MATH 401 HOMEWORK 1

PROBLEM 1

(i) SHOW THAT $x\delta'(x) + \delta(x) = 0$.

(ii) LET $f(x) = H(x-1)\sin x$ WHERE $H(z)$ IS THE HEAVYSIDE FUNCTION. FIND THE DERIVATIVE OF THE GENERALIZED FUNCTION $f(x)$.

(iii) EXPRESS $\delta(x^2 - a^2)$ AS A SUM OF DELTA FUNCTIONS

(iv) SHOW THAT $f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x)$.

PROBLEM 2

DEFINE $Lu \equiv u'' - k^2 u$ ON $0 < x < 1$ WITH $k > 0$.

(i) FIND THE GREEN'S FUNCTION FOR Lu WITH $u(0) = u(1) = 0$.

(ii) FIND AN INTEGRAL REPRESENTATION FOR THE SOLUTION TO

$$Lu = f(x), \quad 0 < x < 1$$

$$u(0) = 1, \quad u(1) = 0$$

(NOTICE INHOMOGENEOUS BC here).

(iii) SUPPOSE THAT $f(x)$ DEPENDS ON A PARAMETER ε AS

$$f(x) \equiv \frac{1}{\varepsilon}\,\mathrm{sech}^2\left(\frac{x - 1/2}{\varepsilon}\right).$$

FIND A GOOD APPROXIMATION WHEN $\varepsilon \to 0^+$ FOR THE SOLUTION TO

$$Lu = f(x) \quad \text{ON} \quad 0 < x < 1$$

WITH $u(0) = 0, \quad u(1) = 0$.

PROBLEM 3

CONSIDER $Lu \equiv u'' - k^2 u$ WITH $k > 0$.

(i) FIND THE GREEN'S FUNCTION FOR Lu ON $-\infty < x < \infty$ WITH $u \to 0$ AS $|x| \to \infty$.

(ii) FIND THE GREEN'S FUNCTION FOR Lu ON $0 \le x < \infty$ WITH $u(0) = 0$ AND $u \to 0$ AS $x \to \infty$.

PROBLEM 4

(i) FIND THE GREEN'S FUNCTION FOR

$$Lu \equiv (xu')' - \frac{1}{x}u = f(x) \quad \text{ON} \quad 0<x<1$$

WITH $u(0)$ FINITE, $u(1)=0$.

(ii) LET $U(r,\varphi)$ WITH r, φ BEING POLAR COORDINATES SATISFYING

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\varphi\varphi} = \delta(r-r_0)\cos\varphi \quad \text{IN} \quad 0\le r\le 1,\ 0\le\varphi\le 2\pi, \text{ WITH } 0<r_0<1,$$

AND WITH

$$U=0 \text{ ON } r=1; \quad U \text{ BOUNDED AS } r\to 0$$

$$U,\ U_\varphi \ 2\pi \text{ PERIODIC IN } \varphi.$$

FIND AN EXPLICIT SOLUTION FOR $U(r,\varphi)$ IN TERMS OF $\cos\varphi$ AND YOUR 1-D GREEN'S FUNCTION IN PART (i).

PROBLEM 5 DEFINE $Lu \equiv u'' + k^2 u$ ON $0\le x\le a$ WITH $a>0$ AND $k>0$.

(i) FIND THE GREEN'S FUNCTION FOR Lu WITH $u'(0)=u'(a)=0$.

(ii) SHOW THAT THE GREEN'S FUNCTION IN (i) DOES NOT EXIST WHEN $k=n\pi/a$ WITH $n=1,2,\ldots$

(iii) SUPPOSE THAT $Lu=f(x)$ ON $0\le x\le a$

WITH $u'(0)=0$ AND $u'(a)=1$.

LET $k=n\pi/a$. FIND THE CONDITION ON $f(x)$ REQUIRED FOR THIS PROBLEM TO HAVE A SOLUTION.

(HINT: NOTICE THAT u HAS INHOMOGENEOUS BOUNDARY CONDITIONS).

MATH 401 HW # 1 SOLUTIONS

PROBLEM 1

(i) SHOW THAT $x\delta'(x) + \delta(x) = 0$.

LET ϕ BE ANY TEST FUNCTION. THEN INTEGRATING BY PARTS:

$$(x\delta', \phi) = \int_{-\infty}^{\infty} \delta' x \phi \, dx = x\phi\delta\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)\,(x\phi)'\,dx.$$

SINCE ϕ HAS COMPACT SUPPORT, $\phi(\pm\infty) = 0$ AND

$$(x\delta', \phi) = -\int_{-\infty}^{\infty} \delta(x)\,[x\phi' + \phi]\,dx = -\phi(0) \quad \text{BY SIFTING PROPERTY.}$$

THUS $$(x\delta', \phi) = -\phi(0) = \int_{-\infty}^{\infty} (-\delta(x))\,\phi\,dx. = (-\delta, \phi).$$

WE CONCLUDE THAT $x\delta' = -\delta$ IN SENSE OF DISTRIBUTIONS.

(ii) FIND GENERALIZED DERIVATIVE OF $f(x) = H(x-1)\sin x$.

LET ϕ BE ANY TEST FUNCTION. THEN INTEGRATING BY PARTS

$$(f'(x), \phi(x)) = -(f(x), \phi'(x)) = -(H(x-1)\sin x, \phi'(x)) = -\int_1^{\infty} \sin x\, \phi'(x)\,dx.$$

NOW INTEGRATING BY PARTS,

$$(f'(x), \phi(x)) = -\sin x\,\phi(x)\Big|_1^{\infty} + \int_1^{\infty} \cos x\,\phi(x)\,dx = +\sin(1)\,\phi(1) + \int_1^{\infty} H(x-1)\cos x\,\phi(x)\,dx$$

HENCE $$(f'(x), \phi(x)) = (\delta(x-1)\sin x, \phi(x)) + (H(x-1)\cos x, \phi(x))$$
$$= (\delta(x-1)\sin x + H(x-1)\cos x, \phi(x)).$$

WE CONCLUDE THAT $f'(x) = \delta(x-1)\sin x + H(x-1)\cos x$ IN DISTRIBUTION SENSE.

(iii) EXPRESS $\delta(x^2 - a^2)$ AS A SUM OF DELTA FUNCTIONS.

WE RECALL THAT IF $f(x_k) = 0$, $f'(x_k) \neq 0$ FOR $k = 1, \ldots, n$ THEN $\delta(f(x)) = \sum_{k=1}^{n} \frac{\delta(x - x_k)}{|f'(x_k)|}$.

HERE $n = 2$ AND $f(x) = x^2 - a^2$ WITH $x_1 = a$, $x_2 = -a$ AND

$|f'(x_j)| = 2a$. THUS YIELDS $$\delta(x^2 - a^2) = \frac{1}{2a}\,[\delta(x+a) + \delta(x-a)].$$

(iv) SHOW THAT $f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x)$.

LET $\phi(x)$ BE ANY TEST FUNCTION. WE INTEGRATE BY PARTS AND USE SIFTING PROPERTY

$$(f(x)\delta'(x), \phi(x)) = \int_{-\infty}^{\infty} \delta'(x) f(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} \delta(x)\,(f\phi)'\,dx = -\int_{-\infty}^{\infty} \delta(x)\,(f\phi' + f'\phi)\,dx$$
$$= -f'(0)\,\phi(0) - f(0)\,\phi'(0). = -f'(0)\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx - f(0)\,\phi'(0).$$

BUT $\phi'(0) = \int_{-\infty}^{\infty} \delta(x)\,\phi'(x)\,dx = -\int_{-\infty}^{\infty} \delta'(x)\,\phi(x)\,dx$. THUS $(f\delta', \phi) = -(\delta f'(0), \phi) + (\delta' f(0), \phi)$

THIS YIELDS $$f(x)\,\delta'(x) = -\delta(x) f'(0) + \delta'(x)\,f(0)$$

PROBLEM 2 $\quad Lu \equiv u'' - k^2 u$ ON $0<x<1$ WITH $k>0$.

(i) FIND G-FUNCTION FOR Lu WITH $u(0)=u(1)=0$.

SINCE L IS SL OPERATOR AND BC ARE SEPARATED, WE HAVE THAT THE G-FUNCTION $v(t;x)$ SATISFIES

$$\begin{cases} v'' - k^2 v = \delta(t-x) \quad \text{ON} \quad 0<t<1 \\ v(0) = v(1) = 0. \end{cases}$$

THE TWO SOLUTIONS ARE SPAN $\{\cosh[k(t-x)], \sinh[k(t-x)]\}$.

BY INSPECTION TO SATISFY THE BC WE TAKE

$$v = \begin{cases} c_1 \sinh[kt], & 0<t<x \\ c_2 \sinh[k(t-1)], & x<t<1 \end{cases}$$

NOW IMPOSING CONTINUITY ACROSS $t=x$ WE HAVE

$$v = \begin{cases} c \sinh[kt] \sinh[k(x-1)], & 0<t<x \\ c \sinh[kx] \sinh[k(t-1)], & x<t<1. \end{cases}$$

NOW THE JUMP CONDITION IS $\quad v'(x^+) - v'(x^-) = 1$.

THUS $\quad c_1\left(\sinh(kx)\; k\cosh[k(t-1)]\Big|_{t=x} - \sinh[k(x-1)]\; k\cosh(kt)\Big|_{t=x}\right) = 1.$

HENCE $\quad ck\left[\sinh(kx)\cosh(k(x-1)) - \sinh(k(x-1))\cosh(kx)\right] = 1.$

SINCE $\quad \sinh(A-B) = \sinh A \cosh B - \sinh B \cosh A \quad$ WE HAVE

$$ck \sinh[kx - k(x-1)] = 1 \longrightarrow c = \frac{1}{k\sinh k}.$$

WE CONCLUDE THAT

$$v = \frac{1}{k\sinh k}\begin{cases} \sinh(kt)\sinh(k(x-1)), & 0<t<x \\ \sinh(kx)\sinh(k(t-1)), & x<t<1. \end{cases} \qquad (*)$$

(ii) NOW FIND INTEGRAL REPRESENTATION FOR SOLUTION TO

$$Lu = f(x), \quad 0<x<1$$
$$u(0) = 1, \quad u(1) = 0.$$

WE HAVE BY LAGRANGE $\quad (v, Lu) = (vu' - uv')\Big|_0^1 + (u, Lv).$

SINCE $v=0$ AT $t=0,1$ AND $Lv = \delta(t-x)$ WE GET $\quad (v, f) = +u(0)v'(0) + u(x).$

$$u(x) = -v'(0) + \int_0^1 v f\, dt. \qquad \left(v' \equiv v_t(t;x)\Big|_{t=0}.\right)$$

NOW $\quad v'(0) = \dfrac{1}{k\sinh k}\, k\cosh(kt)\Big|_{t=0} \sinh(k(x-1)) = \dfrac{\sinh[k(x-1)]}{\sinh k}.$

WE CONCLUDE THAT

$$u(x) = -\frac{\sinh[\kappa(x-1)]}{\sinh \kappa} + \int_0^1 v(t;x)\, f(t)\, dt \qquad (++)$$

WHERE $v(x)$ IS DEFINED BY (*).

(iii) NOW SUPPOSE THAT

$$f(x) = \frac{1}{\varepsilon} \operatorname{sech}^2\left(\frac{x-1/2}{\varepsilon}\right)$$

FIND A GOOD APPROXIMATION TO THE SOLUTION OF

$$Lu = f(x) \text{ on } 0<x<1; \quad u(0)=1, \; u(1)=0$$

AS $\varepsilon \to 0^+$. WE OBSERVE THAT SINCE $\operatorname{sech}^2 z = \dfrac{4}{(e^z+e^{-z})^2}$ THAT

$$\lim_{\substack{\varepsilon\to 0^+ \\ x \neq 1/2}} f(x) = 0, \quad f(0) = \frac{1}{\varepsilon} > 0 \text{ AND } \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \operatorname{sech}^2\left(\frac{x-1/2}{\varepsilon}\right) dx$$

$$= \int_{-\infty}^{\infty} \operatorname{sech}^2 z\, dz < \infty.$$

THUS $\quad f(x) \xrightarrow{\varepsilon \to 0} \left(\int_{-\infty}^{\infty} \operatorname{sech}^2 z\, dz\right) \delta(x - 1/2)$.

NOW SINCE $d/dz \tanh z = \operatorname{sech}^2 z$ WE HAVE $\displaystyle\int_{-\infty}^{\infty} \operatorname{sech}^2 z\, dz = \tanh z \Big|_{-\infty}^{\infty} = 2.$

HENCE $\quad f(x) \to 2\delta(x - 1/2)$ AS $\varepsilon \to 0$.

WE THEN USE (++) FROM PART (ii) TO OBTAIN AS $\varepsilon \to 0$ THAT

$$u(x) = -\frac{\sinh[\kappa(x-1)]}{\sinh\kappa} + \int_0^1 v(t;x)\, 2\delta(t-1/2)\, dt.$$

$$\to \quad u(x) = -\frac{\sinh[\kappa(x-1)]}{\sinh\kappa} + 2v(1/2; x).$$

where FROM (*)

$$v(1/2;x) = \frac{1}{\kappa \sinh\kappa}\begin{cases} \sinh(\kappa/2)\sinh(\kappa(x-1)) & \text{IF } 1>x>1/2 \\ \sinh(\kappa x)\sinh\left(\kappa\left(\frac{1}{2}-1\right)\right) & \text{IF } 0<x<1/2. \end{cases}$$

PROBLEM 3 CONSIDER $Lu \equiv u'' - k^2 u$ WITH $k > 0$.

(i) FIND THE G-FUNCTION FOR Lu ON $-\infty < x < \infty$ WITH $u \to 0$ AS $|x| \to \infty$.

SINCE L IS A SL OPERATOR, THE G-FUNCTION $V(t;x)$ SATISFIES

$$V'' - k^2 V = \delta(t-x), \quad -\infty < t < \infty$$

$$V \to 0 \text{ AS } t \to \pm\infty.$$

FOR $V'' - k^2 V = 0$ THEN $V = \text{SPAN}\{e^{-kt}, e^{kt}\}$. TO HAVE DECAY AS $t \to \pm\infty$ WE HAVE

$$V = \begin{cases} c_1 e^{tk}, & -\infty < t < x \\ c_2 e^{-tk}, & x < t < \infty \end{cases}.$$

NOW TO HAVE CONTINUITY AT $t = x$ WE PUT

$$V = \begin{cases} c e^{kt} e^{-xk}, & -\infty < t < x \\ c e^{-tk} e^{xk}, & x < t < \infty \end{cases}$$

WHERE c IS FOUND FROM JUMP CONDITION $V'(x^+) - V'(x^-) = 1$.

THUS $-kc e^{-tk} e^{xk}\Big|_{t=x} - \left(ck e^{kt} e^{-xk} \right)\Big|_{t=x} = 1 \to c = -\dfrac{1}{2k}$.

THUS $$V = -\frac{1}{2k} \begin{cases} e^{k(t-x)}, & -\infty < t < x \\ e^{-k(t-x)}, & x < t < \infty \end{cases}$$

WE CONCLUDE THAT $$V(t;x) = -\frac{1}{2k} e^{-k|t-x|}.$$

(ii) NOW FIND G-FUNCTION FOR Lu WITH $0 < x < \infty$, $u(0) = 0$ AND $u \to 0$ AS $x \to \infty$.

THE G-FUNCTION SATISFIES

$$V'' - k^2 V = \delta(t-x), \quad 0 < t < \infty$$

$$V = 0 \text{ AT } t = 0, \quad V \to 0 \text{ AS } t \to \infty$$

SO $$V = \begin{cases} c_1 [e^{kt} - e^{-kt}], & 0 < t < x \\ c_2 e^{-kt}, & t > x. \end{cases}$$

IMPOSING CONTINUITY $$V = \begin{cases} c(e^{kt} - e^{-kt}) e^{-kx}, & 0 < t < x \\ c e^{-kt} (e^{kx} - e^{-kx}), & t > x. \end{cases}$$

NOW $V'(x^+) - V'(x^-) = 1$ DETERMINES c. SO $-ck e^{-kx}(e^{kx} - e^{-kx}) - c(ke^{kx} + ke^{-kx})e^{-kx} = 1$

THIS YIELDS $-2ck = 1$ SO $c = -1/2k$. HENCE $$V = -\frac{1}{2k} \begin{cases} e^{k(t-x)} - e^{-k(t+x)}, & 0 < t < x \\ e^{-k(t-x)} - e^{-k(t+x)}, & t > x \end{cases}$$

WE CONCLUDE THAT $$V(t;x) = -\frac{1}{2k} \left(e^{-k|t-x|} - e^{-k(t+x)} \right) \text{ ON } t > 0.$$

PROBLEM 4

(i) FIND THE G-FUNCTION FOR
$$Lu \equiv (xu')' - \frac{1}{x} u = f(x) \quad \text{ON } 0<x<1 \quad \text{WITH } u(0) \text{ FINITE AND } u(1)=0.$$

SOLUTION SINCE L IS A SL OPERATOR WITH SEPARATED BC WE HAVE THAT THE G-FUNCTION SATISFIES
$$\left.\begin{aligned} &(t v')' - \frac{1}{t} v = \delta(t-x) \quad \text{ON } 0<t<1 \\ &\text{WITH } \quad v=0 \text{ AT } t=0,1. \end{aligned}\right\} \quad (*)$$

FOR $t \neq x$ WE HAVE $t v'' + v' - \frac{1}{t} v = 0 \rightarrow t^2 v'' + t v' - v = 0$ EULER'S EQUATION.

WE PUT $v = t^r$ SO THAT $r(r-1) + r - 1 = 0 \rightarrow r^2 = 1$ SO $r = \pm 1$.

THUS $v = \text{SPAN}\{t, 1/t\}$. TO SATISFY THE BC WE WRITE
$$v = \begin{cases} c_1 t, & 0<t<x \\ c_2(t - 1/t), & x<t<1 \end{cases}$$

NEXT IMPOSING CONTINUITY AT $t = x$,
$$v = \begin{cases} c\, t\,(x - 1/x), & 0<t<x \\ c\,(t - 1/t)\, x, & x<t<1 \end{cases}$$

WHERE c IS FOUND FROM JUMP CONDITION $v'(x^+) - v'(x^-) = \frac{1}{x}$.

NOW THIS GIVES $c\left[\left(1 + \frac{1}{x^2}\right) x - \left(x - \frac{1}{x}\right)\right] = \frac{1}{x} \longrightarrow c = \frac{1}{2}$.

THUS
$$v = \frac{1}{2}\begin{cases} t\,(x - 1/x), & 0<t<x \\ x\,(t - 1/t), & x<t<1. \end{cases} \quad (+)$$

(ii) NOW LET $\tilde{u}(r,\varphi)$ SATISFY, FOR $0<r_0<1$,
$$\tilde{u}_{rr} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\varphi\varphi} = \delta(r-r_0)\cos\varphi \quad \text{IN } 0 \le r \le 1,\ 0 \le \varphi \le 2\pi$$
$\tilde{u} = 0$ ON $r=1$; $\tilde{u}$ BOUNDED AS $r \to 0$; $\tilde{u}, \tilde{u}_\varphi$ 2π PERIODIC.

SOLVE FOR $\tilde{u}(r,\varphi)$ IN TERMS OF SOLUTION IN (i).

WRITE $\tilde{u}(r,\varphi) = u(r)\cos\varphi$. WE WILL OBTAIN $u(r)$. UPON SUBSTITUTING WE OBTAIN THAT
$$u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u = \delta(r-r_0).$$

MULTIPLYING BY r:
$$Lu \equiv (r u_r)_r - \frac{1}{r}u = r\,\delta(r-r_0) = r_0\,\delta(r-r_0)$$
$u=0$ AT $r=1$, u BOUNDED AS $r \to 0$.

ALL WE NEED IS TO REPLACE $t \mapsto r$ AND $x \mapsto r_0$ IN (+) AND MULTIPLY BY r_0
$$\Longrightarrow u = \frac{r_0}{2}\begin{cases} r\,(r_0 - 1/r_0), & 0<r<r_0 \\ r_0\,(r - 1/r), & r_0<r<1 \end{cases}$$

PROBLEM 5 DEFINE $Lu \equiv u'' + k^2 u$ ON $0 \le x \le a$ WITH $a > 0$ AND $k > 0$.

(i) FIND GREEN'S FUNCTION FOR Lu WITH $u'(0) = u'(a) = 0$.

SOLUTION SINCE L IS A SL OPERATOR AND WE HAVE WELL-SEPARATED BC

WE HAVE THAT THE G-FUNCTION V SATISFIES

$$\left.\begin{aligned} LV \equiv V'' + k^2 V = \delta(t-x) \quad &\text{ON} \quad 0 < t < a \\ V' = 0 \quad \text{AT} \quad t = 0, a & \end{aligned}\right\} \quad (*)$$

FOR $t \neq x$ $V = \text{SPAN}\{\cos(k(t-x)), \sin(k(t-x))\}$ AND SO SATISFYING THE BC

$$V = \begin{cases} C_1 \cos(kt), & 0 < t < x \\ C_2 \cos(k(t-a)), & x < t < a. \end{cases}$$

NOW IMPOSING CONTINUITY AT $t = x$ WE HAVE

$$V = \begin{cases} C \cos(kt) \cos(k(x-a)), & 0 < t < x \\ C \cos(k(t-a)) \cos(kx), & x < t < a. \end{cases}$$

WHERE C IS FOUND BY JUMP CONDITION $V'(x^+) - V'(x^-) = 1$.

WE OBTAIN $-Ck \sin(k(x-a)) \cos(kx) + Ck \sin(kx) \cos(k(x-a)) = 1$.

SINCE $\sin(A-B) = \sin A \cos B - \sin B \cos A$ WE GET

$$Ck \sin[kx - k(x-a)] = 1 \rightarrow Ck \sin(ka) = 1.$$

WE CONCLUDE THAT $C = \dfrac{1}{k \sin(ka)}$

HENCE

$$V(t;x) = \frac{1}{k \sin(ka)} \begin{cases} \cos(kt) \cos(k(x-a)), & 0 < t < x \\ \cos(k(t-a)) \cos(kx), & x < t < a. \end{cases}$$

(ii) WHEN $k = n\pi/a$ WITH $n = 1, 2, \ldots$ WE HAVE THAT V IS UNDEFINED. AT THESE VALUES, $\exists$ A NONTRIVIAL HOMOGENEOUS SOLUTION TO $(*)$ GIVEN BY

$$\phi = \cos[n\pi t/a]$$

BY LAGRANGE IF V EXISTS WE HAVE $(\phi, LV) = (V, L\phi) \Rightarrow \phi(x) \equiv 0$

WHICH IS A CONTRADICTION. NO G-FUNCTION WHEN $k = n\pi/a$.

(iii) LET $Lu \equiv u'' + k^2 u = f(x)$ ON $0 < x < a$ WITH $u'(0) = 0$ AND $u'(a) = 1$, WITH $k = \frac{n\pi}{a}$.

THE HOMOGENEOUS PROBLEM HAS A NONTRIVIAL SOLUTION $\phi = \cos(n\pi x/a)$.

USE LAGRANGE: $(\phi, Lu) = (\phi u' - \phi' u)\Big|_0^a + (u, L\phi) \Rightarrow (\phi, f) = \phi(a)$, SINCE $L\phi = 0$ WITH $\phi'(0) = \phi'(a) = 0$ AND $u'(0) = 0$. FOR A SOLUTION TO EXIST WE THUS REQUIRE $(\phi, f) = \phi(a) \longrightarrow \displaystyle\int_0^a \cos\left(\frac{n\pi x}{a}\right) f(x)\, dx = \cos(n\pi) = (-1)^n$.