Math 400: Midterm 2 2025 (40 points): Michael J. Ward

Instructions: One page (two-sided) of handwritten or *LaTexed* notes is allowed. No other aids are permitted.

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1. (15 points) Consider the steady-state temperature distribution $u(r,\theta)$ in a disk of radius $a>0$ satisfying

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0\,,\quad 0\le r\le a\,,\quad 0\le\theta\le 2\pi\,,$$
$$u,\, u_\theta \quad 2\pi \text{ periodic in } \theta\,;\qquad u\,,u_r \quad \text{bounded as } r\to 0\,.$$
$$u(a,\theta)=\begin{cases}1 & \text{if } 0\le\theta\le\pi\\ \\ -1 & \text{if } \pi<\theta<2\pi\end{cases}\,.$$

   (a) (6 points) By separating variables, show that for some coefficients b_n that you are to find, we have

$$u(r,\theta)=\sum_{n=1}^{\infty} b_n\left(\frac{r}{a}\right)^n \sin(n\theta)\,.$$

   (b) (9 points) Sum the eigenfunction expansion in part (a).

2. (10 points) Let $D>0$ be a constant, and suppose that $u(x,t)$ satisfies

$$u_t=Du_{xx}\,,\quad 0\le x<\infty\,,\quad t\ge 0\,;\qquad u(x,0)=0\,,$$
$$Du_x(0,t)=-t^{1/2}\,;\quad u,u_x\to 0 \quad \text{as} \quad x\to\infty \quad \text{for} \quad t \text{ fixed}\,.$$

   By solving the PDE using the Laplace transform show that $u(x,t)$ can be written as

$$u(x,t)=\int_0^t \mathcal{F}(x,\tau)\,d\tau\,,$$

   where you are to find the function $\mathcal{F}(x,\tau)$. (The only Laplace transform results you need are: $\mathcal{L}(t^p)=\frac{\Gamma(p+1)}{s^{p+1}}$, $\Gamma(z+1)=z\Gamma(z)$, $\Gamma(1)=1$, $\Gamma(1/2)=\sqrt{\pi}$, $\mathcal{L}\left[\text{erfc}\left(\lambda/(2\sqrt{t})\right)\right]=e^{-\lambda\sqrt{s}}/s$ for $\lambda\ge 0$, and the convolution formula $\mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)\,d\tau\right]=\hat{F}(s)\hat{G}(s)$.)

3. (15 points) (Short Answer questions: Legendre Polynomials)

   (a) (8 points) Find the explicit solution to the following 3-D Laplace's equation **outside a sphere of radius** a with azimuthal symmetry, and where θ is the polar angle:

$$\Delta u=0\,,\quad u(a,\theta)=\cos(2\theta)\,,\quad u\to 0\,,\quad \text{as } r\to\infty\,.$$

   (Recall: $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)$ and that the first few Legendre polynomials $P_n(x)$ are

$$P_0(x)=1\,,\quad P_1(x)=x\,,\quad P_2(x)=\frac{1}{2}\left(3x^2-1\right)\,,\quad P_3(x)=\frac{1}{2}\left(5x^3-3x\right)\,.$$

   (b) (3 points) Let n be a non-negative integer and let $P_n(x)$ be the Legendre polynomial of degree n. Calculate (with a clear reason provided) the integral I_n defined by

$$I_n\equiv\int_0^\pi P_n(\cos\theta)\,\sin\theta\,d\theta\,.$$

   (c) (4 points) Consider the 3-D Poisson equation **inside a sphere of radius** $a>0$ with a flux boundary condition on $r=a$ given in terms of the polar angle θ by

$$\Delta u=F\,,\quad 0\le r\le a\,;\qquad \frac{\partial u}{\partial r}=2\cos^2(\theta)-1\,,\quad \text{on}\quad r=a\,.$$

   Determine the value of the constant F that is required for this problem to have a solution. (Hint: parts (a) and (b) will help you make this a really short calculation).

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   Determine the value of the constant F that is required for this problem to have a solution. (Hint: parts (a) and (b) will help you make this a really short calculation).

MATH 400 MIDTERM 2

PROBLEM 1

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0, \quad 0 \le r \le a, \quad 0 \le \varphi \le 2\pi$$

u, u_φ 2π periodic in φ; u, u_r BOUNDED AS $r \to 0$

$$u(a,\varphi) = \begin{cases} 1 & \text{IN } 0 \le \varphi \le \pi \\ -1 & \text{IN } \pi < \varphi < 2\pi. \end{cases} \equiv f(\varphi)$$

(a) BY SEPARATION OF VARIABLES

$$u(r,\varphi) = A_0 + \sum_{n=1}^{\infty} (r/a)^n \left[A_n \cos n\varphi + B_n \sin n\varphi \right].$$

WE HAVE $\quad A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi)\, d\varphi = 0$

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \cos(n\varphi)\, d\varphi = \frac{1}{\pi} \left[\int_0^{\pi} \cos(n\varphi)\, d\varphi - \int_{\pi}^{2\pi} \cos(n\varphi)\, d\varphi \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{n} \sin(n\varphi) \Big|_0^{\pi} - \frac{1}{n} \sin(n\varphi) \Big|_{\pi}^{2\pi} \right] = 0.$$

NOW BY SYMMETRY

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \sin(n\varphi)\, d\varphi = \frac{2}{\pi} \int_0^{\pi} \sin(n\varphi)\, d\varphi = \frac{2}{\pi} \left(\frac{-1}{n} \cos(n\varphi) \Big|_0^{\pi} \right)$$

SO $$B_n = -\frac{2}{\pi n} \left[(-1)^n - 1 \right] = \begin{cases} \frac{4}{\pi n} & \text{IF } n \text{ IS ODD} \\ 0 & \text{IF } n \text{ IS EVEN}. \end{cases}$$

$$u(r,\varphi) = \sum_{\substack{n=1 \\ n \text{ ODD}}}^{\infty} \frac{4}{\pi n} (r/a)^n \sin(n\varphi).$$

(b) LET $z = \frac{r}{a} e^{i\varphi}$ SO THAT

$$u(r,\varphi) = \frac{4}{\pi} \operatorname{IM} \left(\sum_{m=1}^{\infty} \frac{z^{2m-1}}{2m-1} \right).$$

RECALL $\quad \frac{1}{1-z} = 1 + z + z^2 + \cdots \quad$ IF $|z| < 1$

$$\rightarrow \quad -\log(1-z) = z + z^2/2 + z^3/3 + \cdots \quad \text{IF } |z| < 1$$

$$z \mapsto -z \Rightarrow -\log(1+z) = -z + z^2/2 - z^3/3 + \cdots$$

SUBTRACT $\quad \log(1+z) - \log(1-z) = 2\left(z + z^3/3 + \cdots \right) \quad$ IF $|z| < 1$

SO $u(r,\varphi) = \frac{2}{\pi}\, \mathrm{IM}\left[\, \mathrm{LOG}(1+z) - \mathrm{LOG}(1-z)\right] = \frac{2}{\pi}\left[\mathrm{ARG}\left(1+z\right) - \mathrm{ARG}\left(1-z\right)\right]$

NOW SINCE $z = z_R + i z_I = \frac{r}{a}\cos\varphi + \frac{ir}{a}\sin\varphi$ SATISFIES $-1 < z_R < 1$

FOR $r < a$, WE HAVE $\mathrm{ARG}(1+z) \in (-\pi/2, \pi/2)$, $\mathrm{ARG}(1-z) \in (-\pi/2, \pi/2)$

SO THAT WE CAN COMBINE $\mathrm{ARG}(1+z)$, $\mathrm{ARG}(1-z)$ TO GET

$$u(r,\varphi) = \frac{2}{\pi}\, \mathrm{ARG}\left(\frac{1+z}{1-z}\right) = \frac{2}{\pi}\, \mathrm{TAN}^{-1}\left(\frac{\mathrm{IM}\left(\frac{1+z}{1-z}\right)}{\mathrm{RE}\left(\frac{1+z}{1-z}\right)}\right) \quad (*)$$

NOW $w = w_R + i w_I = \frac{1+z}{1-z} = \frac{(1+z)\overline{(1-z)}}{|1-z|^2} = \frac{1+z-\bar z - z\bar z}{|1-z|^2} = \frac{1-(r/a)^2 + 2i\,\mathrm{IM}\,z}{|1-z|^2}$

SO $\frac{w_I}{w_R} = \frac{2(r/a)\sin\varphi}{1-r^2/a^2} = \frac{2ar\sin\varphi}{a^2-r^2}$.

$$u(r,\varphi) = \frac{2}{\pi}\,\mathrm{TAN}^{-1}\left(\frac{w_I}{w_R}\right) = \frac{2}{\pi}\,\mathrm{TAN}^{-1}\left(\frac{2ar\sin\varphi}{a^2-r^2}\right) \quad \square$$

CHECK: IF $r \to a^-$ AND $0 < \varphi < \pi \to \sin\varphi > 0 \to u(r,\varphi) \to \frac{2}{\pi}\mathrm{TAN}^{-1}(\infty) = 1$

IF $r \to a^-$ AND $\pi < \varphi < 2\pi \to \sin\varphi < 0 \to u(r,\varphi) \to \frac{2}{\pi}\mathrm{TAN}^{-1}(-\infty) = -1$.

PROBLEM 2 $u_t = D u_{xx}$, $0 \le x < \infty$, $t \ge 0$

$u(x,0) = 0$; $D u_x(0,t) = -t^{1/2}$, $u, u_x \to 0$ AS $x \to \infty$, t FIXED.

SOLUTION LET $\hat{u}(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt$. BY TAKING LAPLACE

TRANSFORMS:
$$D\hat{u}_{xx} = s\hat{u} - u(x,0) = s\hat{u}.$$
$$D\hat{u}_x(0,s) = -\mathcal{L}(t^{1/2}) = -\frac{\Gamma(3/2)}{s^{3/2}}, \quad \hat{u} \to 0 \text{ AS } x \to \infty.$$

SO $\hat{u} = \mathrm{SPAN}\left\{ e^{-\sqrt{s/D}\,x}, e^{\sqrt{s/D}\,x}\right\}$. FOR BOUNDEDNESS $\hat{u} = A e^{-\sqrt{s/D}\,x}$

NOW $\hat{u}_x\big|_{x=0} = A\left(-\sqrt{s/D}\right) = -\frac{\Gamma(3/2)}{D s^{3/2}} \to A = \frac{\Gamma(3/2)}{\sqrt{D}\, s^2}$

SO $\hat{u}(x,s) = \frac{\Gamma(3/2)}{\sqrt{D}} \frac{1}{s^2} e^{-\sqrt{s/D}\,x}$.

TO INVERT, USE CONVOLUTION:

$$\hat{u}(x,s) = \frac{\Gamma(3/2)}{\sqrt{D}}\left(\frac{1}{s}\right)\frac{e^{-\sqrt{s/D}\,x}}{s}.$$

BUT $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$ AND $\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{s/D}\,x}}{s}\right] = \mathrm{ERFC}\left(x/2\sqrt{Dt}\right)$

HENCE BY CONVOLUTION PROPERTY.

$$u(x,t) = \frac{\Gamma(3/2)}{\sqrt{D}}\int_0^t \mathrm{ERFC}\left(x/\sqrt{2D\tau}\right)d\tau.$$

$$\rightarrow F(x,\tau) = \frac{\Gamma(3/2)}{\sqrt{D}}\,\mathrm{ERFC}\left(x/2\sqrt{D\tau}\right) \quad \text{WITH } \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}.$$

PROBLEM 3

(a)
$$u(r,\varphi) = \sum_{n=0}^{\infty} A_n\left(\frac{a}{r}\right)^{n+1}P_n(\cos\varphi)$$

$$u(a,\varphi) = \cos(2\varphi) = \cos^2\varphi - (1-\cos^2\varphi) = 2\cos^2\varphi - 1 = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi).$$

Let $x = \cos\varphi$: $\quad 2x^2 - 1 = \sum_{n=0}^{\infty} A_n P_n(x) \rightarrow A_n = 0$ IF $n \neq 0, 2$.

$$\rightarrow 2x^2 - 1 = A_0P_0(x) + A_2P_2(x) = A_0 + A_2\left[\frac{3x^2}{2} - \frac{1}{2}\right] = A_0 - \frac{A_2}{2} + \frac{3x^2}{2}A_2.$$

SO $2 = 3A_2/2$, $\quad -1 = A_0 - A_2/2$

$$\rightarrow A_2 = 4/3, \quad A_0 = \tfrac{1}{2}(4/3) - 1 = -1/3.$$

$$\rightarrow u(r,\varphi) = -\frac{1}{3}\left(\frac{a}{r}\right)P_0(\cos\varphi) + \frac{4}{3}\left(\frac{a}{r}\right)^3 P_2(\cos\varphi)$$

(b) LET $I_n = \int_0^{\pi} P_n(\cos\varphi)\sin\varphi\, d\varphi$ FOR $n = 0, 1, \ldots$

let $x = \cos\varphi$, $dx = -\sin\varphi\, d\varphi$ $\rightarrow I_n = \int_{+1}^{-1} P_n(x)(-dx) = \int_{-1}^{1} P_n(x)\,dx.$

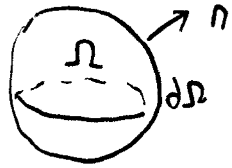
BY ORTHOGONALITY PROPERTY OF SL EIGENFUNCTIONS

$$\int_{-1}^{1} \underbrace{P_0(x)}_{=1} P_n(x)\,dx = 0 \text{ FOR } n = 1, 2\ldots \rightarrow I_n = 0 \text{ FOR } n = 1, 2\ldots$$

FOR $n = 0$ WE CALCULATE $I_0 = \int_{-1}^{1} P_0(x)\,dx = 2.$

$$I_n = \begin{cases} 0 & \text{IF } n = 1, 2\ldots \\ 2 & \text{IF } n = 0 \end{cases}$$

(c) WE LET Ω BE SPHERE OF RADIUS a WITH BOUNDARY $\partial\Omega$.



THEN $\partial_n u = \nabla u \cdot \hat{n} = \dfrac{\partial u}{\partial r}$ ON $r = a$.

BY DIV. THEOREM, WE HAVE "COMPATABILITY CONDITION"

$$\int_\Omega \Delta u \, d\underline{x} = \int_\Omega \nabla \cdot (\nabla u) \, d\underline{x} = \int_{\partial\Omega} \nabla u \cdot \hat{n} \, ds = \int_{\partial\Omega} \frac{\partial u}{\partial r}\bigg|_{r=a} ds \quad (*)$$

LET u SATISFY $\begin{cases} \Delta u = F & \text{IN } \Omega \\ u_r = g(\varphi) = 2\cos^2\varphi - 1 & \text{ON } r = a \end{cases}$

BY $(*)$ AND RECALLING $dS = a^2 \sin\varphi \, d\varphi \, d\phi$ WE MUST HAVE

$$\int_\Omega F \, d\underline{x} = \int_{\partial\Omega} g \, ds \rightarrow F \text{ VOLUME}(\Omega) = \int_0^{2\pi} \int_0^{\pi} a^2 g(\varphi) \sin\varphi \, d\varphi \, d\phi$$

SO $$F \frac{4\pi a^3}{3} = 2\pi a^2 \int_0^\pi g(\varphi) \sin\varphi \, d\varphi. \quad (+)$$

BUT FROM PART (a) $$g(\varphi) = 2\cos^2\varphi - 1 = A_0 P_0(\cos\varphi) + A_2 P_2(\cos\varphi)$$

WITH $A_0 = -1/3$, $A_2 = 4/3$.

SO $$F \frac{4\pi a^3}{3} = 2\pi a^2 \left[\int_0^\pi \left(\frac{-1}{3}\right) P_0(\cos\varphi) \sin\varphi \, d\varphi + \int_0^\pi \frac{4}{3} P_2(\cos\varphi) \sin\varphi \, d\varphi \right]$$

BY PART (b) ONLY FIRST TERM IS NON-ZERO AND SO

$$F \frac{4\pi a^3}{3} = 2\pi a^2 \left(\left(\frac{-1}{3}\right) \underbrace{\int_0^\pi P_0(\cos\varphi) \sin\varphi \, d\varphi}_{=2} \right)$$

$$F \frac{4\pi a^3}{3} = -\frac{4\pi a^2}{3}$$

$$\rightarrow F = -1/a.$$