE TENSION IN THE STRING T IS CONSTANT WE GET

$$u_{tt} = c^2 u_{xx} + Q$$

WHERE $c^2 = T/\rho_0$. HERE $c = \sqrt{\frac{T}{\rho_0}}$ HAS UNITS OF length/TIME.

- IF THE BODY FORCE IS GRAVITY THEN $Q = -g$.

NOW WE DERIVE THE WAVE EQUATION FOR ACOUSTICS, i.e. SOUND PROPAGATION. THE COMPRESSIBLE NAVIER STOKES EQUATION NEGLECTING VISCOSITY IS

$$
(NS)\begin{cases} \rho_t + \nabla\cdot[\rho\, \underline{u}] = 0 \\ \underline{u}_t + \underline{u}\cdot\nabla\underline{u} = -\dfrac{1}{\rho}\nabla P. \end{cases}
\qquad
\begin{aligned} \rho &: \text{density} \\ P &: \text{pressure} \\ \underline{u} &: \text{velocity.} \end{aligned}
$$

WE WILL ASSUME A <u>CONSTITUTIVE</u> <u>LAW</u> THAT

$$\rho = \rho(P) \qquad (\text{density depends on pressure}).$$

ASSUMING THAT PRESSURE IS NEARLY CONSTANT AT P_0 AND THAT $|\underline{u}|$ IS SMALL WE USE TAYLOR SERIES: WE WRITE $P = P_0 + \hat{P}$ WITH $\hat{P}$ SMALL

$$\rho = \rho(P_0 + \tilde{P}) = \rho(P_0) + \rho'(P_0)\hat{P} + \cdots$$

THIS GIVES THAT DENSITY ρ IS NEARLY UNIFORM WITH VALUE $\rho(P_0) \equiv \rho_0$ AND THAT THE PERTURBATION SATISFIES

$$\tilde{\rho} = \rho'(P_0)\tilde{P}.$$

FROM (NS) WE GET

$$\frac{\partial}{\partial t}[\rho_0 + \tilde{\rho}] + \nabla\cdot[(\rho_0 + \tilde{\rho})\,\underline{u}] = 0$$

$$\underbrace{\underline{u}_t + \underline{u}\cdot\nabla\underline{u}}_{\longleftarrow \text{Neglect} \rightarrow} = -\frac{1}{\rho_0 + \tilde{\rho}}\nabla[P_0 + \hat{P}] \cong -\frac{\nabla\hat{P}}{\rho_0}.$$

THIS GIVES THE APPROXIMATE SYSTEM

$$\frac{\partial}{\partial t}\tilde{\rho} + \rho_0\nabla\cdot\underline{u} = 0 \qquad \text{AND} \qquad \underline{u}_t = -\frac{1}{\rho_0}\nabla\hat{P}.$$

NOW DIFFERENTIATE THE FIRST EQUATION WRT t: $\hat{\rho}_{tt} + \rho_0\nabla\cdot[\underline{u}_t] = 0$, AND COMBINE WITH SECOND EQUATION TO GET $\hat{\rho}_{tt} = \nabla^2\hat{P}$. HOWEVER, $\tilde{\rho} = \rho'(P_0)\tilde{P}$ SO THAT THE WAVE EQUATION FOR PRESSURE PERTURBATIONS IS $\tilde{P}_{tt} = c^2\nabla^2\tilde{P}$ WITH $c = 1/\sqrt{\rho'(P_0)}$.

WAVE EQUATION ON INFINITE INTERVALS

WE CONSIDER $u_{tt} = c^2 u_{xx}$

WE CAN FACTOR THIS AS $(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$

THEN $(\partial_t - c\partial_x)(u_t + c u_x) = 0 \rightarrow u = \phi(x - ct)$ IS A SOLUTION.

ALTERNATIVELY, WE CAN FACTOR $(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$

WHICH YIELDS $u_t - c u_x = 0$ OR $u = \psi(x + ct)$.

BY SUPERPOSITION THE SOLUTION CAN BE WRITTEN

AS $$u = \underbrace{\phi(x-ct)}_{\text{Right-moving wave}} + \underbrace{\psi(x+ct)}_{\text{left-moving wave}} \qquad \phi, \psi \text{ ARBITRARY}$$

THIS IS CALLED <u>D'ALEMBERT'S</u> REPRESENTATION OF THE SOLUTION TO THE WAVE EQUATION.

- NOT USEFUL IF BOUNDARY CONDITIONS APPLIED ON $x = 0, L$ SINCE IT IS THEN DIFFICULT TO DETERMINE ϕ AND ψ. $\rightarrow$ USE EIGENFUNCTION EXPANSIONS.
- ON AN INFINITE INTERVAL THIS IS VERY USEFUL.

<u>EXAMPLE</u> FIND THE SOLUTION TO

$$u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty, \ t > 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

NOW $u(x,t) = \phi(x - ct) + \psi(x + ct)$.

THEN $$\phi(x) + \psi(x) = f(x)$$
$$-c\phi'(x) + c\psi'(x) = g(x)$$

WE INTEGRATE TO OBTAIN $-\phi(x) + \psi(x) = \frac{1}{c}\int_0^x g(\eta)\,d\eta + A$.

THEN ADDING THAT EQUATION TO $\phi(x) + \psi(x) = f(x)$

WE OBTAIN THAT
$$\psi(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(\lambda)\,d\lambda + A/2$$
$$\phi(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(\lambda)\,d\lambda - A/2$$

THEN
$$u(x,t) = \phi(x-ct) + \psi(x+ct) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\lambda)\,d\lambda$$

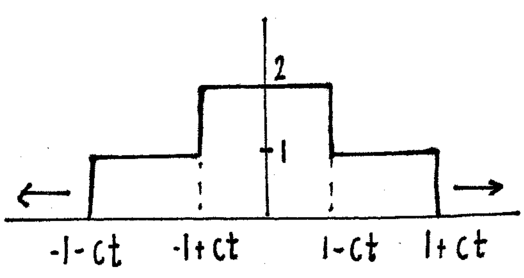
IS THE SOLUTION OF THE WAVE EQUATION. NOTICE THAT WLOG WE CAN TAKE $A = 0$.

EXAMPLE 1 SUPPOSE THAT $g(x) = 0$ AND $f(x) = \begin{cases} 2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

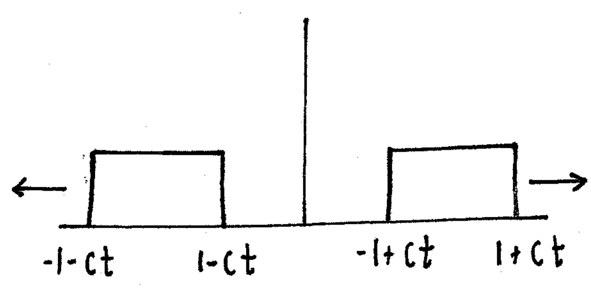
THEN $f(x+ct) = \begin{cases} 2, & -1 < x+ct < 1 \longrightarrow -1-ct < x < 1-ct \\ 0, & \text{otherwise} \end{cases}$

$f(x-ct) = \begin{cases} 2, & -1+ct < x < 1+ct \\ 0, & \text{otherwise} \end{cases}$

THEN $u(x,t)$ FOR $t < 1/c$



$u(x,t)$ FOR $t > 1/c$



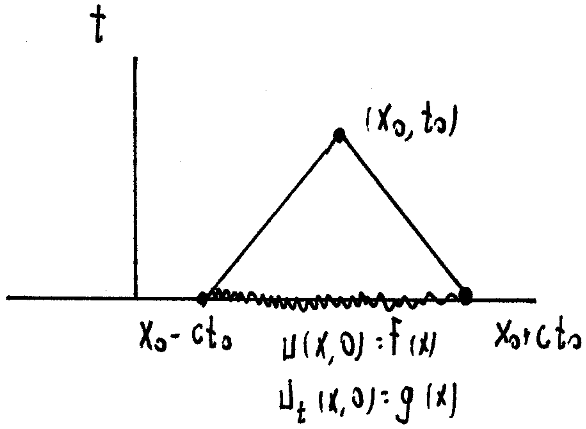
DOMAIN OF DEPENDENCE AND RANGE OF INFLUENCE

CONSIDER $u_{tt} = c^2 u_{xx}$

$u(x,0) = f(x), \quad u_t(x,0) = g(x) \leftarrow$ INITIAL "DATA"

WE RECALL BY D'ALEMBERT'S FORMULA

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\lambda)\, d\lambda .$$



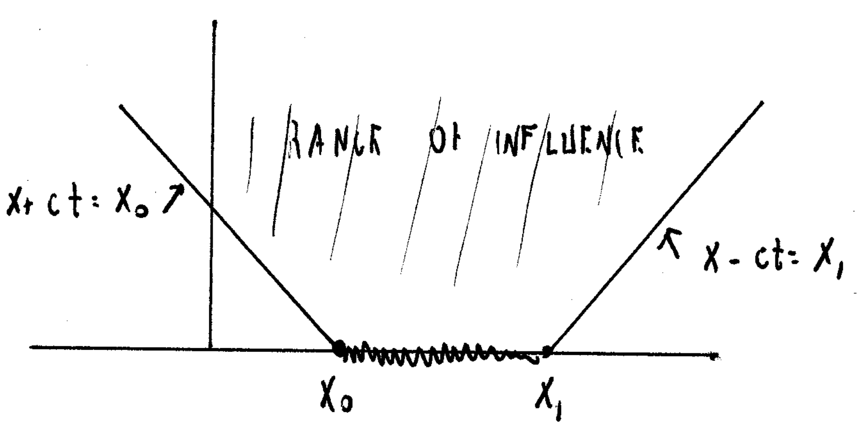
DOMAIN OF DEPENDENCE

WHAT PART OF INITIAL DATA $f(x)$ AND $g(x)$ INFLUENCE THE SOLUTION AT POINT (x_0, t_0)? NOTICE

$$u(x_0,t_0) = \frac{1}{2}\left[f(x_0+ct_0) + f(x_0-ct_0) \right] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(\lambda)\, d\lambda .$$

OBSERVE THAT SOLUTION AT (x_0,t_0) DEPENDS ON "DATA" IN $x_0 - ct_0 < x < x_0 + ct_0$.

THE <u>RANGE OF INFLUENCE</u> OF INITIAL DATA $f(x)$ AND $g(x)$ IS THE REGION OF THE (x,t) PLANE INFLUENCED BY ANY CHANGES IN $f(x)$ AND/OR $g(x)$ OVER SOME REGION OF X-AXIS.



- IF INITIAL DATA CHANGED BETWEEN $x_0 < x < x_1$ IT INFLUENCES THE SOLUTION $u(x,t)$ IN THE SHADED REGION.

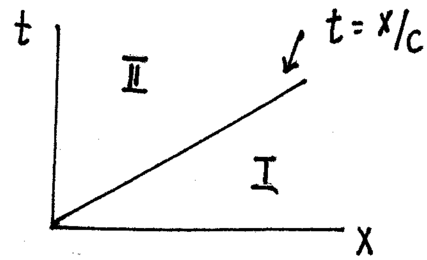
EXAMPLE I FIND THE SOLUTION $u(x,t)$ TO THE FOLLOWING WAVE EQUATION IN THE QUARTER PLANE $x>0,\ t>0$.

$$u_{tt} = c^2 u_{xx} \qquad x>0,\ t>0$$

$$u(0,t) = h(t), \quad u(x,0) = g(x), \quad u_t(x,0) = f(x).$$

THE GENERAL SOLUTION IS $u(x,t) = F(x-ct) + G(x+ct)$.

THERE ARE TWO DIFFERENT REGIONS:



SATISFYING THE DATA GIVES

$$(1) \qquad F(-ct) + G(ct) = h(t), \quad t>0$$

$$(2) \qquad F(x) + G(x) = g(x), \quad x>0$$

$$(3) \qquad -cF'(x) + cG'(x) = f(x), \quad x>0.$$

NOW (2) AND (3) YIELD F AND G FOR POSITIVE ARGUMENTS.

WE OBTAIN $\quad -cF(x) + cG(x) = \int_0^x f(\lambda)\,d\lambda$

HENCE
$$-F(x) + G(x) = \frac{1}{c}\int_0^x f(\lambda)\,d\lambda$$
$$F(x) + G(x) = g(x)$$

THIS YIELDS
$$(4) \qquad F(x) = \frac{g(x)}{2} - \frac{1}{2c}\int_0^x f(\lambda)\,d\lambda, \quad x>0$$

$$(5) \qquad G(x) = \frac{g(x)}{2} + \frac{1}{2c}\int_0^x f(\lambda)\,d\lambda, \quad x>0.$$

HENCE
$$(6) \quad F(x-ct) = \frac{1}{2}g(x-ct) - \frac{1}{2c}\int_0^{x-ct} f(\lambda)\,d\lambda \qquad x-ct>0$$

$$(7) \quad G(x+ct) = \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_0^{x+ct} f(\lambda)\,d\lambda$$

THIS GIVES
$$u(x,t) = \frac{1}{2}\left[g(x-ct) + g(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f(\lambda)\,d\lambda \quad \text{REGION I.}$$
$$x-ct>0$$

NOW FROM (1) WE OBTAIN

$$F(-ct) = h(t) - G(ct) \qquad t > 0$$

SO $$F(z) = h\left(-z/c\right) - G(-z), \quad z < 0$$

THEREFORE
$$(8) \quad F(x-ct) = h(t - x/c) - G(ct - x), \quad x - ct < 0$$
$$(9) \quad G(x+ct) = \frac{1}{2} g(x+ct) + \frac{1}{2c}\int_0^{x+ct} f(\lambda)\, d\lambda$$

HENCE $$(10) \quad F(x-ct) = h(t - x/c) - \frac{1}{2} g(ct-x) - \frac{1}{2c}\int_0^{ct-x} f(\lambda)\, d\lambda$$

NOW ADDING (9) AND (10) WE OBTAIN

$$u(x,t) = h(t - x/c) + \frac{1}{2}\left[g(x+ct) - g(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} f(\lambda)\, d\lambda$$

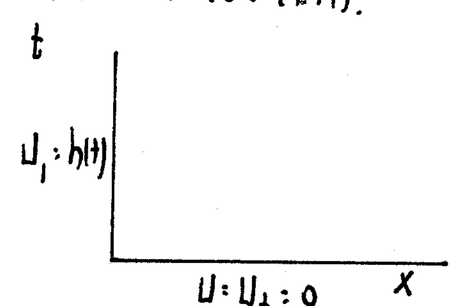
THIS IS THE SOLUTION IN REGION II WHERE $t > x/c$. □

THERE IS A SECOND APPROACH TO SOLVE THIS PROBLEM USING A REFLECTION PRINCIPLE. WE CONSIDER

$$(11) \begin{cases} u_{tt} = c^2 u_{xx}, & x > 0,\ t > 0 \\ u(0,t) = h(t); & u(x,0) = g(x),\ u_t(x,0) = f(x) \end{cases}$$

WE FIRST DECOMPOSE (11) INTO TWO SEPARATE PROBLEMS.
LET $u_1(x,t)$ SOLVE

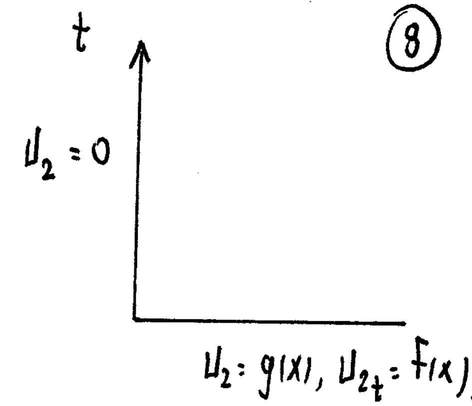
$$(12) \begin{cases} u_{1tt} = c^2 u_{1xx} & \text{IN } x > 0,\ t > 0 \\ u_1(0,t) = h(t); & u_1(x,0) = u_{1t}(x,0) = 0 \end{cases}$$



THE SOLUTION IS SIMPLY

$$u_1(x,t) = \begin{cases} h(t - x/c) & \text{IN } 0 \le x \le ct \\ 0 & \text{IN } x > ct. \end{cases} \quad (13)$$

WE LET $u_2(x,t)$ SOLVE



$$(13)\quad \begin{cases} u_{2tt} = c^2 u_{2xx} \quad \text{IN } x>0,\ t>0 \\ u_2(0,t) = 0\ ;\quad u_2(x,0) = g(x),\ u_{2t}(x,0) = f(x) \end{cases}$$

ONCE WE WILL FIND $u_2(x,t)$ THE SOLUTION TO (11) IS SIMPLY

$$u(x,t) = u_1(x,t) + u_2(x,t).$$

WE NOW SOLVE FOR u_2 BY <u>METHOD OF REFLECTION</u>. WE "EXTEND" THE INITIAL DATA TO $x<0$ BY REQUIRING IT TO BE AN <u>ODD FUNCTION</u>.

$$g_{odd}(x) = \begin{cases} g(x) & \text{IF } x>0 \\ -g(-x) & \text{IF } x<0 \end{cases}, \quad f_{odd}(x) = \begin{cases} f(x) & \text{IF } x>0 \\ -f(-x) & \text{IF } x<0. \end{cases} \quad (14)$$

THEN ON $-\infty < x < \infty$ WE HAVE BY D'ALEMBERT'S FORMULA THAT

$$u_2(x,t) = \frac{1}{2}\left[g_{odd}(x+ct) + g_{odd}(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_{odd}(s)\,ds. \quad (15)$$

NOTICE CRUCIALLY THAT WITH THIS ODD EXTENSION $u_2(0,t) = 0$ IS AUTOMATICALLY SATISFIED. FINALLY, WE CALCULATE (15) IN $x \geq 0,\ t \geq 0$.

- <u>IF $0 < x < ct$</u> WE HAVE $x - ct < 0$ SO WE WRITE $g_{odd}(x-ct) = -g(ct-x)$

$$u_2(x,t) = \frac{1}{2}\left[g(x+ct) + \left(-g(ct-x)\right)\right] + \frac{1}{2c}\int_0^{x+ct} f(s)\,ds + \frac{1}{2c}\int_{x-ct}^{0} -f(-s)\,ds.$$

IN SECOND INTEGRAL REPLACING $s \to -\tilde{s}$ WE GET

$$u_2(x,t) = \frac{1}{2}\left[g(x+ct) - g(ct-x)\right] + \frac{1}{2c}\int_0^{x+ct} f(s)\,ds + \frac{1}{2c}\int_{ct-x}^{0} f(\hat{s})\,d\hat{s}.$$

COMBINING GIVES $\quad u_2(x,t) = \frac{1}{2}\left[g(x+ct) - g(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} f(s)\,ds.$

THEN WITH $u(x,t) = u_1(x,t) + u_2(x,t)$ WE OBTAIN OUR EARLIER RESULT.

- IF $x > ct$, THEN $u_1 \equiv 0$ AND IN (15) $x+ct>0$, $x-ct>0$ SO $g_{odd} \to g$ AND $f_{odd} \to f$. THUS $\quad u_2(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f(s)\,ds.$

REMARK THE IDEA OF METHOD OF REFLECTION IS TO EXTEND THE INITIAL DATA TO $X<0$ EITHER AS ODD (IF WE WANT $U=0$ ON $X=0$, FOR $t>0$) OR EVEN (IF WE WANT $U_X=0$ ON $X=0$ FOR $t>0$). THEN WE CAN USE D'ALEMBERTS SOLUTION AND ONLY NEED TO DETERMINE WHAT IT REDUCES TO IN THE REGIONS $0<X<ct$ AND $X>ct$.

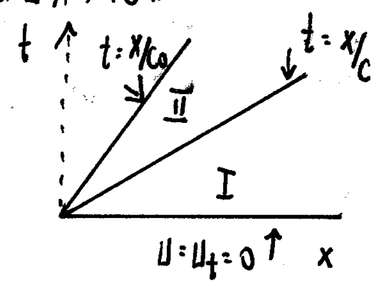
EXAMPLE (DOPPLER SHIFT: EASY VERSION).

LET $C_0<C$, WE WANT TO SOLVE THE WAVE EQUATION

$$U_{tt}=c^2 U_{xx} \quad \text{IN} \quad C_0 t<X<\infty,\ t>0$$

WITH $U(C_0t,t)=\cos(w_0 t)$ "THE SIGNAL"

$U(X,0)=U_t(X,0)=0$ "INITIALLY AMBIENT".



THERE ARE 2 REGIONS TO ANALYZE. IN REGION I AHEAD OF THE WAVE WE HAVE $U\equiv 0$. IN REGION II WE LOOK FOR A RIGHT-MOVING WAVE SOLUTION OF THE FORM $U(X,t)=F(X-ct)$ WHERE F IS TO BE FOUND. IMPOSING $U(C_0t,t)=\cos(w_0 t)=F[(C_0-C)t]$ WE DEFINE $Z\equiv(C_0-C)t$ AND CONCLUDE THAT

$$F(Z)\equiv\cos\left(\frac{W_0}{C_0-C}Z\right).$$

NOW LET $Z\equiv X-ct$ WE GET $U=F(X-ct)$ SO THAT

$$U(X,t)=\cos\left(\frac{W_0}{C_0-C}(X-Ct)\right)=\cos\left(w\left(t-X/C\right)\right) \quad \text{IF} \quad C_0t<X<Ct$$

WHERE THE TEMPORAL FREQUENCY W IS $W\equiv\dfrac{CW_0}{C-C_0}=\dfrac{W_0}{(1-C_0/C)}>W_0$ SINCE $0<C_0/C<1$. THIS MEANS FREQUENCY OF APPROACHING SIGNAL IS LARGER. THIS IS THE DOPPLER EFFECT.

EXAMPLE BY USING LAPLACE TRANSFORMS SHOW THAT THE SOLUTION TO

$$u_{tt} = c^2 u_{xx} + \delta(x) F(t), \quad -\infty < x < \infty, \ t > 0$$

$$u(x,0) = u_t(x,0) = 0$$

IS

$$u(x,t) = \begin{cases} 0 & \text{IF } 0 < t < |x|/c \\ \dfrac{1}{2c}\displaystyle\int_0^{t-|x|/c} F(s)\,ds & \text{IN } t > |x|/c \end{cases}$$

SOLUTION NOW let $U(x,s) = \mathcal{L}[u(x,t)]$.

$$\rightarrow \quad s^2 U - u_t(x,0) - s\,u(x,0) = c^2 U_{xx} + \delta(x) F(s) \quad \text{WITH } F(s) = \mathcal{L}(F(t)).$$

USING $u_t(x,0) = 0$, $u(x,0) = 0$ WE GET

$$U_{xx} - \frac{s^2}{c^2} U = -\frac{1}{c^2}\delta(x) F(s) \quad \text{IN } -\infty < x < \infty.$$

WE WANT U BOUNDED AS $x \to \pm\infty$, U CONTINUOUS ACROSS $x = 0$ AND SATISFY JUMP CONDITION: $U_x|_+ - U_x|_- = -\frac{1}{c^2} F(s)$.

SO
$$U = \begin{cases} A e^{-sx/c} & \text{IF } x > 0 \\ A e^{sx/c} & \text{IF } x < 0. \end{cases}$$

JUMP CONDITION GIVES $-2sA/c = -\frac{1}{c^2} F(s)$ SO $A = \frac{1}{2cs}$.

THIS GIVES $\quad U(x,s) = F(s)\,\dfrac{1}{2sc}\, e^{-s|x|/c}.$

RECALLING $\mathcal{L}[H_\lambda(t)] = e^{-\lambda s}/s$ WE HAVE BY CONVOLUTION FORMULA

$$u(x,t) = \frac{1}{2c}\int_0^t H_\lambda(\tau) F(t-\tau)\,d\tau \quad \text{WHERE } \lambda = |x|/c.$$

NOW
$$H_\lambda(\tau) = \begin{cases} 0 & \text{IF } 0 < \tau < \lambda = |x|/c \\ 1 & \text{IF } \tau > \lambda = |x|/c \end{cases}.$$

SO IF $t < |x|/c$ WE GET $u(x,t) = 0$. IF $t > |x|/c$ WE HAVE $\quad u(x,t) = \dfrac{1}{2c}\displaystyle\int_{|x|/c}^{t} F(t-\tau)\,d\tau$. NOW CHANGE

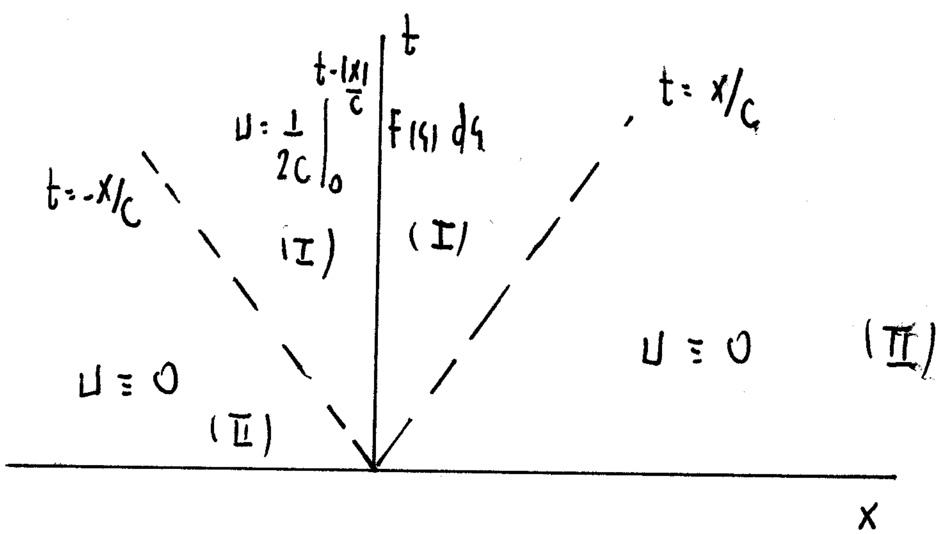
VARIABLES WITH $t - \tau = \xi$, so $dt = -d\xi$.

IF $\tau = t \rightarrow \xi = 0$

$\tau = |x|/c \rightarrow \xi = t - |x|/c$

$$u(x,t) = -\frac{1}{2c}\int_{t-|x|/c}^{0} F(\xi)\,d\xi. \qquad \text{IF } t > |x|/c$$

SO
$$u(x,t) = \begin{cases} \dfrac{1}{2c}\displaystyle\int_0^{t-|x|/c} F(\xi)\,d\xi & \text{IF } t > |x|/c \quad (\mathrm{I}) \\ 0 & \text{IF } 0 < t < |x|/c \quad (\mathrm{II}) \end{cases}$$



MOVING SOURCE PROBLEM (DOPPLER EFFECT - HARDER VERSION)

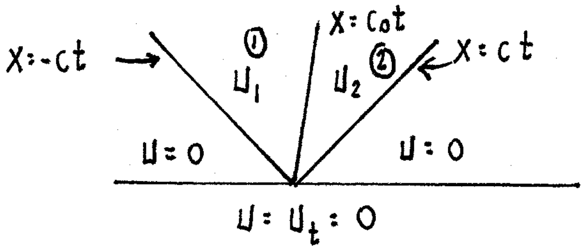
WE CONSIDER THE WAVE EQUATION WITH A CONCENTRATED SOURCE OF STRENGTH $f(t)$ CENTERED ON A MOVING CURVE $x = c_0 t$ WITH $c_0 < c$.

THE MODEL IS

$$\frac{1}{c^2} u_{tt} = u_{xx} + f(t)\,\delta(x - c_0 t) \qquad -\infty < x < \infty, \quad t \geq 0$$

WE ASSUME THAT $u(x,0) = 0$ AND $u_t(x,0) = 0$. NOTE: $\delta(x - c_0 t) = 0$ IF $x \neq c_0 t$.

TO SOLVE THIS PROBLEM WE LET $u_1(x,t) = g(x + ct)$ AND $u_2 = h(x - ct)$ BE THE SOLUTION TO THE LEFT AND RIGHT OF $x = c_0 t$.



NOW TO JOIN THE SOLUTION ACROSS $x = c_0 t$ WE IMPOSE THAT u IS CONTINUOUS BUT THAT u_x HAS A JUMP ACROSS $x = c_0 t$.

TO DERIVE THE JUMP CONDITION WE INTEGRATE FROM $x = c_0 t - \varepsilon$ TO $x = c_0 t + \varepsilon$ TO GET WITH $\varepsilon > 0$ THAT

$$\frac{1}{c^2} \int_{c_0 t - \varepsilon}^{c_0 t + \varepsilon} u_{tt}\, dx = \int_{c_0 t - \varepsilon}^{c_0 t + \varepsilon} u_{xx}\, dx + \int_{c_0 t - \varepsilon}^{c_0 t + \varepsilon} f(t)\,\delta(x - c_0 t)\, dx$$

WE ASSUME u_t, u_{tt} ARE CONTINUOUS ACROSS $x = c_0 t$. BUT THEN

$$\lim_{\varepsilon \to 0} \left[\frac{1}{c^2} \int_{c_0 t - \varepsilon}^{c_0 t + \varepsilon} u_{tt}\, dx = u_x(c_0 t + \varepsilon, t) - u_x(c_0 t - \varepsilon, t) + f(t) \right]$$

THIS YIELDS WITH $\frac{d^2}{dt^2} \frac{1}{c^2} \int_{c_0 t - \varepsilon}^{c_0 t + \varepsilon} u\, dx \to 0$ AS $\varepsilon \to 0$ THAT

$$u_x(c_0 t^+, t) - u_x(c_0 t^-, t) = -f(t)$$

JUMP CONDITION

OR EQUIVALENTLY $\quad u_{2x}(c_0 t, t) - u_{1x}(c_0 t, t) = -f(t)$.

CONTINUITY $\longrightarrow$ $u_2(c_0 t, t) - u_1(c_0 t, t) = 0$

FOR EXAMPLE SUPPOSE THAT

$$f(t) = A\cos(\omega t).$$

IN REGION 2 WE HAVE

$$u_2(x,t) = h(x - ct)$$

WHILE IN REGION 1 WE HAVE $u_1(x,t) = g(x + ct)$.

NOW BY THE CONTINUITY CONDITION

$$(1) \qquad g((c_0 + c)t) = h((c_0 - c)t)$$

AND BY THE JUMP CONDITION:

$$(2) \qquad h'((c_0 - c)t) - g'((c_0 + c)t) = -f(t) = -A\cos(\omega t).$$

NOW WE SOLVE (1) AND (2) TOGETHER. WE DIFFERENTIATE (1) WITH RESPECT TO t TO OBTAIN

$$(c_0 + c)\, g'((c_0 + c)t) = (c_0 - c)\, h'((c_0 - c)t).$$

THEN SUBSTITUTING INTO (2) TO OBTAIN

$$h'[(c_0 - c)t] - \frac{(c_0 - c)}{(c_0 + c)} h'[(c_0 - c)t] = -A\cos(\omega t).$$

LET $\Lambda = (c_0 - c)t$ OR $t = \Lambda/(c_0 - c)$ SO THAT

$$h'(\Lambda) - \frac{(c_0 - c)}{(c_0 + c)} h'(\Lambda) = \frac{2c}{c_0 + c} h'(\Lambda) = -A\cos\left(\frac{\omega\Lambda}{c_0 - c}\right).$$

THIS YIELDS THAT $h'(\Lambda) = -\dfrac{A(c_0 + c)}{2c}\cos\left(\dfrac{\omega\Lambda}{c_0 - c}\right)$.

NOW INTEGRATE: $h(\Lambda) = -\dfrac{A(c_0 + c)(c_0 - c)}{2c\omega}\sin\left(\dfrac{\omega\Lambda}{c_0 - c}\right)$.

THIS YIELDS THAT $h(\Lambda) = -\dfrac{A(c_0^2 - c^2)}{2c\omega}\sin\left(\dfrac{\omega\Lambda}{c_0 - c}\right)$

WE WRITE THIS AS

$$h(\eta) = \frac{A(c_0^2 - c^2)}{2cw} \sin\left(\frac{w\eta}{c - c_0}\right).$$

OR (3) $u_2(x,t) = h(x - ct) = \frac{A(c_0^2 - c^2)}{2cw} \sin\left(\frac{w(x-ct)}{c - c_0}\right).$

THEN USING $h((c_0 - c)t) = g((c_0 + c)t)$ WE LET $\xi = (c_0 + c)t$

SO THAT $g(\xi) = h\left(\frac{c_0 - c}{c_0 + c} t\right) = -\frac{A(c_0^2 - c^2)}{2cw} \sin\left(\frac{w}{c_0 - c} \frac{c_0 - c}{c_0 + c} \xi\right)$

SO $$g(\xi) = -\frac{A(c_0^2 - c^2)}{2cw} \sin\left(\frac{w}{c_0 + c} \xi\right).$$

OR EQUIVALENTLY

(4) $u_1(x,t) = g(x + ct) = -\frac{A(c_0^2 - c^2)}{2cw} \sin\left(\frac{w}{c_0 + c}(x + ct)\right)$

<u>REMARK</u> • SINCE $0 < c_0 < c$ IN (3) THE FREQUENCY IS EFFECTIVELY $w_2 = \frac{w}{(c - c_0)/c} = \frac{w}{1 - c_0/c} > w$ (higher)

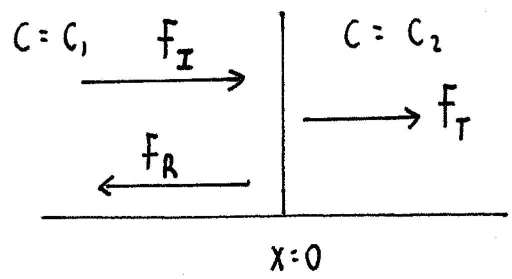
HENCE WE GET <u>higher frequency</u> IN direction that the source is moving. THIS IS THE DOPPLER EFFECT.

• IN REGION I THE FREQUENCY IS

$$w_1 = wc/_{c_0 + c} = w/(1 + c_0/c) < w.$$

THIS MEANS THAT FREQUENCY IS LOWER WHEN SOURCE IS ADVANCING <u>AWAY</u> FROM REGION I.

REFLECTION AT AN INTERFACE



$$u_{tt} = c^2 u_{xx} \qquad c^2 = T/\rho$$

IF $C_1 > C_2$ THEN SECOND string FOR $X > 0$ IS heavier

ASSUME AN INCIDENT WAVE ON THE LEFT, i.e. $F_I(X - C_1 t)$ is GIVEN.

THEN, WE HAVE WITH F_R BEING REFLECTED WAVE AND F_T BEING TRANSMITTED WAVE THAT:

$$u(x,t) = \begin{cases} F_I(x - c_1 t) + F_R(x + c_1 t), & x \leq 0 \\ F_T(x - c_2 t), & x \geq 0. \end{cases} \qquad (1)$$

NOW u AND u_x ARE CONTINUOUS ACROSS THE INTERFACE $X = 0$

$$F_I(-c_1 t) + F_R(c_1 t) = F_T(-c_2 t) \qquad (2)$$

$$F_I'(-c_1 t) + F_R'(c_1 t) = F_T'(-c_2 t) \qquad (3)$$

FOR (3), INTEGRATE WRT t TO GET

$$-\frac{1}{c_1} F_I(-c_1 t) + \frac{1}{c_1} F_R(c_1 t) = -\frac{1}{c_2} F_T(-c_2 t). \qquad (4)$$

NOW (2) AND (4) ARE TWO ALGEBRAIC EQUATIONS FOR F_R AND F_T. THIS CAN BE SOLVED TO OBTAIN

$$F_T(-c_2 t) = \frac{2 c_2}{c_1 + c_2} F_I(-c_1 t). \qquad (5)$$

SO THAT AFTER SWITCHING ARGUMENT: $F_T(x - c_2 t) = \dfrac{2 c_2}{c_1 + c_2} F_I\left(\dfrac{c_1}{c_2}(x - c_2 t)\right)$

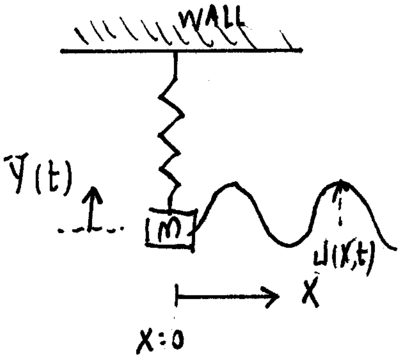
SIMILARLY, $F_R(x + c_1 t) = \left(\dfrac{c_2 - c_1}{c_1 + c_2}\right) F_I\left(-(x + c_1 t)\right)$.

THEN $u(x,t)$ IS GIVEN IN (1).

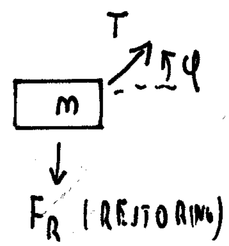
<u>NOTE</u>: IF $C_1 \gg C_2$ TRANSMITTED WAVE AMPLITUDE IS VERY SMALL (string is essentially immobile FOR $X > 0$)

EXAMPLE (RADIATIVE DAMPING)

WE CONSIDER A MASS-SPRING SYSTEM WITH NO DAMPING THAT IS ATTACHED AT $X=0$ TO A SEMI-INFINITE STRING. WHAT WE WILL SHOW IS THAT OSCILLATIONS IN THE SPRING CAN BE DAMPED OUT DUE TO THE COUPLING OWING TO GENERATING WAVES THAT PROPAGATE DOWN THE STRING THAT "RADIATE" ENERGY AWAY FROM THE SPRING.



NOW FOR A FREE BODY DIAGRAM CENTERED AT MASS WE HAVE FOR DEVIATIONS FROM EQUILIBRIUM:

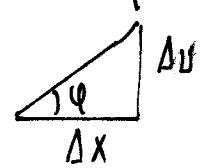


$$m\ddot{Y} = -\kappa Y + T\sin\varphi$$

WHERE T = tension in cable

BUT $T\sin\varphi \cong u_x(0,t)\,T$.

FOR φ SMALL



$\dfrac{\Delta u}{\Delta x} = \tan\varphi \cong \sin\varphi$ φ SMALL.

SO THIS GIVES THE COUPLED PROBLEM:

$$(*)\quad \begin{cases} u_{tt} = c^2 u_{xx} \quad \text{IN } x>0,\ t>0 \\ u(0,t) = Y(t), \quad u \to 0 \text{ AS } x \to +\infty \text{ FOR EACH } t>0. \\ \ddot{Y} + \omega^2 Y = \dfrac{T}{m}\, u_x(0,t) \end{cases}$$

where $\omega = \sqrt{\kappa/m}$, $c = \sqrt{T/\rho}$ ρ: line density of cable

T: tension of cable.

WE CHOOSE AS INITIAL CONDITION:

$$u(x,0) = 0 \qquad u_t(x,0) = 0$$

$$Y(0) = 0, \quad Y'(0) = 1 \leftarrow \text{IMPULSIVE MOTION OF MASS}$$

NOW TAKE LAPLACE TRANSFORM IN TIME:

LET $\hat{U}(x,s) = \mathcal{L}[U(x,t)]$, $\hat{Y}(s) = \mathcal{L}[y(t)]$.

WE GET $\hat{U}_{xx} - \frac{s^2}{c^2}\hat{U} = 0$ IN $x > 0$; $\hat{U}(0,s) = \hat{Y}(s)$

$\hat{U} \to 0$ AS $x \to \infty$

$$s^2 \hat{Y} - 1 + \omega^2 \hat{Y} = \frac{T}{m}\hat{U}_x(0,s)$$

WE OBTAIN $\hat{U}(x,s) = \hat{Y}(s)e^{-sx/c}$ SO THAT $\hat{U}_x(0,s) = -(s/c)\hat{Y}(s)$

WE GET FOR $\hat{Y}(s)$: $(s^2 + \omega^2)\hat{Y} - 1 = -\frac{Ts}{mc}\hat{Y}$

SO THAT $$\hat{Y}(s) = \frac{1}{s^2 + \frac{Ts}{cm} + \omega^2} = \frac{1}{\left(s + \frac{T}{2cm}\right)^2 + \omega^2 - T^2/4c^2m^2}.$$

NOW SUPPOSE $\omega > T/2cm$ SO THAT DEFINING

$$\Delta \equiv \sqrt{\omega^2 - T^2/4c^2m^2} \rightarrow \hat{Y}(s) = \frac{1}{\Delta}\left(\frac{\Delta}{(s + T/2cm)^2 + \Delta^2}\right)$$

BY THE SHIFT THEOREM

$$Y(t) = \mathcal{L}^{-1}[\hat{Y}] = \frac{1}{\Delta}e^{-Tt/2cm}\sin(\Delta t)$$

FINALLY, INVERTING $\hat{U}(x,s)$ USING $\mathcal{L}[H_\lambda(t)f(t-c)] = e^{-cs}\hat{F}(s)$

WE OBTAIN

$$U(x,t) = Y(t - x/c)\,H_{x/c}(t) = \begin{cases} Y(t - x/c) & \text{IF } t > x/c \\ 0 & \text{IF } 0 < t < x/c. \end{cases}$$

<u>INTERPRETATION</u>

(i) IF $\omega > T/2cm$, I.E. IF SPRING CONSTANT IS LARGE ENOUGH $\left(\omega = \sqrt{\frac{k}{m}}\right)$ THEN THE SPRING LOSES ENERGY, AND ITS OSCILLATIONS ARE DAMPED, DUE TO A WAVE THAT PROPAGATES DOWN THE STRING. THIS IS RADIATIVE DAMPING.

(ii) OBSERVE THAT WITHOUT THE COUPLING THERE WOULD BE NO DAMPED OSCILLATIONS IN THE SPRING.

CULTURAL!

IMPEDANCE MATCHING: 3-LAYER MATERIAL

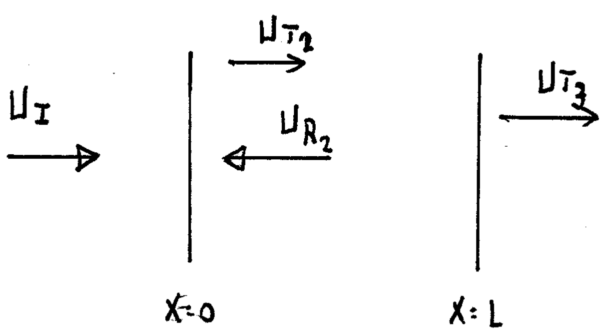
WE CONSIDER

$$u_{tt} = c^2 u_{xx} \quad \text{WITH} \quad c : \begin{cases} c_1, & x < 0 \\ c_2, & 0 < x < L \\ c_3, & x > L \end{cases}$$

WE ASSUME THAT THERE IS A PLANE-WAVE INCIDENT ON $x = 0$ FROM THE LEFT WITH FREQUENCY ω SO THAT

$$u_I(x) = \exp\left(i\omega\left(t - x/c_1\right)\right) \quad \text{FOR} \quad x \leq 0$$

OUR GOAL IS, FOR A SPECIFIED ω, TO CHOOSE L FOR THE SECOND LAYER AND FIND WHAT WAVESPEED c_2 TO TAKE SO THAT THERE IS NO REFLECTED WAVE IN $x \leq 0$. OUR DESIRED PICTURE IS:



THIS MEANS THAT $u(x,t)$ HAS FORM (WHERE T_2, R_2, T_3 TO BE FOUND):

$$u(x,t) = \begin{cases} u_I \equiv \exp\left(i\omega\left(t - x/c_1\right)\right) & \text{IN } x \leq 0 \\ T_2 \exp\left(i\omega\left(t - x/c_2\right)\right) + R_2 \exp\left(i\omega\left(t + x/c_1\right)\right) & \text{IN } 0 \leq x \leq L \\ T_3 \exp\left(i\omega\left(t - x/c_3\right)\right) & \text{IN } x \geq L. \end{cases}$$

OUR CONDITIONS ARE THAT u AND u_x ARE CONTINUOUS ACROSS $x = 0$ AND $x = L$. THIS GIVES 4 EQUATIONS IN 3 UNKNOWNS T_2, R_2, T_3, AND SO FOR THIS PROBLEM TO HAVE A SOLUTION THERE MUST BE A CONSTRAINT ON c_2 AND L. OUR GOAL IS TO FIND THIS CONSTRAINT.

SOLUTION: IF THERE IS NO REFLECTION IN $X<0$ WE GET THAT U HAS THE FORM:

$$U = \begin{cases} \exp\left(i\omega\left(t - x/c_1\right)\right) & X<0 \\ T_2 \exp\left(i\omega\left(t - x/c_2\right)\right) + R_2 \exp\left(i\omega\left(t + x/c_2\right)\right) & 0<X<L \\ T_3 \exp\left(i\omega\left(t - x/c_3\right)\right) & X>L. \end{cases}$$

NOW WE MUST IMPOSE CONTINUITY OF U AND U_X AT $X=0, L$. THIS GIVES,

$$(1) \begin{cases} 1 = T_2 + R_2 \longrightarrow U \text{ CONTINUOUS ACROSS } X=0 \\ -\dfrac{c_2}{c_1} = R_2 - T_2 \longrightarrow U_X \text{ CONTINUOUS ACROSS } X=0 \end{cases}$$

$$(2) \begin{cases} T_2 \exp\left(\dfrac{-i\omega L}{c_2}\right) + R_2 \exp\left(\dfrac{i\omega L}{c_2}\right) = T_3 \exp\left(\dfrac{-i\omega L}{c_3}\right) \longrightarrow U \text{ CONTINUOUS ACROSS } X=0 \\ -T_2 \exp\left(\dfrac{-i\omega L}{c_2}\right) + R_2 \exp\left(\dfrac{i\omega L}{c_2}\right) = -\dfrac{c_2}{c_3} T_3 \exp\left(-i\omega L/c_3\right) \longrightarrow U \text{ CONTINUOUS ACROSS } X=L. \end{cases}$$

SOLVING FOR R_2 AND T_2 WE GET,

$$R_2 = \frac{c_1 - c_2}{2c_1}, \quad T_2 = \frac{c_1 + c_2}{2c_1}.$$

NOW SUBSTITUTE R_2 AND T_2 INTO (2) AND DIVIDE TO ELIMINATE T_3.

THIS GIVES,

$$\frac{T_2 e_- + R_2 e_+}{-T_2 e_- + R_2 e_+} = -\frac{c_3}{c_2} \qquad \text{WHERE} \quad e_\pm = \exp\left(\pm i\omega L/c_2\right).$$

NOW THIS GIVES,

$$(c_1 + c_2) e_- + (c_1 - c_2) e_+ = \frac{c_3}{c_2}\left[(c_1 + c_2) e_- - (c_1 - c_2) e_+\right].$$

NOW THIS YIELDS,

$$c_2 (c_1 + c_2) e_- + c_2 (c_1 - c_2) e_+ = c_3 (c_1 + c_2) e_- - c_3 (c_1 - c_2) e_+$$

THEREFORE,

$$(c_2 - c_3)(c_1 + c_2)\, e_- = -(c_2 + c_3)(c_1 - c_2)\, e_+.$$

SOLVING FOR THE RATIO e_+/e_+ GIVES

$$\frac{e_+}{e_-} = \frac{(c_3 - c_2)}{(c_3 + c_2)} \frac{(c_1 + c_2)}{(c_1 - c_2)}.$$

HOWEVER, WE HAVE $e_+/e_- = \exp\left(2 i w L/c_2\right)$. THIS GIVES

$$\exp\left(\frac{2 i w L}{c_2}\right) = \left(\frac{c_3 - c_2}{c_3 + c_2}\right)\left(\frac{c_1 + c_2}{c_1 - c_2}\right).$$

TAKING REAL AND IMAGINARY PARTS WE GET THAT WE MUST HAVE

$$\cos\left(\frac{2 w L}{c_2}\right) = \left(\frac{c_3 - c_2}{c_3 + c_2}\right)\left(\frac{c_1 + c_2}{c_1 - c_2}\right),$$

AND $$\sin\left(\frac{2 w L}{c_2}\right) = 0.$$

ONE POSSIBLE SOLUTION IS $2 w L/c_2 = \pi$ SO THAT $L = \pi c_2/2w$.

THEN $\cos\left(2 w L/c_2\right) = -1$ AND WE MUST HAVE

$$\left(\frac{c_3 - c_2}{c_3 + c_2}\right)\left(\frac{c_1 + c_2}{c_1 - c_2}\right) = -1.$$

EXPANDING OUT GIVES $(c_3 - c_2)(c_1 + c_2) = (c_2 - c_1)(c_3 + c_2)$ SO THAT $c_3 c_1 - c_2 c_1 - c_2^2 + c_3 c_2 = c_2 c_3 - c_1 c_3 - c_1 c_2 + c_2^2$. THIS YIELDS THAT $2 c_2^2 = 2 c_1 c_3$ OR $c_2 = \sqrt{c_1 c_3}$.

IN CONCLUSION, GIVEN A FREQUENCY w AND WAVE SPEEDS c_1 AND c_3 IF WE CONSTRUCT THE MIDDLE LAYER SO THAT IT HAS A WAVE-SPEED $c_2 = \sqrt{c_1 c_3}$ AND HAS WIDTH $L = \pi c_2/2w$ THEN THERE WILL BE <u>NO</u> <u>REFLECTED</u> WAVE IN $x < 0$.

CULTURAL!

MOVING SOURCE: DOPPLER SHIFT (ALTERNATIVE BUT HARDER DERIVATION)

WE NOW CONSIDER A MOVING SOURCE PROBLEM WHERE $u(x,t)$ SATISFIES

$$u_{tt} = c^2 u_{xx} + \delta(x - c_0 t)\, F(t), \qquad -\infty < x < \infty,\ t > 0$$

$$u(x,0) = u_t(x,0) = 0$$

WHERE $0 < c_0 < c$. WE WILL NOW SOLVE THIS PROBLEM USING LAPLACE TRANSFORMS BUT ON SYSTEM OBTAINED AFTER A GALILEAN CHANGE OF VARIABLES THAT FOLLOWS THE SOURCE.

LET $\begin{matrix} x' = x - c_0 t \\ t' = t \end{matrix}$ AND $V(x', t') = u[c_0 t + x', t']$

NOW
$$u_t = V_{t'} + V_{x'}\, dx'/dt = V_{t'} - c_0 V_{x'}$$

$$u_{tt} = V_{t't'} + V_{t'x'}(-c_0) - c_0 V_{t'x'} + c_0^2 V_{x'x'}$$

$$u_{xx} = V_{x'x'}$$

SUBSTITUTING WE OBTAIN

$$V_{t't'} - 2c_0 V_{t'x'} + c_0^2 V_{x'x'} = c^2 V_{x'x'} + \delta(x')\, F(t')$$

WHICH CAN BE WRITTEN AS

$$V_{x'x'}(c^2 - c_0^2) = V_{t't'} - 2c_0 V_{t'x'} - \delta(x')\, F(t')$$

$$V = 0 \quad V_{t'} = 0 \quad \text{ON} \quad t' = 0.$$

NOW TAKE LAPLACE TRANSFORM: $\bar{V}(x', s) = \mathcal{L}[V(x', t')]$.

THIS YIELDS THAT

$$\bar{V}_{x'x'}(c^2 - c_0^2) = s^2 \bar{V} - 2c_0 s \bar{V}_{x'} - \delta(x')F(s)$$

OR EQUIVALENTLY,

$$\bar{V}_{x'x'} + \frac{2c_0 s}{c^2 - c_0^2}\bar{V}_{x'} - \frac{s^2}{c^2 - c_0^2}\bar{V} = -\frac{\delta(x')}{c^2 - c_0^2}F(s)$$

WHERE WE WANT $\bar{V} \to 0$ AS $|x| \to \infty$. THE CONTINUITY CONDITION AND JUMP CONDITION IS

$$\bar{V} \text{ CONTINUOUS ACROSS } x' = 0$$

$$\bar{V}_{x'}\Big|_+ - \bar{V}_{x'}\Big|_- = -F(s)/(c^2 - c_0^2)$$

NOW FIND SOLUTION TO HOMOGENEOUS EQUATION. PUT $\bar{V} = e^{rx'}$ TO GET

$$r^2 + \frac{2c_0 s}{c^2 - c_0^2} r - \frac{s^2}{c^2 - c_0^2} = 0$$

COMPLETE SQUARE OR USE QUADRATIC FORMULA:

$$r_\pm = \frac{-c_0 s}{c^2 - c_0^2} \pm \frac{1}{2}\sqrt{\frac{4c_0^2 s^2}{(c^2 - c_0^2)^2} + \frac{4s^2}{(c^2 - c_0^2)}} = \frac{-c_0 s}{c^2 - c_0^2} \pm \sqrt{\frac{s^2}{(c^2 - c_0^2)^2}\left[c_0^2 + (c^2 - c_0^2)\right]}$$

$$r_\pm = -\frac{c_0 s}{c^2 - c_0^2} \pm \frac{sc}{(c^2 - c_0^2)} \rightarrow r_+ = \frac{s(c - c_0)}{c^2 - c_0^2} = \frac{s}{c + c_0} > 0$$

$$r_- = \frac{-s(c_0 + c)}{c^2 - c_0^2} = -\frac{s}{c - c_0} < 0$$

THEREFORE,

$$\bar{V} = \begin{cases} A e^{s/(c + c_0)\, x'} & \text{IF } x' < 0 \\ A e^{-s/(c - c_0)\, x'} & \text{IF } x' > 0. \end{cases}$$

BY THE JUMP CONDITION WE FIND A AS:

$$-\frac{As}{c-c_0} - \frac{As}{c+c_0} = -\frac{F(s)}{c^2-c_0^2} \rightarrow A = \frac{\bar{F}(s)}{2sc}$$

THEREFORE, WE HAVE:

$$\bar{V}(x',s) = \frac{F(s)}{2c}\left\{\begin{array}{ll} \frac{1}{s}\, e^{sx'/(c+c_0)} & \text{IF } x'<0 \\ \frac{1}{s}\, e^{-sx'/(c-c_0)} & \text{IF } x'>0 \end{array}\right\} (*)$$

WE NEED ONLY INVERT TO FIND $V(x',t')$ AND THEN PUT $x' = x - c_0 t$ AND $t'=t$ INTO THE EXPRESSION. (RECALL IF $\lambda \geq 0$: $\mathcal{L}^{-1}\left[\frac{e^{-\lambda s}}{s}\right] = H_\lambda(t')$)

WE GET

$$V(x',t') = \frac{1}{2c}\int_0^{t'} F(t'-\tau)\, H_{-x'/(c+c_0)}(\tau)\, d\tau \qquad \text{IF } x'<0$$

WHICH GIVES

$$V(x',t') = \left\{\begin{array}{ll} 0 & \text{IF } 0 \leq t' < -x'/(c+c_0) \\ \frac{1}{2c}\int_{-x'/(c+c_0)}^{t'} F(t'-\tau)\, d\tau & \text{IF } t' > -x'/(c+c_0) \end{array}\right.$$

FOR $x'>0$. NOW LET $\xi = (t'-\tau)$ IN INTEGRAL TO GET FOR $x'<0$ THAT

$$V(x',t') = \left\{\begin{array}{ll} 0 & \text{IF } 0 \leq t' < -x'/(c+c_0) \\ \frac{1}{2c}\int_0^{t'+x'/(c+c_0)} F(\xi)\, d\xi & \text{IF } t' > -x'/(c+c_0) \end{array}\right.$$

NOW REPLACING $x' = x - c_0 t$ AND $t'=t$ WE OBTAIN THAT IF $x - c_0 t < 0$

$$u(x,t) = \left\{\begin{array}{ll} 0 & \text{IF } 0 \leq t \leq -\frac{(x-c_0 t)}{c+c_0} \rightarrow x+ct<0 \\ \frac{1}{2c}\int_0^{t+\frac{(x-c_0t)}{c+c_0}} F(\xi)\, d\xi & \text{IF } t > -\frac{(x-c_0 t)}{c+c_0} \end{array}\right.$$

THEREFORE IF $X - C_0 t < 0$ WE HAVE

$$u(x,t) = \begin{cases} 0 & \text{if } \quad x + ct < 0 \quad (\mathrm{I}) \\ \dfrac{1}{2c}\displaystyle\int_0^{\frac{x+ct}{c+c_0}} f(s)\, ds & \text{if } \quad x + ct \geq 0 \quad (\mathrm{II}) \end{cases} \quad (1)$$

WE CAN DO THE SAME PROCEDURE WITH FORMULA IN (X) FOR $x' = x - c_0 t > 0$. WE HAVE FOR $x' > 0$

$$v(x', t') = \frac{1}{2c}\int_0^{t'} f(t'-\tau)\, H_{\frac{x'}{c-c_0}}(\tau)\, d\tau = \begin{cases} 0 & \text{if } \quad 0 \leq t' \leq \dfrac{x'}{c-c_0} \\ \dfrac{1}{2c}\displaystyle\int_{x'/c-c_0}^{t'} f(t'-\tau)\, d\tau & \text{if } \quad t' > x'/c-c_0 \end{cases}$$

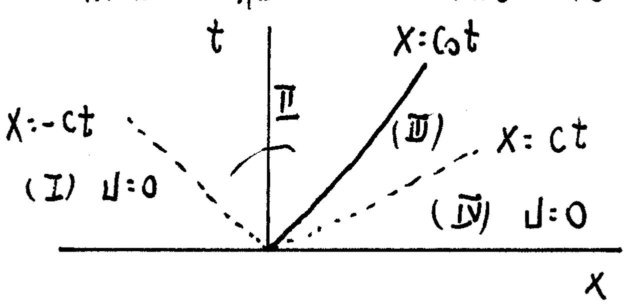
CHANGE VARIABLES $s = t' - \tau$ TO GET FOR $x' > 0$ THAT

$$v(x', t') = \begin{cases} 0 & \text{if } \quad 0 \leq t' \leq x'/(c-c_0) \\ \dfrac{1}{2c}\displaystyle\int_0^{t' - x'/c-c_0} f(s)\, ds & \text{if } \quad t' \geq \dfrac{x'}{c-c_0}. \end{cases}$$

NOW PUT $x' = x - c_0 t$ AND $t' = t$. WE OBTAIN FOR $x - c_0 t > 0$ THAT

$$u(x,t) = \begin{cases} 0 & \text{if } \quad x - ct > 0 \quad (\overline{\mathrm{IV}}) \\ \dfrac{1}{2c}\displaystyle\int_0^{(ct-x)/(c-c_0)} f(s)\, d\tau & \text{if } \quad x - ct < 0 \quad (\mathrm{III}) \end{cases}$$

WE HAVE THE FOLLOWING PICTURE IN (t, x) PLANE WHEN $0 < c_0 < c$.



$$u(x,t) = \begin{cases} \dfrac{1}{2c}\displaystyle\int_0^{\frac{x+ct}{c+c_0}} f(s)\, ds, & \text{in } \quad -ct < x < c_0 t \\ \dfrac{1}{2c}\displaystyle\int_0^{(ct-x)/c-c_0} f(s)\, ds, & \text{in } \quad c_0 t < x < ct. \end{cases}$$