MATH 400: MIDTERM November 2nd, 2015 (M. J. WARD)

Maximum=50 Points: Closed Book and Closed Notes; 55 minutes

<u>PROBLEM 1:</u> (15 Points) Consider the first-order PDE for $u = u(x,y)$ given by

$$xu_x - yu_y = u^2 , \qquad \text{with} \quad u = 1 \quad \text{on} \quad x = y^2 .$$

(i) Determine the characteristics and an explicit form for the solution $u(x,y)$.

(ii) Plot the data curve and the characteristics. Determine the curve in the (x,y) plane upon which u is infinite.

<u>PROBLEM 2:</u> (20 Points) Consider the traffic flow equation with traffic density $\rho(x,t)$ satisfying

$$\rho_t + 2(1-\rho)\rho_x = 0 , \quad -\infty < x < \infty, \quad t > 0 ,$$
$$\rho(x,0) = 1 , \text{ for } 0 < x < 1 ; \qquad \rho(x,0) = 0 , \text{ for } x < 0 \text{ and } x > 1 .$$

(i) For $0 < t < 1$, determine $\rho(x,t)$ in the various regions of the (t,x) plane, inserting a shock and an expansion fan as required. Plot the characteristics, the shock, and the expansion fan in the (t,x) plane for $0 < t < 1$. (Fit any shock with the flux function $q(\rho) = 2\left(\rho - \rho^2/2\right)$.)

(ii) Determine how the solution changes for $t > 1$ and insert a new shock as needed.

(iii) Plot the solution $\rho(x,t)$ versus x for $0 < t < 1$ and then for $t > 1$. Calculate $\int_{-\infty}^{\infty} \rho(x,t)\, dx$ at any time t.

<u>PROBLEM 3:</u> (15 Points) Consider the following eigenvalue problem for $\phi(x)$ and λ:

$$(x+2)\phi'' - \phi' + \frac{\lambda}{(x+2)}\phi = 0 , \quad -1 < x < 2 ; \qquad \phi(-1) = \phi(2) = 0 .$$

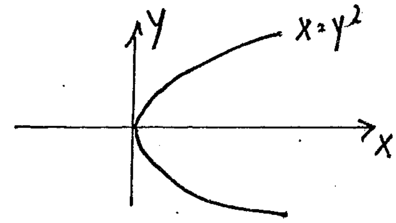
(i) Write this problem in Sturm-Liouville form and state the orthogonality relation.

(ii) Prove (without solving the equation explicitly) that the eigenvalues satisfy $\lambda > 0$.

(iii) Determine the eigenvalues explicity. (Hint: think of Euler's equation).

M400 MIDTERM

PROBLEM 1



WE CONSIDER $\quad xu_x - yu_y = u^2 \quad$ WITH $\quad u=1 \quad$ ON $\quad x=y^2$

(i) WE SOLVE BY CHARACTERISTICS. PARAMETRIZE DATA CURVE

$$y_0 = \tau, \quad x_0 = \tau^2, \quad u_0 = 1, \quad -\infty < \tau < \infty$$

THE CHARACTERISTIC EQUATION ARE

$$\frac{dx}{ds} = x, \quad x(0) = \tau^2; \qquad \frac{dy}{ds} = -y, \quad y(0) = \tau; \qquad \frac{du}{ds} = u^2, \quad u(0) = 1.$$

WE SOLVE TO GET IMPLICIT SOLUTION

$$x = \tau^2 e^s, \quad y = \tau e^{-s}, \quad u = \frac{1}{1-s}$$

ELIMINATING s GIVES CHARACTERISTICS : $\quad e^s = x/\tau^2 = \tau/y \;\rightarrow\; y = \tau^3/x$

CHARACTERISTICS ARE $\quad y = \tau^3/x \rightarrow$ HYPERBOLAE

NOW ELIMINATE τ AND SOLVE FOR s: CONSIDER $y>0$ SINCE $y \mapsto -y$ SYMMETRY.

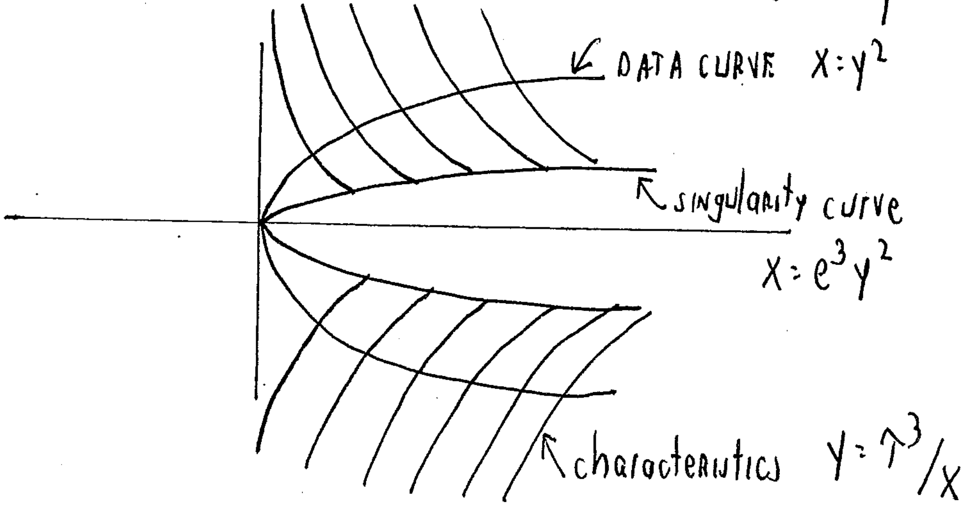
THUS GIVES $\quad x/\tau^2 = e^s = \dfrac{x}{(xy)^{2/3}} = \dfrac{x^{1/3}}{y^{2/3}} \;\longrightarrow\; s = \log\left(\dfrac{x^{1/3}}{y^{2/3}}\right)$

HENCE $\quad s = \dfrac{1}{3}\log\left(x/y^2\right)$

THUS YIELDS THE SOLUTION $\quad u = \dfrac{1}{1-s} \quad$ OR $\quad u = \dfrac{1}{1 - \frac{1}{3}\log(x/y^2)}$

NOTICE THAT $u = \infty$ i.e. BLOWS UP ON $\quad \dfrac{x}{y^2} = e^3 \;\rightarrow\; y^2 = xe^{-3}.$

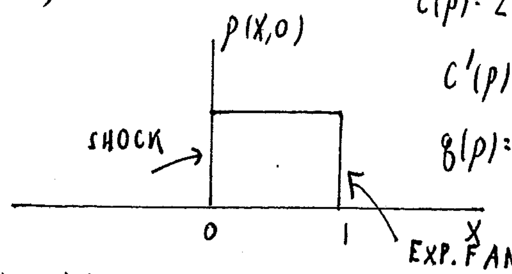
HENCE THE SINGULARITY CURVE IS $\quad y^2 = xe^{-3}.$



SOLUTION IS DEFINED IN $x \geq 0$ EXCEPT FOR REGION where $x > e^3y^2$.

PROBLEM 2 CONSIDER $p_t + 2(1-p)p_x = 0, \quad -\infty < x < \infty, \ t > 0$

$$p(x,0) = \begin{cases} 0, & x<0 \text{ OR } x>1 \\ 1, & 0<x<1 \end{cases}$$



$C(p) = 2(1-p)$

$C'(p) < 0$

$g(p) = 2(p-p^2)$

(i) SINCE $c'(p) < 0$ BUT $c(p) > 0$ FOR $0 < p < 1$ AND $0 < p(x,0) \leq 1$

IT FOLLOWS THAT WAVE STEEPEN AT THE BACK AND AN EXPANSION FAN IS NEEDED AT THE RIGHT.

METHOD OF CHARACTERISTICS:

ON $dx/dt = 2(1-p), \quad dp/dt = 0.$

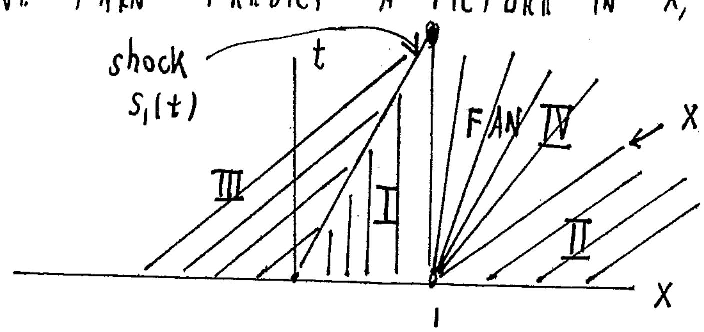
REGION I $\quad 0 < \xi < 1 \qquad \dfrac{dx}{dt} = 2(1-p), \quad \dfrac{dp}{dt} = 0, \quad x(0) = \xi, \ p(0) = 1$

SO $p = 1$ AND $x = \xi$.

REGION II $\quad \xi > 1 \qquad \left.\begin{aligned} \frac{dx}{dt} &= 2(1-p), \quad x(0) = \xi \\ \frac{dp}{dt} &= 0, \quad p(0) = 0 \end{aligned}\right\} \longrightarrow \begin{aligned} p &= 0 \\ x &= 2t + \xi. \end{aligned}$

REGION III $\quad \xi < 0$ (SAME AS REGION I) SO $p = 0, \ x = 2t + \xi$.

WE THEN PREDICT A PICTURE IN x, t PLANE AS SHOWN



SHOCK JOINS REGION I + III

FAN CONNECTS REGION I + II

EXPANSION FAN: REGION IV $\quad$ LET $p = F\left(\dfrac{x-1}{t}\right) \qquad p_t = -F'(\lambda)\dfrac{(x-1)}{t^2}, \quad p_x = F'(\lambda)$

SO PDE GIVES $\quad c(F) = \lambda = 2(1-F) \rightarrow F = 1 - \lambda/2 \qquad \lambda = (x-1)/t$

EXPANSION FAN REGION IV IS $\quad p(x,t) = 1 - (x-1)/2t, \quad 1 < x < 2t+1.$

THE <u>FIRST SHOCK CONNECTS</u> REGION I AND III. RECALL $q(\rho) = 2(\rho - \rho^2/2)$

WE CALCULATE WITH $\dfrac{ds_1}{dt} = \dfrac{q(\rho_{III}) - q(\rho_I)}{\rho_{III} - \rho_I} = \dfrac{q(0) - q(1)}{0-1} = 1, \quad S_1(0) = 0$

HENCE $S_1 = t$ IS FIRST SHOCK

(ii) THE EXPANSION FAN HITS THE SHOCK WHEN $S_1 = 1 \rightarrow t = 1$.

FOR $t > 1$ NEED A NEW SHOCK S_2 CONNECTING REGIONS III AND (IV).

$$\frac{dS_2}{dt} = \frac{q(\rho_{IV}) - q(\rho_{III})}{\rho_{IV} - \rho_{III}} \qquad \rho_{III} = 0, \quad \rho_{IV} = 1 - \frac{(x-1)}{2t}$$

SO $$\frac{dS_2}{dt} = \frac{q(\rho_{IV})}{\rho_{IV}} = 2\left(1 - \frac{\rho_{IV}}{2}\right) = 1 + \frac{(S_2 - 1)}{2t}, \quad S_2(1) = 1.$$

SOLVE $$\frac{dS_2}{dt} - \frac{1}{2t} S_2 = 1 - \frac{1}{2t} \qquad \text{LET } S_2 = V_2 + 1 \text{ SO } \frac{dV_2}{dt} - \frac{1}{2t} V_2 = 1.$$

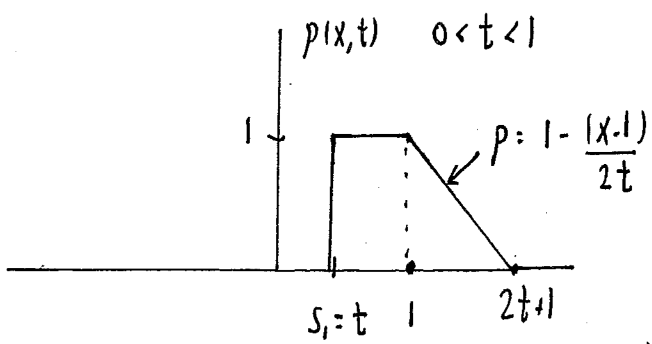
THEN $$(t^{-1/2} V_2)' = t^{-1/2} \rightarrow V_2 t^{-1/2} = 2t^{1/2} + C \rightarrow V_2 = 2t + Ct^{1/2}.$$

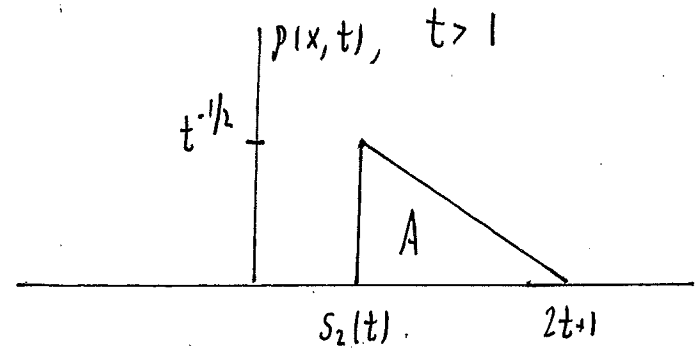
SO $S_2(t) = 2t + 1 + Ct^{1/2}$. NOW $S_2(1) = 1$

SO $S_2(t) = 2t + 1 - 2t^{1/2}$ IS SECOND SHOCK.

AT EDGE OF SHOCK $$\rho_{IV} = 1 - \frac{(S_2 - 1)}{2t} = \frac{1}{\sqrt{t}}.$$

(iii) THEREFORE WE GET THE FOLLOWING PICTURES





NOTICE THAT $$\text{AREA}(A) = \frac{1}{2} t^{-1/2}[2t + 1 - S_2(t)] = \frac{1}{2} t^{-1/2}[2t^{1/2}] = 1.$$

THIS AREA IS SAME AS INITIAL AREA $\int_0^1 \rho(x,0)\, dx = 1$.

PROBLEM 3 CONSIDER $(X+2)\phi'' - \phi' + \frac{\lambda}{(X+2)}\phi = 0$ WITH $\phi(-1) = \phi(2) = 0$.

(i) TO WRITE IN SL FORM FIRST WRITE

$$\phi'' - \frac{1}{(X+2)}\phi' + \frac{\lambda}{(X+2)^2}\phi = 0$$ SO THAT IF WE MULTIPLY BY

$p(X)$ THEN $(p\phi')' + \frac{\lambda p}{(X+2)^2}\phi = 0$ SO THAT $p' = -\frac{1}{(X+2)}p$ IS NEEDED.

THIS GIVES $p(X) = \exp\left[-\int^X \frac{1}{X+2}\,dx\right] = \frac{1}{X+2}$.

$$(*) \quad \left[\frac{1}{(X+2)}\phi'\right]' + \frac{\lambda}{(X+2)^3}\phi = 0$$

SINCE $W(X) = \frac{1}{(X+2)^3}$ WE HAVE $\int_{-1}^{2} \frac{1}{(X+2)^3}\phi_n\phi_m\,dx = 0$ FOR $n \neq m$.

SINCE IT IS A SL PROBLEM WE KNOW λ IS REAL.

(ii) MULTIPLY $(*)$ BY ϕ AND INTEGRATE BY PARTS:

$$\int_{-1}^{2} \phi\left[\frac{1}{(X+2)}\phi'\right]' dx + \lambda\int_{-1}^{2}\frac{\phi^2}{(X+2)^3}\,dx = 0$$

$$\phi\frac{1}{(X+2)}\phi'\Big|_{-1}^{2} - \int_{-1}^{2}\frac{1}{(X+2)}\phi'^2\,dx + \lambda\int_{-1}^{2}\frac{\phi^2}{(X+2)^3}\,dx = 0.$$

SINCE $\phi(-1) = \phi(2) = 0$ THEN $\lambda\int_{-1}^{1}\frac{\phi^2}{(X+2)^3}\,dx = \int_{-1}^{2}\frac{1}{(X+2)}\phi'^2\,dx$.

SINCE $\lambda = 0$ IS <u>NOT</u> AN EIGENVALUE THEN RHS OF EQUATION ABOVE IS POSITIVE AND SO $\lambda > 0$.

(iii) $(X+2)\phi'' - (X+2)\phi' + \lambda\phi = 0$. PUT $\phi = (X+2)^r$ SO THAT $r(r-1) - r + \lambda = 0$

SO $r^2 - 2r + \lambda = 0$ OR $(r-1)^2 = 1-\lambda$. $\lambda > 1$ IS ONLY POSSIBLE RANGE,

SO $r = 1 \pm i\sqrt{\lambda-1} \rightarrow \phi = (X+2)\left[A\cos\left(\sqrt{\lambda-1}\log(X+2)\right) + B\sin\left(\sqrt{\lambda-1}\log(X+2)\right)\right]$

BUT $\phi(-1) = 0$ SO $A = 0$. $\phi(2) = 0$ YIELDS $\sin\left[\sqrt{\lambda-1}\log 4\right] = 0$ OR

$\lambda_n = 1 + n^2\pi^2/(\log 4)^2$, $n = 1, 2, \ldots$