Math 400: Midterm 2 2025 (Sample Problems): Michael J. Ward

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1. (0 points) Consider Laplace's equation for $u = u(r,\theta)$ in the wedge $0 \leq r \leq R$, $0 \leq \theta \leq \alpha$, where (r,θ) are polar coordinates:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0\,, \quad 0 \leq r \leq R\,, \quad 0 \leq \theta \leq \alpha\,,$$
$$u(r,0) = 0\,, \quad u(r,\alpha) = 0\,, \quad u(R,\theta) = 1\,, \quad u \text{ bounded as } r \to 0\,.$$

Here $R > 0$ and $0 < \alpha < 2\pi$.

(a) (0 points) By summing an appropriate eigenfunction expansion, find a compact form for the solution to this PDE.

(b) (0 points) From the solution in part (a), show that the leading order behavior for u as $r \to 0$ is $u \approx Ar^\beta \sin(\beta\theta)$ for some constants A and β. Calculate the constants A and β explicitly. For what values of the wedge-angle α is the partial derivative u_r bounded as $r \to 0$? (Remark: you can still find β even if you are unable to solve part (a)).

2. (0 points) Consider a sphere of radius a with azimuthal symmetry and with surface potential $u(a,\theta) = f(\theta)$. Then, the solution $u(r,\theta)$ to the azimuthally symmetric Laplace's equation, defined both inside and outside the sphere, satisfies

$$\frac{1}{r^2}\left(r^2 u_r\right)_r + \frac{1}{r^2 \sin\theta}\left(\sin\theta\, u_\theta\right)_\theta = 0\,, \quad 0 < r < \infty\,, \quad 0 < \theta < \pi\,,$$
$$u(a,\theta) = f(\theta)\,, \quad 0 < \theta < \pi\,,$$
$$u \text{ bounded as } r \to 0 \text{ and } u \to 0 \text{ as } r \to \infty\,; \quad u \text{ bounded as } \theta \to 0\,,\pi\,.$$

(a) (0 points) Imposing that u is continuous across $r = a$ calculate $u(r,\theta)$ for $r > a$ and then for $0 < r < a$.

(b) (0 points) In terms of $f(\theta)$, give a formula for the surface charge density $\sigma(\theta)$ defined by

$$\sigma(\theta) = \left.\frac{\partial u}{\partial r}\right|_{r=a^+} - \left.\frac{\partial u}{\partial r}\right|_{r=a^-}\,.$$

(c) (0 points) Calculate $\sigma(\theta)$ explicitly when $f(\theta) = \cos(3\theta)$. (Hint: you need the identity $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta)$.) The first few Legendre polynomials $P_n(x)$ are

$$P_0(x) = 1\,, \quad P_1(x) = x\,, \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)\,, \quad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)\,.$$

3. (0 points) Consider the ringing of a hemispherical bell of radius $a > 0$ with azimuthal symmetry. Then, with θ being the polar angle, the deflection of the bell, $u(\theta,t)$, is assumed to satisfy for some constant $c > 0$, the wave equation

$$u_{tt} = \frac{c^2}{a^2 \sin\theta}\left(\sin\theta\, u_\theta\right)_\theta\,, \quad 0 \leq \theta \leq \pi/2\,, \quad t > 0\,,$$
$$u(\theta,0) = f(\theta)\,, \quad u_t(\theta,0) = 0\,, \quad 0 \leq \theta \leq \pi/2\,,$$
$$u(\pi/2,t) = 0\,, \quad u \text{ bounded as } \theta \to \pi\,.$$

(a) (0 points) Determine an infinite series representation for the solution.

(b) (0 points) What are the frequencies of the modes of vibration?

(c) (0 points) If $f(\theta) = \cos^3(\theta) - \cos(\theta)$ simplify your answer in part (a). The Legendre polynomials are given in a previous question.

4. (0 points) Consider the 2-D Laplace's equation for $u = u(r, \theta)$ in the wedge $0 \le r \le a$, $0 \le \theta \le \alpha$, where (r, θ) are polar coordinates:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0\,, \quad 0 \le r \le a\,, \quad 0 \le \theta \le \alpha\,,$$
$$u(r, 0) = 0\,, \quad u(r, \alpha) = 0\,, \quad u(a, \theta) = \theta/\alpha\,, \quad u \text{ bounded as } r \to 0\,.$$

Here $a > 0$ and $0 < \alpha < 2\pi$.

(a) (0 points) Derive an explicit infinite series representation for the solution.

(b) (0 points) Sum the infinite series in part (a).

(c) (0 points) Show that the leading order behavior for u as $r \to 0$ is $u \approx A r^{\beta} \sin(\beta\theta)$ for some constants A and β. Calculate the constants A and β explicitly. For what values of the wedge-angle α is the partial derivative u_r bounded as $r \to 0$? (Remark: you can still find β even if you are unable to solve part (b)).

5. (0 points) Consider the 2-D Laplace's equation for $u = u(r, \theta)$ in the semi-circle $0 \le r \le a$, $0 \le \theta \le \pi$, with $a > 0$ and where (r, θ) are polar coordinates:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0\,, \quad 0 \le r \le a\,, \quad 0 \le \theta \le \pi\,,$$
$$u(r, 0) = 0\,, \quad u(r, \pi) = 0\,, \quad u(a, \theta) = f(\theta)\,, \quad u \text{ bounded as } r \to 0\,.$$

(a) (0 points) Derive an explicit infinite series representation for the solution for an arbitrary $f(\theta)$.

(b) (0 points) By using the identity $2\sin(A)\sin B = \cos(A - B) - \cos(A + B)$ together with a formula for $\sum_{n=1}^{\infty} z^n$ when $|z| < 1$, sum the infinite series in part (a). (Hint: you will have two very similar infinite sums).

6. (0 points) Consider radially symmetric diffusion **exterior to a sphere** of radius $a > 0$:

$$u_t = D\left(u_{rr} + \frac{2}{r} u_r\right)\,, \quad r \ge a\,, \quad t \ge 0\,,$$
$$u(a, t) = t\,; \quad u\,, u_r \text{ bounded as } r \to \infty\,; \qquad u(r, 0) = 0\,.$$

Calculate the total flux $J(t)$ into the sphere defined by

$$J(t) \equiv \int_0^t -D \frac{\partial}{\partial r} u(r, \tau)\Big|_{r=a} d\tau\,.$$

(Recall: $\mathcal{L}(t^p) = \frac{\Gamma(p+1)}{s^{p+1}}$, $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$).

7. (0 points) Let $D > 0$ and $k > 0$ be constants and consider the diffusion problem for $u(x, t)$ given by:

$$u_t = D u_{xx} - k u\,, \quad x > 0\,, \quad t > 0\,; \qquad u(x, 0) = 0\,,$$
$$u(0, t) = t\,; \quad u\,, u_x \to 0 \text{ as } x \to \infty\,, \quad \text{for fixed } t > 0$$

(a) (0 points) Find the solution $u(x, t)$ in terms of a convolution integral.

(b) (0 points) Calculate the local flux $J(t)$ defined by $J(t) = -D u_x(0, t)$. (Recall: $\mathcal{L}(t^p) = \frac{\Gamma(p+1)}{s^{p+1}}$, $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(1) = 1$ and $\mathcal{L}^{-1}\left[e^{-\lambda\sqrt{s}}\right] = \left(\lambda/(2\sqrt{\pi} t^{3/2})\right) e^{-\lambda^2/(4t)}$ for $\lambda > 0$. You should know the convolution and shift theorems for Laplace transforms.)

8. (0 points) (Short Answer questions)

(a) (0 points) In the circular disk $0 < r < a$, $0 \le \theta \le 2\pi$ find the explicit solution to Laplace's equation for $u(r, \theta)$:

$$\Delta u \equiv u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \,, \quad 0 \le r \le a \,, \quad 0 \le \theta \le 2\pi \,,$$
$$u(a, \theta) = 2\cos^2(\theta) + 1 \,, \quad u \text{ bounded as } r \to 0 \,, \quad u \text{ and } u_\theta \;\; 2\pi \text{ periodic in } \theta \,.$$

(Hint: Recall that $\cos^2(\theta) = [1 + \cos(2\theta)]/2$.)

(b) (0 points) Outside the sphere $r > a$, where θ is the polar angle with $0 \le \theta \le \pi$, find the explicit solution to the 3-D Laplace's equation $\Delta u = 0$ with $u = 2\cos^2(\theta) + 1$ on $r = a$, $u \to 0$ as $r \to \infty$, and u is bounded at the poles $\theta = 0, \pi$. (Recall that the first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = (3x^2 - 1)/2$.)

(c) (0 points) Outside the sphere $r > a$, where θ is the polar angle with $0 \le \theta \le \pi$, find the explicit solution to the 3-D Laplace's equation $\Delta u = 0$ with $u = 4\sin^2(\theta) + 1$ on $r = a$, $u \to 0$ as $r \to \infty$, and u is bounded at the poles $\theta = 0, \pi$.

(d) (0 points) Consider the 1-D diffusion equation for $u(x, t)$ with insulated boundaries at $x = 0$ and $x = L > 0$, and with bulk decay, modeled by

$$u_t = u_{xx} - u \,, \qquad 0 \le x \le L \,, \quad t > 0$$
$$u_x(0, t) = u_x(L, t) = 0 \,, \qquad u(x, 0) = x^2 \,.$$

Calculate $\int_0^L u(x, t)\, dx$ explicitly. Also, show that $dM/dt \le 0$ where $M(t) \equiv \int_0^L [u(x, t)]^2 \, dx$.

(e) (0 points) Let Ω be a disk of radius $a > 0$ in 2-D centered at the origin. Let F be a constant and (r, θ) be polar coordinates centered at the origin. Find the value of F that is needed for the following problem to have a solution:

$$\Delta u = F r^2 \quad \text{in } \Omega \,; \qquad \frac{\partial u}{\partial r}(a, \theta) = \cos^2(\theta) \,.$$

(f) (0 points) Let $P_n(x)$ be a Legendre polynomial and let m be a positive integer. Explain why $\int_{-1}^1 x^m P_n(x)\, dx = 0$ whenever $m < n$. Why would it then follow that for any integer $n \ge 1$ we must have $\int_{-1}^1 P_n(x) P'_{n-1}(x)\, dx = 0$?

(g) (0 points) Let $n \ge 1$ be an integer. It is well-known that the Legendre polynomials satisfy the recurrence relation

$$P'_{n-1}(x) = -n P_n(x) + x P'_n(x) \,.$$

Upon multiplying this equation by $P_n(x)$ and integrating over the interval $[-1, 1]$ derive our previous result for $\int_{-1}^1 [P_n(x)]^2 \, dx$. You will need the result in part (f). (Recall also: $P_n(1) = 1$, so by even-odd symmetry $P_n(-1) = (-1)^n$).

(h) (0 points) Suppose that we have expanded $f(x)$ on $[-1, 1]$ in terms of Legendre polynomials as $f(x) = \sum_{n=0}^\infty a_n P_n(x)$ for some coefficients a_n. Determine a formula in terms of a_n for $\int_{-1}^1 [f(x)]^2 \, dx$.