

Math 400: Midterm 1: 2026 (40 points): Michael J. Ward
Instructions: One page (one-side only) of handwritten or *LaTeXed* notes is allowed. No other aids are permitted.

1. (18 points) Consider radially symmetric diffusion inside a sphere of radius a with an inhomogeneous boundary concentration. The concentration $u(r, t)$ satisfies

$$\begin{cases} u_t = D \left(u_{rr} + \frac{2}{r} u_r \right), & 0 \leq r \leq a, \quad t > 0, \\ u(a, t) = 1 - e^{-kt}; \quad u, u_r \text{ bounded as } r \rightarrow 0, \\ u(r, 0) = 0 \text{ for } 0 \leq r \leq a, \end{cases}$$

where $D > 0$ and $k > 0$ are constants.

- (a) (5 points) Show that by substituting $u(r, t) = 1 - e^{-kt} + v(r, t)$, you obtain a PDE problem with homogeneous boundary conditions. Find the PDE for $v(r, t)$ as well as its boundary and initial conditions.
- (b) (10 points) Look for a solution in the form $v(r, t) = \sum_{n=1}^{\infty} b_n(t) \Phi_n(r)$, where $\Phi_n(r)$ are the eigenfunctions of the underlying Sturm-Liouville eigenvalue problem to be found. Derive and solve the ODE's for the coefficients $b_n(t)$ in an explicit form as you can (no integrals).
- (c) (3 points) Define the flux out of the sphere by $J(t) \equiv -4\pi a^2 D \frac{\partial u}{\partial r} \big|_{r=a}$. Derive an infinite series representation for $J(t)$. If the dimensions of u are moles/Length³ what are the dimensions of J ?
2. (17 points) Consider time-dependent diffusion inside half a disk of radius a , given in polar coordinates by $0 \leq r \leq a$ and $0 \leq \theta \leq \pi$, for which all three boundaries are insulating. In polar coordinates we have that $u(r, \theta, t)$ satisfies

$$\begin{cases} u_t = D \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right), & 0 \leq r \leq a, \quad 0 \leq \theta \leq \pi, \quad t > 0, \\ u_r(a, \theta, t) = 0, \quad 0 \leq \theta \leq \pi; \quad u_\theta(r, 0, t) = 0, \quad 0 \leq r \leq a; \quad u_\theta(r, \pi, t) = 0, \quad 0 \leq r \leq a, \\ u, u_r \text{ bounded as } r \rightarrow 0, \\ u(r, \theta, 0) = r + f(r) \cos(\theta), \quad 0 \leq r \leq a, \quad 0 \leq \theta \leq \pi, \end{cases}$$

where $D > 0$ is a constant.

- (a) (4 points) Ignoring the initial condition for the moment, by using separation of variables find the appropriate eigenfunctions in both the angular and radial directions.
- (b) (10 points) By exploiting the form of the initial condition, determine an infinite series representation for the solution for an arbitrary $f(r)$. You do not need to calculate integrals involving any Bessel functions.
- (c) (3 points) From your separation of variables solution, calculate the steady-state value $\lim_{t \rightarrow \infty} u(r, \theta, t)$. Does this value confirm what you expected intuitively?
3. (5 points) A few simple problems:

- (a) (2 points) (Inside a disk) Let $a > 0$. Find the solution $u(r, \theta)$, in which u is 2π periodic in θ , to

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} - 9u = 0, \quad 0 \leq r \leq a; \quad u(a, \theta) = 1 + \cos \theta, \quad u \text{ bounded as } r \rightarrow 0.$$

- (b) (3 points) (Sturm-Liouville). Let $a(x)$ and $b(x)$ be smooth functions on $0 \leq x \leq 1$ and assume that $b(x) > 0$ on $0 \leq x \leq 1$. Consider the eigenvalue problem for eigenpairs λ and $\Phi(x)$ of

$$\Phi'' + a(x)\Phi' + \lambda b(x)\Phi = 0, \quad 0 \leq x \leq 1, \quad \text{with } \Phi(0) = 0, \quad \Phi'(1) = -\Phi(1).$$

Prove that λ must be real-valued and satisfies $\lambda \geq 0$.

PROBLEM 1

$$u_t = D \left(u_{rr} + \frac{2}{r} u_r \right) \quad 0 \leq r \leq a, \quad t > 0$$

$$u(a, t) = 1 - e^{-\kappa t} ; \quad u, u_r \text{ BOUNDED AS } r \rightarrow 0$$

$$u(r, 0) = 0 \quad 0 \leq r \leq a.$$

(a) PUT $u(r, t) = 1 - e^{-\kappa t} + v(r, t)$. THEN $u(r, 0) = v(r, 0) = 0$, $u(a, t) = 1 - e^{-\kappa t} + v(a, t)$,

SO THAT $v(a, t) = 0$. BUT $u_t = \kappa e^{-\kappa t} + v_t$ SO THAT

$$\kappa e^{-\kappa t} + v_t = D \left(v_{rr} + \frac{2}{r} v_r \right)$$

$$\rightarrow \begin{cases} v_t = D \left(v_{rr} + \frac{2}{r} v_r \right) - \kappa e^{-\kappa t} \\ v(a, t) = 0, \quad v(r, 0) = 0, \quad v, v_r \text{ BOUNDED AS } r \rightarrow 0. \end{cases}$$

(b) THE EIGENFUNCTIONS FOR UNDERLYING SL PROBLEM ARE

$$\Phi'' + \frac{2}{r} \Phi' + \lambda \Phi = 0$$

$$\rightarrow (r^2 \Phi')' + \lambda r^2 \Phi = 0$$

$$\Phi(a) = 0, \quad \Phi, \Phi' \text{ BOUNDED AS } r \rightarrow 0.$$

$$\text{SO } W = r^2.$$

$$\text{PUT } \Phi = f(r)/r \rightarrow f'' + \lambda f = 0 \rightarrow f(r) = \sin(n\pi r/a) \quad \lambda_n = n^2 \pi^2 / a^2.$$

$$f(a) = 0$$

$$\text{SO } \Phi_n = \frac{\sin(n\pi r/a)}{r}, \quad \lambda_n = n^2 \pi^2 / a^2.$$

$$\text{PUT } v(r, t) = \sum_{n=1}^{\infty} b_n(t) \Phi_n(r) \quad \text{EXPAND } 1 = \sum_{n=1}^{\infty} c_n \Phi_n(r)$$

$$\text{SO } \sum_{n=1}^{\infty} b_n' \Phi_n = D \sum_{n=1}^{\infty} b_n \left(\Phi_n'' + \frac{2}{r} \Phi_n' \right) - \kappa e^{-\kappa t} \sum_{n=1}^{\infty} c_n \Phi_n$$

$$= -D \sum_{n=1}^{\infty} b_n \lambda_n \Phi_n + \sum_{n=1}^{\infty} (-c_n e^{-\kappa t} \kappa) \Phi_n$$

$$\rightarrow \sum_{n=1}^{\infty} (b_n' + D b_n \lambda_n) \Phi_n = \sum_{n=1}^{\infty} (-c_n e^{-\kappa t} \kappa) \Phi_n$$

BY ORTHOGONALITY

$$\begin{cases} b_n' + D \lambda_n b_n = -c_n \kappa e^{-\kappa t} \text{ AND } 0 = v(r, 0) = \sum_{n=1}^{\infty} b_n(0) \Phi_n(r) \\ b_n(0) = 0 \end{cases}$$

NOW TO FIND C_n WE USE IBP:

$$C_n = \frac{\int_0^a \frac{r^2 \sin\left(\frac{n\pi r}{a}\right) dr}{\int_0^a \sin^2\left(\frac{n\pi r}{a}\right) dr} = \frac{2}{a} \int_0^a r \sin\left(\frac{n\pi r}{a}\right) dr = \frac{2}{a} \left[-\frac{r a}{n\pi} \cos\left(\frac{n\pi r}{a}\right) \Big|_0^a \right]$$

THIS GIVES $C_n = \frac{2a}{n\pi} (-1)^{n+1}$.

NOW SOLVE FOR b_n : HOMOGENEOUS SOLUTION $u_{nh} = e^{-D\lambda_n t}$.

FOR PARTICULAR PUT $b_{np} = A e^{-\mu t} \rightarrow -\mu A + D\lambda_n A = -C_n \mu \rightarrow A = \frac{C_n \mu}{\mu - D\lambda_n}$.

SO $b_{np} = \frac{C_n \mu}{\mu - D\lambda_n} e^{-\mu t}$. ASSUME $\mu \neq D\lambda_n$ FOR ANY n .

SO SATISFYING $b_n(0) = 0 \rightarrow u(r, t) = 1 - e^{-\mu t} + \sum_{n=1}^{\infty} b_n(t) \Phi_n(r)$

where $b_n(t) = \frac{C_n \mu}{\mu - D\lambda_n} (e^{-\mu t} - e^{-D\lambda_n t})$, $C_n = \frac{2a}{n\pi} (-1)^{n+1}$, $\Phi_n = \frac{\sin(n\pi r/a)}{r}$

AND $\lambda_n = n^2 \pi^2 / a^2$.

(C) DEFINE FLUX $J = -4\pi a^2 D \partial u / \partial r \Big|_{r=a}$. LET $[..]$ = DIMENSION OF.

$[a^2] = \text{LENGTH}^2$, $[D] = \text{LENGTH}^2 / \text{TIME}$, $[u] = \text{MOLE} / \text{LENGTH}^3$, $[u_r] = \frac{\text{MOLE}}{\text{LENGTH}^4}$.

SO $[J] = (\text{LENGTH}^2) (\text{LENGTH}^2) / \text{TIME} \cdot \text{MOLE} / \text{LENGTH}^4 \rightarrow [J] = \text{MOLE} / \text{TIME}$

J REPRESENTS HOW MUCH "MATERIAL" TRANSPORTED ACROSS SURFACE OF SPHERE.

NOW $J(t) = -4\pi a^2 D \sum_{n=1}^{\infty} \frac{C_n \mu}{\mu - D\lambda_n} (e^{-\mu t} - e^{-D\lambda_n t}) \Phi_n'(a)$. BUT $\Phi_n'(a) = \frac{n\pi}{a} \cos(n\pi r/a) \Big|_a$

SO $\Phi_n'(a) = n\pi (-1)^n / a^2$. NOW $C_n \Phi_n'(a) = \frac{n\pi (-1)^n}{a^2} \frac{2a}{n\pi} (-1)^{n+1} = -\frac{2}{a}$.

THIS GIVES $J(t) = 8\pi a D \sum_{n=1}^{\infty} \frac{1}{(1 - D\lambda_n/\mu)} (e^{-\mu t} - e^{-D\lambda_n t})$

PROBLEM 2

(a) SEPARATE VARIABLE. $u(r, \varphi, t) = T(t) \Phi(r) \Psi(\varphi)$.

$$\rightarrow \frac{T'}{T} = \frac{\Phi'' + 1/r \Phi'}{\Phi} + \frac{1}{r^2} \frac{\Psi''}{\Psi} = -\lambda. \rightarrow T' = -\lambda T \rightarrow T = e^{-\lambda t}$$

$$\text{NOW } \left(\frac{r^2 \Phi'' + r \Phi'}{\Phi} + \lambda r^2 \right) = -\frac{\Psi''}{\Psi} = \sigma. \rightarrow \begin{cases} \Psi'' + \sigma \Psi = 0 \\ \Psi'(0) = \Psi'(\pi) = 0 \end{cases}$$

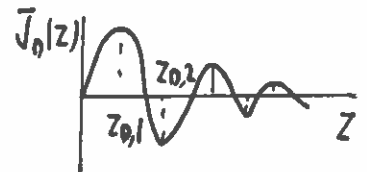
$$\text{WE HAVE } \begin{cases} \Psi_0 = 1, \sigma_0 = 0 \\ \Psi_n = \cos(n\varphi), \sigma_n = n^2, n=1, 2, \dots \end{cases}$$

$$\text{NOW } \begin{cases} r^2 \Phi'' + r \Phi' + (\lambda r^2 - n^2) \Phi = 0 \\ \Phi'(a) = 0, \Phi \text{ BOUNDED AS } r \rightarrow 0. \end{cases}$$

• NOW IF $n=1, 2, \dots$ THEN $\Phi_n(r) = J_n(\sqrt{\lambda} r)$.

$$\text{FIXED } \Phi_n'(a) = 0 \rightarrow \sqrt{\lambda} a = z_{n,m}, m=1, 2, \dots$$

$$\rightarrow \Phi_{n,m}(r) = J_n(\sqrt{\lambda_{n,m}} r) \text{ WITH } J_n'(z_{n,m}) = 0$$

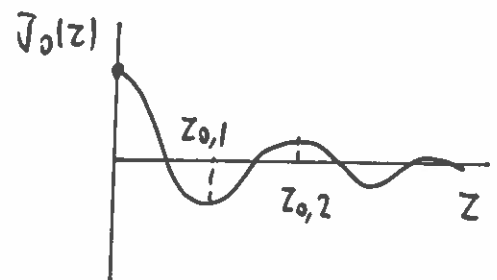


• IF $n=0$ THEN WE HAVE $\Phi_0 = J_0(\sqrt{\lambda} r)$.

• NOTICE $\exists$ ZERO EIGENVALUE $\lambda_{0,0} = 0, \Phi_{0,0} = 1$

$$\text{AND } \Phi_{0,m}(r) = J_0(\sqrt{\lambda_{0,m}} r) \quad m=1, 2, \dots$$

$$\lambda_{0,m} = \frac{z_{0,m}^2}{a^2}$$



(b) NOW EXPLOITING FORM OF INITIAL CONDITION WE LOOK FOR

$$U(r, \varphi, t) = V(r, t) + (\cos \varphi) W(r, t)$$

WE HAVE

$$V(r, t) = c_0 + \sum_{m=1}^{\infty} c_m e^{-D \lambda_{0,m}^2 t} J_0(\sqrt{\lambda_{0,m}} r)$$

$$W(r, t) = \sum_{m=1}^{\infty} d_m e^{-D \lambda_{1,m}^2 t} J_1(\sqrt{\lambda_{1,m}} r)$$

WITH $V(r, 0) = r$ WE HAVE AND WEIGHT = r THAT BY

ORTHOGONALITY: $c_0 = \int_0^a r^2 dr / \int_0^a r dr = 2a/3$

$$c_m = \int_0^a r^2 J_0(\sqrt{\lambda_{0,m}} r) dr / \int_0^a r (J_0(\sqrt{\lambda_{0,m}} r))^2 dr$$

SIMILARLY WITH $W(r, 0) = f(r)$

$$d_m = \int_0^a r f(r) J_1(\sqrt{\lambda_{1,m}} r) dr / \int_0^a r (J_1(\sqrt{\lambda_{1,m}} r))^2 dr.$$

(c) NOW AS $t \rightarrow \infty$ $U(r, \varphi, t) \rightarrow c_0 = 2a/3$.

check $U_{ave} = \frac{\int_0^\pi \int_0^a (r + f(r) \cos \varphi) r dr d\varphi}{(\pi a^2/2)} = \frac{\pi \int_0^a r^2 dr}{\pi a^2/2} = \frac{2a}{3} \checkmark$

PROBLEM 3

(a) $U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi} - 9U = 0, \quad 0 \leq r \leq a, \quad 0 \leq \varphi \leq 2\pi$

$U(a, \varphi) = 1 + \cos \varphi; \quad U \text{ BOUNDED AS } r \rightarrow 0.$

PUT $U = f_0(r) + f_1(r) \cos \varphi$ TO GET

$$\left. \begin{aligned} f_0'' + \frac{1}{r} f_0' - 9f_0 &= 0 \\ f_0(a) &= 1, \quad f_0 \text{ BOUNDED AS } r \rightarrow 0 \end{aligned} \right\} \rightarrow f_0(r) = A_0 I_0(3r) \quad \text{WITH } A_0 = \frac{1}{I_0(3a)}$$

$$\left. \begin{aligned} f_1'' + \frac{1}{r} f_1' - \left(9 + \frac{1}{r^2}\right) f_1 &= 0 \\ f_1(a) &= 1, \quad f_1 \text{ BOUNDED AS } r \rightarrow 0 \end{aligned} \right\} \rightarrow f_1(r) = A_1 I_1(3r) \quad \text{WITH } A_1 = 1/I_1(3a)$$

SO $U(r, \varphi) = \frac{I_0(3r)}{I_0(3a)} + \cos \varphi \frac{I_1(3r)}{I_1(3a)}$

(b) IF WE CAN PUT IN SL FORM $\rightarrow \Lambda$ REAL

MULTIPLY BY p : $p \Phi'' + ap \Phi' + \Lambda p b \Phi = 0.$

CHOOSE $p' = ap$ SO $p = \exp\left(\int_0^x a(t) dt\right) > 0$ ON $[0, 1]$

THEN $(x) \left\{ \begin{aligned} (p \Phi')' + \Lambda w(x) \Phi &= 0, \quad 0 < x < 1 \\ \Phi(0) &= 0, \quad \Phi'(1) = -\Phi(1). \end{aligned} \right.$

SINCE $w(x) = p(x) b(x) > 0$ ON $[0, 1] \rightarrow (x)$ HAS SL FORM $\rightarrow \Lambda$ REAL.

MOREOVER, SINCE $q(x) = 0$ AND WE HAVE ROBIN BC OF RIGHT SIGN

$\Lambda \geq 0. \quad \square$

EXTRA (NOT NEEDED) PROVE $\Lambda \geq 0$. MULTIPLY BY Φ AND INTEGRATE

$\int_0^1 \Phi (p \Phi')' dx + \Lambda \int_0^1 w \Phi^2 dx = 0 \rightarrow \Lambda \int_0^1 w \Phi^2 dx = - \int_0^1 (p \Phi')' \Phi dx.$

USE IBP: $\Lambda \int_0^1 w \Phi^2 dx = - \Phi \Phi' p \Big|_0^1 + \int_0^1 p (\Phi')^2 dx = - \Phi(1) \Phi'(1) p(1) + \int_0^1 p \Phi'^2 dx.$

BUT $\Phi'(1) = -\Phi(1) \rightarrow \Lambda \int_0^1 w \Phi^2 dx = p(1) (\Phi(1))^2 + \int_0^1 p \Phi'^2 dx \geq 0$

$\rightarrow \Lambda \geq 0.$