M400 SAMPLE MIDTERM PROBLEMS

PROBLEM 1 (a) BY SEPARATION OF VARIABLES

$$u(r,\varphi) = \sum_{n=1}^{\infty} A_n \left(\frac{r}{R}\right)^{n\pi/\alpha} \sin\left(n\pi\varphi/\alpha\right)$$

NOW $u(R,\varphi) = 1 = \sum_{n=1}^{\infty} A_n \sin\left(n\pi\varphi/\alpha\right) \rightarrow A_n = \frac{2}{\alpha}\int_0^\alpha \sin\left(\frac{n\pi\varphi}{\alpha}\right) d\varphi$

SO $A_n = -\frac{2}{\alpha}\left(\frac{\alpha}{n\pi}\right)\cos\left(\frac{n\pi\varphi}{\alpha}\right)\Big|_0^\alpha = -\frac{2}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 4/n\pi & \text{IF } n = \text{ODD} \\ 0 & \text{IF } n = \text{even} \end{cases}$

SO $$u(r,\varphi) = \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{1}{2m-1}\left(\frac{r}{R}\right)^{(2m-1)\pi/\alpha} \sin\left(\pi(2m-1)\varphi/\alpha\right) \qquad (*)$$

LET $z = (r/R)^{\pi/\alpha} e^{i\pi\varphi/\alpha}$. THEN $u(r,\varphi) = \frac{4}{\pi}\,\mathrm{IM}\left(\sum_{m=1}^{\infty} z^{2m-1}\right)$

BUT $-\log(1-z) = z + z^2/2 + z^3/3 + \cdots$

$-\log(1+z) = -z + z^2/2 - z^3/3 + \cdots$

THIS YIELDS, $\log(1+z) - \log(1-z) = 2\left(z + z^3/3 + \cdots\right)$

WE GET $u(r,\varphi) = \frac{2}{\pi}\,\mathrm{IM}\left[\log(1+z) - \log(1-z)\right]$

$u(r,\varphi) = \frac{2}{\pi}\left[\mathrm{ARG}(1+z) - \mathrm{ARG}(1-z)\right]$

BUT $w_1 = 1+z$ SATISFIES $\mathrm{RE}\, w_1 > 0$ AND IF $w_2 = 1-z$, $\mathrm{RE}\, w_2 > 0$.

SO $u(r,\varphi) = \frac{2}{\pi}\,\mathrm{ARG}\left(\frac{1+z}{1-z}\right) = \frac{2}{\pi}\,\mathrm{TAN}^{-1}\left(\frac{\mathrm{IM}\left(\frac{1+z}{1-z}\right)}{\mathrm{RE}\left(\frac{1+z}{1-z}\right)}\right)$

NOW let $w = w_R + i\,w_I = \frac{1+z}{1-z}$. WE HAVE $u(r,\varphi) = \frac{2}{\pi}\,\mathrm{TAN}^{-1}\left(\frac{w_I}{w_R}\right)$.

MULTIPLY BY CONJUGATE: $w = \frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1-|z|^2 + (z-\bar{z})}{|1-z|^2} = \frac{1-|z|^2 + 2i\,\mathrm{IM}z}{|1-z|^2}$

NOW $w_I/w_R = \frac{2\,\mathrm{IM}(z)}{1-|z|^2} = \frac{2(r/R)^{\pi/\alpha}\sin(\pi\varphi/\alpha)}{1-(r/R)^{2\pi/\alpha}}$.

THEREFORE, $$u(r,\varphi) = \frac{2}{\pi}\,\mathrm{TAN}^{-1}\left(\frac{2(r/R)^{\pi/\alpha}\sin(\pi\varphi/\alpha)}{1-(r/R)^{2\pi/\alpha}}\right).$$

(b) FROM INFINITE SERIES IN (*) THE $m=1$ TERM GIVES

$$u(r,\varphi) \simeq \frac{4}{\pi}\left(\frac{r}{R}\right)^{\pi/\alpha} \sin\left(\pi\varphi/\alpha\right) \quad \text{AS } r\to 0. \;\to A = \frac{4}{\pi} R^{-\pi/\alpha},\; B = \pi/\alpha.$$

DIFFERENTIATING $\quad u_r \simeq \frac{4}{\pi}\left(\frac{\pi}{\alpha R}\right)\left(\frac{r}{R}\right)^{\pi/\alpha - 1} \sin\left(\pi\varphi/\alpha\right)$ AS $r\to 0$

WHICH IS BOUNDED ONLY IF $\pi/\alpha - 1 \geq 0 \;\to\; \alpha \leq \pi.$

PROBLEM 2

(a) WE HAVE

$$u(r,\varphi) = \left\{ \begin{array}{ll} \sum_{n=0}^{\infty} A_n (a/r)^{n+1} P_n(\cos\varphi) & \text{IN } r \geq a \\ \sum_{n=0}^{\infty} A_n (r/a)^{n} P_n(\cos\varphi) & \text{IN } 0 \leq r \leq a \end{array} \right\} \quad (+)$$

WHERE WE IMPOSED CONTINUITY AT $r=a$. NOW SET $u(a,\varphi) = f(\varphi)$

SO THAT $\quad f(\varphi) = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi)$

SO THAT $$A_n = \frac{(2n+1)}{2}\int_0^{\pi} f(\varphi) P_n(\cos\varphi)\sin\varphi \, d\varphi. \quad (*)$$

(b) NOW $$u_r\big|_{r=a^+} = \sum_{n=0}^{\infty} -(n+1)\frac{A_n}{a} P_n(\cos\varphi)$$
$$u_r\big|_{r=a^-} = \sum_{n=0}^{\infty} \frac{n A_n}{a} P_n(\cos\varphi)$$

SO $$\sigma(\varphi) = u_r\big|_{r=a^+} - u_r\big|_{r=a^-} = \sum_{n=0}^{\infty} -\frac{A_n}{a}\left[2n+1\right] P_n(\cos\varphi)$$

THIS GIVES FROM (*) $$\sigma(\varphi) = -\frac{1}{2a}\sum_{n=0}^{\infty}(2n+1)^2 P_n(\cos\varphi)\int_0^{\pi} f(\varphi) P_n(\cos\varphi)(\sin\varphi)\,d\varphi$$

(c) NOW let $f(\varphi) = \cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi)$.

LET $x = \cos\varphi$ SO THAT $\quad 4x^3 - 3x = \sum_{n=0}^{\infty} A_n P_n(x) \to A_n = 0$ EXCEPT FOR $n = 1, 3$.

WE HAVE $\quad 4x^3 - 3x = A_1 P_1(x) + A_3 P_3(x)$

$$\to \quad 4x^3 - 3x = A_1 x + \frac{A_3}{2}(5x^3 - 3x) = \frac{5A_3}{2}x^3 + x\left(A_1 - \frac{3A_3}{2}\right)$$

SO $4 = 5A_3/2$ AND $-3 = A_1 - 3A_3/2 \to A_3 = 8/5,\; A_1 = -3 + \frac{3}{2}\left(\frac{8}{5}\right) = -3 + \frac{12}{5} = -\frac{3}{5}$

SO $A_1 = -3/5,\; A_3 = 8/5$ IN (+) WITH $A_n = 0$ IF $n \notin \{1,3\}$.

PROBLEM 3

(a) SEPARATE VARIABLES AS $u(\varphi,t) = T(t)\,\psi(\varphi)$ TO GET

$$\frac{a^2 T''}{c^2 T} = \frac{1}{\sin\varphi}\,\frac{[\sin\varphi\,\psi']'}{\psi} = -\lambda_n \quad\longrightarrow\quad T'' + \omega_n^2 T = 0$$

WITH $\omega_n = \sqrt{\lambda_n}\, c/a$

SO THE SL PROBLEM IS $\longrightarrow T_n = A_n\cos(\omega_n t) + B_n\sin(\omega_n t)$

$$\begin{cases} [\sin\varphi\,\psi']' + \lambda_n \sin\varphi\,\psi = 0 \quad \text{IN } 0 \le \varphi \le \pi/2 \\ \psi(\pi/2) = 0, \quad \psi \text{ BOUNDED AS } \varphi \to 0^+ \end{cases}$$

THIS GIVES $\psi_n(\varphi) = P_n(\cos\varphi)$ WITH n = ODD.

$\lambda_n = n(n+1)$

SO WE HAVE $$u(\varphi,t) = \sum_{\substack{n=1\\ n\text{ ODD}}}^{\infty} [A_n\cos(\omega_n t) + B_n\sin(\omega_n t)]\, P_n(\cos\varphi) \quad \text{WITH } \omega_n = \frac{c}{a}\sqrt{n(n+1)}$$

NOW $$u(\varphi,0) = f(\varphi) = \sum_{\substack{n=1\\ n\text{ ODD}}}^{\infty} A_n P_n(\cos\varphi)$$

$$\longrightarrow \quad A_n = (2n+1)\int_0^{\pi/2} f(\varphi)\, P_n(\cos\varphi)\sin\varphi\, d\varphi.$$

$$u_t(\varphi,0) = 0 \;\longrightarrow\; B_n = 0.$$

SO $$u(\varphi,t) = \sum_{\substack{n=1\\ n\text{ ODD}}}^{\infty} A_n\cos(\omega_n t)\, P_n(\cos\varphi). \quad \Big\}\ (*)$$

(b) THE FREQUENCIES OF VIBRATION ARE $\omega_n = \frac{c}{a}\sqrt{n(n+1)}$ (n ODD)

(c) IF $f(\varphi) = \cos^3\varphi - \cos\varphi = \sum_{\substack{n=1\\ n\text{ ODD}}}^{\infty} A_n P_n(\cos\varphi)$ WE HAVE $A_n = 0$ EXCEPT FOR $n = 1, 3$.

LET $x = \cos\varphi \longrightarrow x^3 - x = A_1 P_1(x) + A_3 P_3(x) = A_1 x + \frac{A_3}{2}(5x^3 - 3x) = x^3\left(\frac{5A_3}{2}\right) + x\left(A_1 - \frac{3A_3}{2}\right)$

SO $1 = 5A_3/2$, $\;-1 = A_1 - 3A_3/2 \;\longrightarrow\; A_1 = \frac{3}{2}A_3 - 1.$

$\longrightarrow A_3 = 2/5,\quad A_1 = -2/5 \;\longrightarrow\; u(\varphi,t) = -\frac{2}{5}\cos(\omega_1 t)\,P_1(\cos\varphi) + \frac{2}{5}\cos(\omega_3 t)\,P_3(\cos\varphi)$

PROBLEM 4

(a) BY SEPARATING VARIABLES

$$u(r,\varphi) = \sum_{n=1}^{\infty} A_n \left(\frac{r}{a}\right)^{n\pi/\alpha} \sin\left(\frac{n\pi\varphi}{\alpha}\right)$$

WITH $u(a,\varphi) = \frac{\varphi}{\alpha} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi\varphi}{\alpha}\right) \rightarrow A_n = \frac{2}{\alpha}\int_0^{\alpha} \frac{\varphi}{\alpha} \sin\left(\frac{n\pi\varphi}{\alpha}\right) d\varphi$

INTEGRATING BY PARTS

$u = \varphi, \quad du = d\varphi$

$dv = \sin\left(\frac{n\pi\varphi}{\alpha}\right) d\varphi \quad v = -\frac{\alpha}{n\pi}\cos\left(\frac{n\pi\varphi}{\alpha}\right)$

$$\rightarrow A_n = \frac{2}{\alpha^2}\left[-\frac{\varphi\alpha}{n\pi}\cos\left(\frac{n\pi\varphi}{\alpha}\right)\Big|_0^{\alpha} + \frac{\alpha}{n\pi}\int_0^{\alpha}\cos\left(\frac{n\pi\varphi}{\alpha}\right) d\varphi\right]$$

$$= \frac{2}{\alpha^2}\left[-\frac{\alpha^2}{n\pi}(-1)^n + \left(\frac{\alpha}{n\pi}\right)\left(\frac{\alpha}{n\pi}\right)\underbrace{\sin\left(\frac{n\pi\varphi}{\alpha}\right)\Big|_0^{\alpha}}_{=0}\right]$$

$$\rightarrow A_n = \frac{2}{n\pi}(-1)^{n-1}$$

SO $u(r,\varphi) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{r}{a}\right)^{n\pi/\alpha} \sin\left(\frac{n\pi\varphi}{\alpha}\right) \qquad (*)$

(b) LET $z = \left(\frac{r}{a}\right)^{\pi/\alpha} e^{i\pi\varphi/\alpha}$ SO THAT $|z| = (r/a)^{\pi/\alpha} < 1$ IF $0 \le r < a$.

WE GET $u(r,\varphi) = \frac{2}{\pi}\, \mathrm{IM}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} z^n\right)$.

RECALL $\frac{1}{1-z} = 1 + z + z^2 + \ldots \longrightarrow \frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 \ldots$ FOR $|z|<1$

THUS $\mathrm{LOG}(1+z) = z - z^2/2 + z^3/3 + \ldots = \sum_{n=1}^{\infty}(-1)^{n-1} z^n/n$ FOR $|z|<1$

$\rightarrow u(r,\varphi) = \frac{2}{\pi}\,\mathrm{IM}\left[\mathrm{LOG}(1+z)\right]$. BUT $\mathrm{RE}(1+z) > 0$ ALWAYS

AND SO $u(r,\varphi) = \frac{2}{\pi}\tan^{-1}\left(\frac{\mathrm{IM}(1+z)}{\mathrm{RE}(1+z)}\right) = \frac{2}{\pi}\tan^{-1}\left(\frac{\mathrm{IM}(z)}{1+\mathrm{RE}(z)}\right)$

THIS GIVES $u(r,\varphi) = \frac{2}{\pi}\tan^{-1}\left(\frac{(r/a)^{\pi/\alpha}\sin(\pi\varphi/\alpha)}{1+(r/a)^{\pi/\alpha}\cos(\pi\varphi/\alpha)}\right)$.

(c) FROM (*) WE KEEP THE $n=1$ TERM ONLY TO GET

$$u(r,\varphi) \simeq \frac{2}{\pi}\left(\frac{r}{a}\right)^{\pi/\alpha} \sin\left(\pi\varphi/\alpha\right) \quad \text{AS} \quad r\to 0 \quad \to A = \frac{2}{\pi} a^{-\pi/\alpha}, \; B = \pi/\alpha.$$

$$\to \quad u_r \simeq \left(\frac{2}{\pi}\right)\left(\frac{\pi}{a\alpha}\right)\left(\frac{r}{a}\right)^{\pi/\alpha - 1} \sin\left(\pi\varphi/\alpha\right) \quad \text{AS} \quad r\to 0$$

WE CONCLUDE THAT u_r IS BOUNDED AS $r\to 0$ ONLY WHEN $\frac{\pi}{\alpha}-1 \geq 0$

THIS IMPLIES $0 < \alpha \leq \pi$.

PROBLEM 5 BY SEPARATION OF VARIABLES

(a)
$$u(r,\varphi) = \sum_{n=1}^{\infty} A_n \left(\frac{r}{a}\right)^n \sin(n\varphi)$$

SETTING $u(a,\varphi) = f(\varphi) = \sum_{n=1}^{\infty} A_n \sin(n\varphi) \to A_n = \frac{2}{\pi}\int_0^{\pi} f(w)\sin(nw)\,dw$

THEN $$u(r,\varphi) = \frac{2}{\pi}\int_0^{\pi} f(w)\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \sin(n\varphi)\sin(nw)\right)dw.$$

(b) USING $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$ WE GET

$$u(r,\varphi) = \frac{1}{\pi}\int_0^{\pi} f(w)\left[-S_+ + S_-\right] dw$$

WITH
$$S_+ = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\varphi+w)) = \text{RE}\left[\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\varphi+w)}\right]$$
$$S_- = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\varphi-w)) = \text{RE}\left[\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\varphi-w)}\right]$$

DEFINE $Z_+ = (r/a)e^{i(\varphi+w)}$, $Z_- = (r/a)e^{i(\varphi-w)}$ THEN

$$S_+ = \text{RE}\left(\sum_{n=1}^{\infty} Z_+^n\right) = \text{RE}\left(\frac{Z_+}{1-Z_+}\right) \qquad S_- = \text{RE}\left(\sum_{n=1}^{\infty} Z_-^n\right) = \text{RE}\left(\frac{Z_-}{1-Z_-}\right)$$

SO $$u(r,\varphi) = \frac{1}{\pi}\int_0^{\pi} f(w)\left[\text{RE}\left(\frac{Z_-}{1-Z_-}\right) - \text{RE}\left(\frac{Z_+}{1-Z_+}\right)\right] dw \quad (*)$$

NOW $\quad \mathrm{RE}\left(\frac{Z_-}{1-Z_-}\right) = \mathrm{RE}\left(\frac{Z_-\overline{(1-Z_-)}}{|1-Z_-|^2}\right) = \mathrm{RE}\left(\frac{Z_- - |Z_-|^2}{|1-Z_-|^2}\right)$

$$\mathrm{RE}\left(\frac{Z_-}{1-Z_-}\right) = \frac{1}{|1-Z_-|^2}\left(\mathrm{RE}\, Z_- - |Z_-|^2\right)$$

$$= \frac{(r/a)\cos(\varphi-w) - r^2/a^2}{\left(1-\frac{r}{a}\cos(\varphi-w)\right)^2+\left(\frac{r}{a}\right)^2\sin^2(\varphi-w)}$$

$$\mathrm{RE}\left(\frac{Z_-}{1-Z_-}\right) = \frac{(r/a)\cos(\varphi-w) - r^2/a^2}{1+(r/a)^2-2(r/a)\cos(\varphi-w)}$$

SIMILARLY $\quad \mathrm{RE}\left(\frac{Z_+}{1-Z_+}\right) = \dfrac{(r/a)\cos(\varphi+w) - r^2/a^2}{1+(r/a)^2-2(r/a)\cos(\varphi+w)}$

IN (*) WE GET

$$u(r,\varphi) = \frac{1}{\pi}\int_0^{\pi} f(w)\left[\frac{(r/a)\cos(\varphi-w) - r^2/a^2}{1+(r/a)^2-2(r/a)\cos(\varphi-w)} - \frac{[(r/a)\cos(\varphi+w)-r^2/a^2]}{1+(r/a)^2-2(r/a)\cos(\varphi+w)}\right]dw.$$

PROBLEM 6

WE TAKE LAPLACE TRANSFORMS

$$\hat{U}(r,s) = \mathcal{L}[U(r,t)]$$

TO GET $s\hat{U} - U(r,0) = D\left[\hat{U}_{rr} + \frac{2}{r}\hat{U}_r\right]$ WITH $\hat{U}(a,s) = \mathcal{L}[U(a,t)] = 1/s^2$.

SINCE $U(r,0) = 0$ WE GET

$$\hat{U}_{rr} + \frac{2}{r}\hat{U}_r - \frac{s}{D}\hat{U} = 0 \quad \text{IN } r > a \qquad \hat{U} = \frac{1}{r}\,\text{SPAN}\left\{e^{-\sqrt{s/D}\,r},\ e^{\sqrt{s/D}\,r}\right\}$$

$$\hat{U}(a,s) = 1/s^2, \quad \hat{U} \text{ BOUNDED AS } r \to \infty$$

THIS GIVES $\hat{U} = \frac{A}{r} e^{-\sqrt{s/D}\,r}$. THE CONDITION ON $r = a$ GIVES $\frac{1}{s^2} = \frac{A}{a} e^{-\sqrt{s/D}\,a}$.

THIS GIVES $A = (a/s^2)\, e^{+\sqrt{s/D}\,a}$.

WE GET $\hat{U}(r,s) = \left(\frac{a}{r}\right)\frac{1}{s^2} e^{-\sqrt{s/D}\,(r-a)}$ IN $r > a$.

NOW $\hat{U}_r\Big|_{r=a} = -\frac{1}{\sqrt{D}}\frac{1}{s^{3/2}} - \frac{1}{s^2}\left(\frac{a}{r^2}\right)\Big|_{r=a} = -\frac{1}{\sqrt{D}}\frac{1}{s^{3/2}} - \frac{1}{a s^2}$.

MULTIPLYING BY $-D$

$$\longrightarrow \quad -D\hat{U}_r\Big|_{r=a} = \frac{\sqrt{D}}{s^{3/2}} + \frac{D}{a s^2}.$$

NOW USE $\mathcal{L}[t^p] = \Gamma(p+1)/s^{p+1}$ TO GET USING $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2)$, $\Gamma(1/2) = \sqrt{\pi}$,

$$-DU_r\Big|_{r=a} = \frac{\sqrt{D}\,t^{1/2}}{\Gamma(3/2)} + \frac{Dt}{a} = \frac{2\sqrt{D}\,t^{1/2}}{\sqrt{\pi}} + \frac{Dt}{a}$$

NOW INTEGRATING,

$$J(t) = \int_0^t -DU_r\Big|_{r=a}\,d\tau = \int_0^t \left(\frac{2\sqrt{D}}{\sqrt{\pi}}\tau^{1/2} + \frac{D\tau}{a}\right)d\tau$$

SO

$$J(t) = \frac{4}{3}\sqrt{\frac{D}{\pi}}\,t^{3/2} + \frac{Dt^2}{2a}.$$

PROBLEM 7 TAKE LAPLACE TRANSFORM IN TIME.

(a) LET $\hat{U}(x,s) = \mathcal{L}[u(x,t)]$, WE GET

$$s\hat{U} - u(x,0) = D\hat{U}_{xx} - k\hat{U} \quad \text{ON} \quad x>0 \quad \text{WITH} \quad \hat{U}(0,s) = \mathcal{L}[u(0,t)] = 1/s^2.$$

WITH $u(x,0)=0$ WE GET

$$\hat{U}_{xx} - \frac{(k+s)}{D}\hat{U} = 0 \quad \text{IN} \quad x>0 \quad ; \quad \hat{U}(0,s) = 1/s^2.$$

THE SOLUTION IS $\hat{U}(x,s) = \dfrac{1}{s^2} e^{-\sqrt{\frac{k+s}{D}}\,x}$. $(*)$

NOW DECOMPOSE AS

$$\hat{U}(x,s) = \hat{F}(s)\,\hat{G}(s+k) \quad \text{WITH} \quad \hat{F}(s) = \frac{1}{s^2}, \quad \hat{G}(s) = e^{-\sqrt{s/D}\,x}.$$

$\lambda = x/\sqrt{D}$

WE HAVE

$$f(t) = \mathcal{L}^{-1}[\hat{F}(s)] = t$$

$$g(t) = \mathcal{L}^{-1}[\hat{G}(s)] = \frac{x}{2\sqrt{\pi D}\,t^{3/2}} e^{-x^2/4Dt}$$

SO BY SHIFT THEOREM OF LAPLACE TRANSFORM $\mathcal{L}^{-1}[\hat{G}(s+k)] = \dfrac{x}{2\sqrt{\pi D}\,t^{3/2}} e^{-kt} e^{-x^2/4Dt}$.

FINALLY BY CONVOLUTION PROPERTY

$$u(x,t) = \int_0^t f(t-\tau)\frac{x}{2\sqrt{\pi D}\,\tau^{3/2}} e^{-k\tau} e^{-x^2/4D\tau}\,d\tau = \frac{x}{2\sqrt{\pi D}} \int_0^t \frac{(t-\tau)}{\tau^{3/2}} e^{-k\tau} e^{-x^2/4D\tau}\,d\tau.$$

(b) FROM $(*)$ WE CALCULATE $-D\hat{U}_x(0,s) = +\dfrac{D}{s^2}\sqrt{\dfrac{k+s}{D}} = \sqrt{D}\,\dfrac{\sqrt{k+s}}{s^2}$.

NOW WE MUST INVERT $\mathcal{L}^{-1}\left[\dfrac{\sqrt{k+s}}{s^2}\right]$.

WE WRITE $\dfrac{\sqrt{k+s}}{s^2} = \dfrac{(k+s)}{s^2\sqrt{k+s}} = \dfrac{k}{s^2\sqrt{k+s}} + \dfrac{1}{s\sqrt{s+k}}$.

RECALLING $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$, $\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}}\right] = \frac{1}{\sqrt{\pi t}}$, $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$ WE GET BY

SHIFT THEOREM AND CONVOLUTION THAT

$$J(t) = \mathcal{L}^{-1}\left[\sqrt{D}\,\frac{\sqrt{k+s}}{s^2}\right] = k\sqrt{D}\int_0^t (t-\tau)\frac{e^{-k\tau}}{\sqrt{\pi\tau}}\,d\tau + \sqrt{D}\int_0^t \frac{e^{-k\tau}}{\sqrt{\pi\tau}}\,d\tau.$$

PROBLEM 8

(a) IN 2-D IN A DISK

$$u(r,\varphi) = A_0 + \sum_{n=1}^{\infty} (r/a)^n \left[A_n \cos n\varphi + B_n \sin n\varphi \right]$$

WE WANT $u(a,\varphi) = 2\cos^2\varphi + 1 = 2\left[1+\cos(2\varphi)\right]/2 + 1 = \cos(2\varphi) + 2$

SO $u(r,\varphi) = A_0 + A_2 (r/a)^2 \cos(2\varphi)$ WITH $A_0 = 2,\ A_2 = 1$

(b) $u(r,\varphi) = \sum_{n=0}^{\infty} A_n (a/r)^{n+1} P_n(\cos\varphi)$

LET $r = a$ $\quad u = 2\cos^2\varphi + 1 = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi)$

so let $x = \cos\varphi$: $\quad 2x^2 + 1 = \sum_{n=0}^{\infty} A_n P_n(x)$

WE HAVE $A_n = 0 \ \forall n$ EXCEPT $n = 0, 2.$ $\rightarrow 2x^2 + 1 = A_0 P_0(x) + A_2 P_2(x)$

$$2x^2 + 1 = A_0 + \frac{A_2}{2}(3x^2 - 1) = x^2\left(\frac{3A_2}{2}\right) + A_0 - A_2/2.$$

SO $3A_2/2 = 2$ AND $A_0 - A_2/2 = 1$

$\rightarrow A_2 = 4/3 \rightarrow A_0 = 1 + 2/3 = 5/3$

$$\rightarrow u(r,\varphi) = \frac{5}{3}\left(\frac{a}{r}\right) P_0(\cos\varphi) + \frac{4}{3}\left(\frac{a}{r}\right)^3 P_2(\cos\varphi)$$

(c) $u(r,\varphi) = \sum_{n=0}^{\infty} A_n (a/r)^{n+1} P_n(\cos\varphi)$

NOW LET $r = a \rightarrow u = 4\sin^2\varphi + 1 = 4(1-\cos^2\varphi) + 1 = 5 - 4\cos^2\varphi = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi)$

so let $x = \cos\varphi \rightarrow 5 - 4x^2 = A_0 P_0(x) + A_2 P_2(x)$, $A_n = 0$ FOR $n \neq 0, 2$.

$$\rightarrow 5 - 4x^2 = A_0 + \frac{A_2}{2}(3x^2 - 1) = \frac{3A_2}{2}x^2 + A_0 - A_2/2$$

$\rightarrow -4 = 3A_2/2 \rightarrow A_0 - A_2/2 = 5$

$\rightarrow A_2 = -8/3 \qquad A_0 = 5 - 4/3 = 11/3$

$$u(r,\varphi) = \frac{11}{3}\left(\frac{a}{r}\right) P_0(\cos\varphi) - \frac{8}{3}\left(\frac{a}{r}\right)^3 P_2(\cos\varphi).$$

(d) $u_t = u_{xx} - u$ WITH $u_x = 0$ AT $x = 0, L$; $u(x,0) = x^2$.

INTEGRATING $\int_0^L u_t \, dx = \int_0^L u_{xx}\, dx - \int_0^L u \, dx$

$$\rightarrow \frac{d}{dt} M = u_x \Big|_0^L - M \quad \text{WITH} \quad M(t) = \int_0^L u(x,t)\, dx.$$

NOW $dM/dt = -M$, $M(0) = \int_0^L x^2\, dx = L^3/3$

SO $M = (L^3/3)e^{-t}$.

NOW REDEFINE $M(t)$ TO BE $M(t) = \int_0^L [u(x,t)]^2\, dx$.

WE CALCULATE $\frac{dM}{dt} = \int_0^L 2 u u_t \, dx = \int_0^L 2u(u_{xx} - u)\, dx$, USING PDE.

$$\frac{dM}{dt} = 2\int_0^L u\, u_{xx}\, dx - 2\int_0^L u^2\, dx = 2\left[u u_x \Big|_0^L - \int_0^L u_x^2\, dx\right] - 2\int_0^L u^2\, dx.$$

BUT $u_x = 0$ AT $x = 0, L$ SO THAT

$$\frac{dM}{dt} = -2\left[\int_0^L (u^2 + u_x^2)\, dx\right] \le 0.$$

(e) USE DIVERGENCE THEOREM

$$\int_\Omega \nabla\cdot(\nabla u)\, d\underline{x} = \int_\Omega \Delta u \, d\underline{x} = \int_{\partial\Omega} \nabla u \cdot \hat{n}\, ds = \int_{\partial\Omega} \frac{\partial u}{\partial r}\Big|_{r=a} ds$$

SO $\int_\Omega F r^2\, d\underline{x} = F\int_0^{2\pi}\int_0^a r^2 (r\, dr\, d\varphi) = \int_0^{2\pi} \cos^2\varphi \, \underbrace{a\, d\varphi}_{= ds}$

$$\rightarrow 2\pi F\left(a^4/4\right) = \frac{a}{2}\int_0^{2\pi}(1 + \cos 2\varphi)\, d\varphi = \pi a.$$

solve FOR F to get $F = 2/a^3$

(f) CONSIDER $\int_{-1}^1 x^m P_n(x)\, dx$ WITH $0 < m < n$.

THEN $x^m = \sum_{j=0}^m a_j P_j(x)$ FOR SOME a_j.

$$\rightarrow \int_{-1}^1 x^m P_n\, dx = \sum_{j=0}^m a_j \int_{-1}^1 P_j P_n\, dx = 0 \quad \text{SINCE} \int_{-1}^1 P_j P_n\, dx = 0$$

FOR $j = 0, \dots, m < n$ BY ORTHOGONALITY.

SINCE $P_{n-1}'(x)$ IS A POLYNOMIAL OF degree $n-2$ WE HAVE $P_{n-1}'(x) = \sum_{j=0}^{n-2} a_j P_j(x) \longrightarrow$ WE CONCLUDE IN SAME WAY THAT $\int_{-1}^1 P_n P_{n-1}'\, dx = 0$.

(a) STARTING WITH

$$P_{n-1}'(x) = -n P_n(x) + x P_n'(x)$$

MULTIPLY BY $P_n(x)$, INTEGRATE FROM $-1 < x < 1$, AND USE $\int_{-1}^{1} P_n P_{n-1}' \, dx = 0$ BY (F). WE OBTAIN,

$$\int_{-1}^{1} P_n P_{n-1}' \, dx = -n \int_{-1}^{1} [P_n(x)]^2 \, dx + \int_{-1}^{1} x \, P_n(x) P_n'(x) \, dx$$

$$0 = -n \int_{-1}^{1} P_n^2 \, dx + \int_{-1}^{1} \frac{x}{2} \left[P_n^2 \right]' \, dx$$

USE IBP,

$$n \int_{-1}^{1} P_n^2 \, dx = \frac{x}{2} [P_n(x)]^2 \Big|_{-1}^{1} - \int_{-1}^{1} \frac{1}{2} P_n^2 \, dx$$

$$\rightarrow (n + 1/2) \int_{-1}^{1} P_n^2 \, dx = \frac{[P_n(1)]^2}{2} - \frac{(-1)[P_n(-1)]^2}{2} = [P_n(1)]^2$$

SINCE $(P_n(1))^2 = (P_n(-1))^2$. (RECALL $P_n(1) = 1$ SO $P_n(-1) = \begin{cases} -1 & \text{IF } n = \text{ODD} \\ 1 & \text{IF } n = \text{even} \end{cases}$

SO

$$\left(n + \frac{1}{2} \right) \int_{-1}^{1} P_n^2 \, dx = 1 \rightarrow \int_{-1}^{1} P_n^2 \, dx = \frac{2}{2n+1}.$$

(b) let $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$.

THEN

$$[f(x)]^2 = \left(\sum_{n=0}^{\infty} a_n P_n(x) \right) \left(\sum_{m=0}^{\infty} a_m P_m(x) \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n a_m P_n P_m.$$

INTEGRATING AND USING ORTHOGONALITY $\int_{-1}^{1} P_n P_m \, dx = 0$ IF $n \neq m$,

$$\int_{-1}^{1} [f(x)]^2 \, dx = \sum_{n=0}^{\infty} a_n^2 \int_{-1}^{1} P_n^2 \, dx = \sum_{n=0}^{\infty} a_n^2 \left(\frac{2}{2n+1} \right)$$

SO

$$\int_{-1}^{1} [f(x)]^2 \, dx = \sum_{n=0}^{\infty} \frac{2 a_n^2}{2n+1}.$$