Math 400: Midterm 1: 2025 (40 points): Michael J. Ward

Instructions: One page (two-sided) of handwritten or *LaTexed* notes is allowed. No other aids are permitted.

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1. (18 points) Consider forced vibrations of a string of length π that is pinned at $x = 0$ and $x = \pi$. The small amplitude vertical deflection $u(x,t)$ of the string is assumed to satisfy the forced wave equation

$$
\begin{aligned}
&u_{tt} = c^2 u_{xx} + \cos(\omega t)\,, \quad 0 \leq x \leq \pi\,, \quad t \geq 0\,,\\
&u(0,t) = 0\,, \quad u(\pi,t) = 0\,, \quad \text{for } t \geq 0\,,\\
&u(x,0) = 0\,, \quad u_t(x,0) = 1\,, \quad \text{for } 0 \leq x \leq \pi\,.
\end{aligned}
$$

Here $c > 0$ and $\omega > 0$ are constants. We aim to obtain an infinite series representation for the solution $u(x,t)$ in the form $u(x,t) = \sum_{n=1}^{\infty} B_n(t)\Phi_n(x)$ where $\Phi_n(x)$ are the appropriate eigenfunctions.

   (a) (12 points) Determine $\Phi_n(x)$ and derive a differential equation for $B_n(t)$ where you are to specify the initial values $B_n(0)$ and $B_n'(0)$. Give your result in an explicit a form as you can.

   (b) (2 points) For what values of ω will resonance occur?

   (c) (4 points) For **non-resonant** values of ω calculate the solution $B_n(t)$ explicitly for each $n = 1, 2, \ldots$.

2. (18 points) We will determine the axially symmetric steady-state temperature distribution $u(r,z)$ in a finite cylinder where both the bottom of the cylinder and its lateral side are insulating while the top of the cylinder has a non-uniform temperature. The PDE to be solved for $u(r,z)$ is:

$$
\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + u_{zz} = 0\,, \quad 0 \leq r \leq a\,, \quad 0 \leq z \leq H\,,\\
&u_z = 0 \quad \text{on } z = 0\,,\ 0 \leq r \leq a\,; \quad u = f(r) \quad \text{on } z = H\,,\ 0 \leq r \leq a\,,\\
&u_r = 0 \quad \text{on } r = a\,,\ 0 \leq z \leq H\,; \quad u\,,\ u_r \text{ bounded as } r \to 0\,.
\end{aligned}
$$

   (a) (12 points) Find an infinite series representation for the solution $u(r,z)$ for an arbitrary smooth function $f(r)$.

   (b) (3 points) From your soluton in part (a) calculate the total flux out of the top surface given by $\int_0^a r u_z(r,z)|_{z=H}\, dr$. Here the notation in the integrand means to evaluate $u_z(r,z)$ on $z = H$. Is there an alternative, and simpler way, not involving infinite series, to reach the same conclusion?

   (c) (3 points) Determine $u(r,z)$ explicitly from your result in part (a) when $f(r) = 1$ on $0 \leq r \leq a$.

3. (4 points) Two simple problems:

   (a) (2 points) (Outside a sphere) Let $a > 0$. Find the explicit solution $u(r)$ to

$$
u_{rr} + \frac{2}{r}u_r - 9u = 0\,, \quad r > a\,; \quad u(a) = 1\,, \quad u \text{ bounded as } r \to \infty\,.
$$

   (b) (2 points) (Outside a disk) Let $a > 0$. Find the explicit solution $u(r,\theta)$, in which u is 2π periodic in θ, to

$$
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} - 9u = 0\,, \quad r > a\,; \quad u(a,\theta) = \cos\theta\,, \quad u \text{ bounded as } r \to \infty\,.
$$

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PROBLEM 1

$$u_{tt} = c^2 u_{xx} + \cos(\omega t) \quad 0 \le x \le \pi,\ t > 0$$

$$u(0,t) = u(\pi,t) = 0\ ; \quad u(x,0) = 0, \quad u_t(x,0) = 1$$

(a) FOR HOMOG. problem let $u = T(t)\,\Phi(x)$

$$\rightarrow \quad \frac{T''}{c^2 T} = \frac{\Phi''}{\Phi} = -\lambda \quad \rightarrow \quad \Phi'' + \lambda \Phi = 0,\ 0 < x < \pi$$

$$\Phi(0) = \Phi(\pi) = 0.$$

SO $\Phi_n(x) = \sin(nx)$, $\lambda_n = n^2$, $n = 1, 2 \ldots$

FOR FORCED PROBLEM PUT $u(x,t) = \sum_{n=1}^{\infty} B_n(t)\,\Phi_n(x)$.

THEN
$$\sum_{n=1}^{\infty} B_n''\,\Phi_n = c^2 \sum_{n=1}^{\infty} B_n\,\Phi_n'' + \cos(\omega t)$$

$$= c^2 \sum_{n=1}^{\infty} B_n(-n^2)\,\Phi_n + \cos(\omega t)$$

SO
$$\sum_{n=1}^{\infty} \left[B_n'' + n^2 c^2 B_n \right] \Phi_n = \cos(\omega t). \qquad (*)$$

NOW EXPAND $\cos(\omega t) = \sum_{n=1}^{\infty} g_n(t)\,\Phi_n(x) \rightarrow g_n(t) = \dfrac{\int_0^{\pi} \cos(\omega t)\,\Phi_n(x)\,dx}{\underbrace{\int_0^{\pi} \Phi_n^2\,dx}_{(=\pi/2)}}$

SO
$$g_n(t) = \frac{2}{\pi}\cos(\omega t) \int_0^{\pi} \sin(nx)\,dx$$

$$g_n(t) = \frac{2}{\pi}\cos(\omega t)\left[-\frac{1}{n}\cos(nx)\Big|_0^{\pi} \right] = \frac{2}{\pi n}\cos(\omega t)\left[1 - (-1)^n\right]$$

THEREFORE $\quad B_n'' + n^2 c^2 B_n = g_n(t) \quad$ FOR $n = 1, 2 \ldots$

WE WRITE THIS AS

$$(1) \qquad B_n'' + \omega_n^2 B_n = d_n \cos(\omega t) \qquad n = 1, 2 \ldots$$

WITH $\omega_n = nc$ AND
$$d_n = \frac{2}{\pi n}\left[1 - (-1)^n\right] = \begin{cases} 0 & \text{if } n = \text{even} \\ \dfrac{4}{\pi n} & \text{if } n = \text{ODD}. \end{cases}$$

TO FIND INITIAL CONDITIONS

$$u(x,0) = 0 = \sum_{n=1}^{\infty} B_n(0)\sin(nx) \longrightarrow B_n(0)=0$$

$$u_t(x,0) = 1 = \sum_{n=1}^{\infty} B_n'(0)\sin(nx)$$

By orthogonality $\quad B_n'(0) = \dfrac{\int_0^{\pi} \sin(nx)\,dx}{\int_0^{\pi} \sin^2(nx)\,dx} = \left(\dfrac{2}{\pi}\right)\dfrac{[1-(-1)^n]}{n}$

SO

$$(2)\begin{cases} B_n'' + w_n^2 B_n = d_n \cos(wt) \\ B_n(0)=0, \quad B_n'(0) = d_n \end{cases} \qquad \text{WITH } d_n = \begin{cases} 0, & n = \text{even} \\ \dfrac{4}{\pi n}, & n = \text{ODD.} \end{cases}$$

AND $w_n = cn$.

(b) RESONANCE WILL OCCUR IF

$w = w_n$ FOR SOME <u>ODD</u> INDEX $n = 1,3,5,\ldots$

THIS IS BECAUSE $d_n \neq 0$ IF n = ODD BUT $d_n = 0$ IF n = even.

(c) SUPPOSE $w \neq w_n$. WE NOW SOLVE (2).

THE HOMOGENEOUS SOLUTION IS $B_{nH} = A_n\cos(w_n t) + D_n \sin(w_n t)$.

THE PARTICULAR SOLUTION HAS $B_{np} = K\cos(wt)$. TO FIND K WE SUBSTITUTE $\rightarrow -Kw^2 + Kw_n^2 = d_n \rightarrow K = \dfrac{d_n}{w_n^2 - w^2}$.

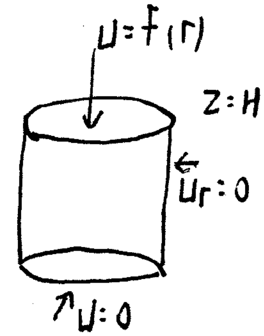
SO $$B_n = A_n\cos(w_n t) + D_n\sin(w_n t) + \frac{d_n}{w_n^2 - w^2}\cos(wt)$$

$$B_n(0) = 0 \rightarrow A_n = -d_n/(w_n^2 - w^2)$$

$$B_n'(0) = d_n \rightarrow w_n D_n = d_n \rightarrow D_n = d_n/w_n$$

SO $$B_n(t) = \frac{d_n}{w_n^2 - w^2}\left[\cos(wt) - \cos(w_n t)\right] + \frac{d_n}{w_n}\sin(w_n t).$$

PROBLEM 2



(3)

$$u_{rr} + \frac{1}{r} u_r + u_{zz} = 0 \quad \text{IN} \quad 0 \le r \le a, \quad 0 \le z \le H$$

(a) WE WRITE $u(r,z) = \Phi(r) Z(z)$. WE SEEK SL IN RADIAL DIRECTION.

$$\rightarrow \quad \frac{\Phi'' + 1/r\, \Phi'}{\Phi} = -\frac{Z''}{Z} = -\lambda.$$

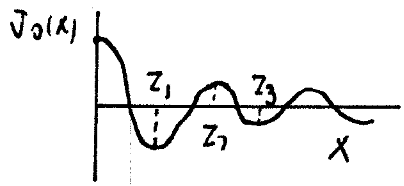
SO
$$\Phi'' + \frac{1}{r}\Phi' + \lambda \Phi = 0$$
$$\Phi'(a) = 0, \quad \Phi \text{ BOUNDED AS } r \to 0.$$

$\Rightarrow$ IF $\lambda > 0$, $\Phi = J_0(\sqrt{\lambda}\, r)$

$(r\Phi')' + \lambda r \Phi = 0$ SL FORM

$\rightarrow$ weight $w = r$.

- $\lambda_0 = 0$, $\Phi_0 = 1$ IS EIGENPAIR.
- IF $\lambda > 0$ $\quad J_0'(\sqrt{\lambda}\, a) = 0 \rightarrow \sqrt{\lambda}\, a = z_n$ WITH $n = 1, 2, \ldots$



$J_0'(z_n) = 0$ gives horizontal tangent points.

$$\rightarrow \quad \lambda_n = z_n^2 / a^2, \quad n = 1, 2, \ldots$$

NOW
$$\begin{cases} Z'' - \lambda_n Z = 0 \\ Z'(0) = 0 \quad \text{BOTTOM WALL.} \end{cases}$$

IF $n = 0 \rightarrow Z_0 = az + b$ BUT $Z_0'(0) = 0 \rightarrow a = 0$.

- SO IF $n = 0$ WHERE $\lambda_0 = 0 \rightarrow Z_0 = 1$
- IF $n = 1, 2 \ldots \quad Z_n(z) = \cosh(\sqrt{\lambda_n}\, z)$.

SO BY SUPERPOSITION

$$u(r,z) = c_0 \Phi_0 Z_0 + \sum_{n=1}^{\infty} c_n \Phi_n Z_n.$$

$$(1) \quad u(r,z) = c_0 + \sum_{n=1}^{\infty} c_n \cosh(\sqrt{\lambda_n}\, z) J_0(\sqrt{\lambda_n}\, r)$$

FINALLY USING $u(r,H) = F(r)$ GIVES

$$F(r) = C_0 + \sum_{n=1}^{\infty} C_n \cosh(\sqrt{\lambda_n}\, H)\, J_0(\sqrt{\lambda_n}\, r)$$

BY ORTHOGONALITY WE KNOW $\int_0^a r\, \Phi_n \Phi_m\, dr = 0$ IF $n \neq m$ AND $n, m \in \{0,1,2,\dots\}$

WHERE $\Phi_0 = 1$ AND $\Phi_n = J_0(\sqrt{\lambda_n}\, r)$.

- TO FIND C_0 MULTIPLY BY $r\,\Phi_0$ AND INTEGRATE WITH $\Phi_0 = 1$

  BY ORTHOGONALITY $\int_0^a r\, \Phi_0 J_0(\sqrt{\lambda_n}\, r)\, dr = 0$ FOR $n = 1, 2 \dots$ AND SO

$$\int_0^a r\, F(r)\, dr = C_0 \int_0^a r\, dr = C_0\, a^2/2$$

  SO $$C_0 = \frac{2}{a^2} \int_0^a r\, F(r)\, dr. \qquad (2)$$

- TO FIND C_n FOR $n \geq 1$ MULTIPLY BY $r\, J_0(\sqrt{\lambda_m}\, r)$ AND INTEGRATE

  THEN $$C_n = \frac{\int_0^a r\, F(r)\, J_0(\sqrt{\lambda_n}\, r)\, dr}{\int_0^a r \left(J_0(\sqrt{\lambda_n}\, r)\right)^2 dr} \; \frac{1}{\cosh(\sqrt{\lambda_n}\, H)} \qquad n = 1, 2 \dots \qquad (3)$$

(RECALL $\int_0^a r\, (J_0(\sqrt{\lambda_n}\, r))^2\, dr = \frac{a^2}{2} \left[J_0(\sqrt{\lambda_n}\, a)\right]^2$ WHEN $J_0'(\sqrt{\lambda_n}\, a) = 0$)

PUTTING (2) AND (3) INTO (1) GIVES $u(r,z) = C_0 + \sum_{n=1}^{\infty} C_n \cosh(\sqrt{\lambda_n}\, z)\, J_0(\sqrt{\lambda_n}\, r)$.

(b) TO FIND TOTAL FLUX CALCULATE $u_z(r,z)\big|_{z=H}$ A)

$$u_z(r,z)\Big|_{z=H} = \sum_{n=1}^{\infty} C_n \sqrt{\lambda_n} \sinh(\sqrt{\lambda_n}\, H)\, J_0(\sqrt{\lambda_n}\, r).$$

MULTIPLY BY r AND INTEGRATING WE USE $\int_0^a r\, J_0(\sqrt{\lambda_n}\, r)\, dr = 0$

TO GET $\int_0^a r\, u_z(r,z)\Big|_{z=H} dr = 0$. THIS IS NOT AT ALL SURPRISING

SINCE BY DIVERGENCE THEOREM $\int_\Omega \partial_n u\, ds = 0$ WHERE Ω IS CYLINDER.

(c) IF $F(r) = 1$ THEN $C_n = 0$ FOR $n = 1, 2 \dots$ AND $C_0 = 1$.

HENCE $u(r,z) = 1$ EVERYWHERE!

PROBLEM 3

(a) $$u_{rr} + \frac{2}{r} u_r - 9u = 0 \quad \text{IN} \quad r \geq a$$

$$u(a) = 1, \quad u \text{ BOUNDED AS } r \to \infty.$$

SOLUTION let $u(r) = \psi(r)/_r \rightarrow \psi'' - 9\psi = 0 \rightarrow \psi = \text{SPAN}\{e^{3r}, e^{-3r}\}$

FOR BOUNDED AS $r \to \infty$ PUT $u(r) = \frac{A}{r} e^{-3r}$.

TO FIND A: SET $u(a) = 1 \rightarrow 1 = \frac{A}{a} e^{-3a}$ SO $A = a e^{3a}$.

THIS GIVES $u(r) = \frac{a}{r} e^{-3(r-a)}$.

(b) $$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} - 9u = 0 \quad \text{IN} \quad r \geq a$$

$$u(a, \varphi) = \cos\varphi \ ; \quad u \text{ BOUNDED AS } r \to \infty$$

SOLUTION SET $u = F(r)\cos\varphi$

WE SUBSTITUTE AND GET

$$F'' + \frac{1}{r} F' - \frac{1}{r^2} F - 9F = 0$$

SO $$r^2 F'' + r F' - (9r^2 + 1) F = 0$$

WITH $F(a) = 1$.

$$F = \text{SPAN}\{K_1(3r), I_1(3r)\}$$

BOUNDED AS $r \to \infty$ GIVES $F = A K_1(3r)$

TO FIND A SET $u(a) = 1 = A K_1(3a)$ SO $A = 1/K_1(3a)$

$$\rightarrow F(r) = \frac{K_1(3r)}{K_1(3a)} \longrightarrow u = \frac{K_1(3r)}{K_1(3a)} \cos\varphi.$$