MATH 401 HOMEWORK 2

FOR PROBLEMS 1, 2, AND 3 BELOW, FIND WHETHER A SOLVABILITY CONDITION IS NEEDED FOR $f(x)$, AND IF SO FIND THIS SOLVABILITY CONDITION. ASSUMING THAT THIS CONDITION IS SATISFIED, CALCULATE THE MODIFIED GREEN'S FUNCTION AND FIND AN INTEGRAL REPRESENTATION FOR THE SOLUTION.

PROBLEM 1 $\quad Lu \equiv u'' = f(x)$, ON $0<x<1$

WITH $u(0)=0$, $u'(1)=u(1)$.

PROBLEM 2 $\quad Lu \equiv u'' + \pi^2 u = f(x)$, ON $0<x<1$

WITH $u(0)=0$, $u(1)=0$.

PROBLEM 3 $\quad Lu \equiv \left[(1-x^2)u'\right]' = f(x)$ ON $-1<x<1$

WITH $u(-1)$ AND $u(1)$ BOUNDED.

PROBLEM 4 CONSIDER THE NON-SELF-ADJOINT PROBLEM

$$\left.\begin{array}{l} Lu \equiv u'' + 2u' + u = f(x) \quad \text{ON } 0<x<1 \\ \text{WITH} \quad u(0)=0 \text{ AND } u(1)=1. \end{array}\right\} \quad (*)$$

FIND AN INTEGRAL REPRESENTATION FOR THE SOLUTION TO $(*)$ IN TWO WAYS.

(i) BY FINDING THE GREEN'S FUNCTION FOR L^*.

(ii) BY FIRST MULTIPLYING $(*)$ BY SOME FUNCTION TO MAKE THE RESULTING PROBLEM SELF-ADJOINT. THEN FIND THE GREEN'S FUNCTION FOR THE NEW PROBLEM.

PROBLEM 5 LET $k>0$ BE ARBITRARY AND CONSIDER THE NON-SELF-ADJOINT PROBLEM

$$Lu \equiv u'' + k^2 u = f(x), \quad \text{ON } 0<x<1$$

WITH $u(0)=u(1)$ AND $u'(0)=-u'(1)$.

FIND A CONDITION ON $f(x)$ FOR THIS PROBLEM TO HAVE A SOLUTION.

(HINT: YOU WILL NEED TO FIND A NONTRIVIAL SOLUTION TO THE ADJOINT PROBLEM).

HOMEWORK # 2 SOLUTIONS M. WARD

PROBLEM 1 $\quad Lu = u'' = f(x) \quad 0<x<1$

$$u(0)=0 \qquad u'(1)=u(1)$$

SOLUTION $\quad (v, Lu) = (u'v - uv')\Big|_0^1 + (u, Lv)$

$$(v, Lu) = -u'(0)v(0) + \Big((u'(1)-u(1))v(1) - u(1)\big(v'(1)-v(1)\big)\Big) + (u, Lv)$$

NOW $u'(1)-u(1)=0$, CHOOSE

$$(*) \quad \begin{array}{ll} Lv = \delta(t-x) & 0<t<1 \\ v(0)=0 & v'(1)-v(1)=0 \end{array}$$

THEN $\quad u(x) = \int_0^1 v(t;x)\, f(t)\, dt.$

THE HOMOGENEOUS ADJOINT PROBLEM IS $Lv_H = 0 \quad v_H(0)=0, \; v_H'(1) = v_H(1)$.

THUS $v_H(t) = c_0 t + c_1 \quad v_H(0)=0 \rightarrow c_1 = 0$ AND $v_H'(1) = v_H(1) \rightarrow c_0$ ARBITRARY. THUS

$$v_H(t) = t \quad \text{SOLVES HOMOGENEOUS ADJOINT EQUATION}$$

ALSO $\quad (v_H, Lv) = (v, Lv_H) = 0$

$v_H(x) = 0$ WHICH IS A CONTRADICTION. HENCE NO SOLUTION TO (*) EXISTS.

TO FIND SOLVABILITY CONDITION USE

$$(v_H, Lu) = (v_H, f) = (u, Lv_H) = 0$$

THUS WE NEED $\quad (v_H, f) = \int_0^1 t\, f(t)\, dt = 0$ FOR A SOLUTION.

ASSUME THE SOLVABILITY CONDITION IS SATISFIED. THE MODIFIED GREENS FUNCTION SATISFIES

$$L\hat{v} = \delta(t-x) + c_1 v_H(t)$$

$$\hat{v} = 0 \text{ AT } t=0; \quad \hat{v}'(1) - \hat{v}(1) = 0 \qquad (v_H, v_H) = 1/3$$

NOW $\quad (v_H, L\hat{v}) = v_H(x) + c_1 (v_H, v_H) = 0 \rightarrow c_1 = -v_H(x)/(v_H, v_H) = -3x$

NOW $\quad \hat{v}'' = c_1 t + \delta(t-x) \quad 0<t<1 \qquad \hat{v}(x^+) = \hat{v}(x^-)$

$$\hat{v} = c_1 t^3/6 + \begin{cases} a_0 t + a_1 & 0<t<x \\ a_2 t + a_3 & x<t<1 \end{cases} \qquad \hat{v}'(x^+) - \hat{v}'(x^-) = 1$$

NOW $\hat{v}(0)=0 \rightarrow a_1 = 0 \quad \hat{v}'(1) = \hat{v}(1) \rightarrow a_3 = c_1/3$.

$$\hat{v}(x^+) = \hat{v}(x^-) \longrightarrow a_2 x + a_3 = a_0 x + a_1 \rightarrow (a_2 - a_0)x = -c_1/3 = x$$

$$\hat{v}'(x^+) - \hat{v}'(x^-) = 1 \longrightarrow a_2 - a_0 = 1 \qquad a_2 - a_0 = 1$$

THUS $\quad a_1 = 0 \quad a_3 = -x \quad a_2 - a_0 = 1$

$$\hat{v} = -\frac{1}{2}xt^3 + \begin{cases} a_0 t & 0<t<x \\ t + a_0 t - x & x<t<1 \end{cases}$$

NOW $(\hat{v}, Lu) = (u, L\hat{v}) = u(x) + c_1(u, v_H)$ $\qquad c_1 = -3x$

SINCE WE CAN ADD ANY MULTIPLE OF $v_H(x)$ TO $u(x)$ WE HAVE

$$u(x) = Ax + \int_0^1 \hat{v}(t;x)\, f(t)\,dt$$

SINCE $\int_0^1 t f(t)\,dt = 0$ AND $\int_0^1 t^3 f(t)\,dt =$ CONSTANT, WE GET

$$u(x) = Bx + \int_x^1 (t-x) f(t)\,dt.$$

PROBLEM 2 $\qquad Lu = u'' + \pi^2 u = f(x) \quad 0<x<1$

$$u(0) = u(1) = 0$$

NOW $(v, Lu) = (u, Lv)$ WHEN $v(0) = v(1) = 0$.

THE HOMOGENEOUS ADJOINT PROBLEM IS

$$Lv = 0 \longrightarrow v_H(t) = \sin(\pi t)$$
$$v(0) = v(1) = 0$$

THE SOLVABILITY CONDITION IS $(v_H, Lu) = 0$ OR $\int_0^1 \sin(t\pi)\, f(t)\,dt = 0$.

ASSUME THAT THIS CONDITION IS SATISFIED. THE MODIFIED GREEN'S FUNCTION SATISFIES

$$L\hat{v} = \delta(t-x) + c_1 v_H(t) \quad 0<t<1 \qquad c_1 = -v_H(x)/(v_H, v_H)$$
$$\hat{v}(0) = \hat{v}(1) = 0 \qquad v_H(t) = \sin(\pi t)$$

NOW $(v_H, v_H) = \int_0^1 \sin^2(\pi t)\,dt = \frac{1}{2}. \Rightarrow c_1 = -2\sin(\pi x)$.

SOLVING WE GET

$$\hat{v} = -\frac{c_1 t}{2\pi}\cos(\pi t) + \begin{cases} A\sin(\pi t) & 0<t<x \\ B\sin(\pi t) + C\cos(\pi t) & x<t<1 \end{cases}$$

NOW $\hat{v}(0) = 0$ IS SATISFIED

$$\hat{v}(1) = 0 \longrightarrow \frac{c_1}{2\pi} + C\cos\pi = 0 \qquad C = \frac{c_1}{2\pi}$$

NOW CONTINUITY $\hat{v}(x^+) = \hat{v}(x^-) \longrightarrow B\sin(\pi x) + C\cos(\pi x) = A\sin(\pi x)$

$$\hat{v}'(x^+) - \hat{v}'(x^-) = 1 \qquad \pi B\cos(\pi x) - C\pi\sin(\pi x) = A\pi\cos(\pi x) + 1$$

THUS
$$\begin{array}{c} \overset{M}{\downarrow} \\ \begin{bmatrix} \pi\cos(\pi X) & -\pi\cos(\pi X) \\ \sin(\pi X) & -\sin(\pi X) \end{bmatrix} \end{array} \begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} C\pi\sin(\pi X)+1 \\ -C\cos\pi X \end{bmatrix}$$

NOTICE THAT MATRIX M IS SINGULAR $\longrightarrow$ A AND B ARE NOT UNIQUELY DETERMINED. (PERHAPS A AND B DO NOT EXIST)

REDUCING WE GET
$$\begin{bmatrix} \cos\pi X & -\cos\pi X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} C\sin(\pi X) + 1/\pi \\ -\dfrac{C}{\cos\pi X} - \dfrac{1}{\pi}\tan\pi X \end{bmatrix}$$

BUT $C = \dfrac{C_1}{2\pi} = -\dfrac{1}{\pi}\sin\pi X$. THUS
$$\begin{bmatrix} \cos\pi X & -\cos\pi X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{\pi}\sin^2(\pi X) + \dfrac{1}{\pi} \\ 0 \end{bmatrix}$$

$$B\cos\pi X - A\cos\pi X = \frac{1}{\pi}\cos^2\pi X \qquad B - A = \frac{1}{\pi}\cos\pi X$$
$$A = B - \frac{1}{\pi}\cos\pi X$$

THUS
$$\hat{v} = -\frac{C_1 t}{2\pi}\cos(\pi t) + \begin{cases} B\sin(\pi t) - \dfrac{1}{\pi}\sin(\pi t)\cos(\pi X) & 0<t<X \\ B\sin(\pi t) - \dfrac{1}{\pi}\sin(\pi X)\cos(\pi t) & X<t<1 \end{cases}$$

NOW $(\hat{v}, Lu) = (u, L^*\hat{v}) = u(X) + C_1(V_H(t), u)$ $\qquad C_1 = -V_H(X)/(V_H, V_H)$.

SINCE WE CAN ADD ANY MULTIPLE OF $V_H(X)$ TO u WE HAVE
$$u(X) = A\,V_H(X) + \int_0^1 \hat{v}(t;X)\,f(t)\,dt$$

SINCE $\int_0^1 t\cos(\pi t)f(t)\,dt$ = CONSTANT AND $\int_0^1 f(t)\sin(\pi t)\,dt = 0$ WAS ASSUMED, WE HAVE
$$u(X) = B\sin(\pi X) - \frac{1}{\pi}\int_0^X \sin(\pi t)\cos(\pi X)f(t)\,dt - \frac{1}{\pi}\int_X^1 \sin(\pi X)\cos(\pi t)f(t)\,dt$$

WHERE B IS ARBITRARY.

PROBLEM 3 $\quad Lu \equiv [(1-x^2)u']' = f(x) \quad -1 < x < 1$

$u(-1)$ AND $u(1)$ FINITE

NOW $\quad (v, Lu) = (u, Lv) + (1-t^2)\left[u'v - uv'\right]\Big|_{-1}^{1}$

WHAT WE NEED ARE $v(-1), v(1), v'(\pm 1)$ FINITE. THE HOMOGENEOUS ADJOINT PROBLEM IS

$$Lv \equiv [v'(1-t^2)]' = 0 \qquad \rightarrow \qquad v' = c/(1-t^2)$$

$$v(-1),\ v(1) \text{ FINITE} \qquad v = \frac{1}{2} c \log\left(\frac{1-t}{1+t}\right) + B \qquad c = 0 \text{ FOR FINITENESS CONDITION.}$$

THUS $\quad v_H(t) = 1$ SOLVES HOMOGENEOUS ADJOINT PROBLEM.

THE SOLVABILITY CONDITION IS

$$(v_H, Lu) = (u, Lv_H) = 0 \quad \longrightarrow \quad \int_{-1}^{1} v_H f\,dt = 0 \quad \text{OR} \quad \int_{-1}^{1} f(t)\,dt = 0$$

ASSUME THAT THIS CONDITION IS SATISFIED. THE MODIFIED GREEN'S FUNCTION SATISFIES

$$L\hat{v} = \delta(t-x) + c_1 v_H(t) \qquad c_1 = -v_H(x)/(v_H, v_H) \qquad c_1 = -1/2$$

$$\hat{v}(\pm 1) \text{ FINITE}$$

NOW $\quad [(1-t^2)\hat{v}']' = -\frac{1}{2} + \delta(t-x) \qquad (1-t^2)\hat{v}' = -\frac{1}{2}t + A$

$$\hat{v}' = -\frac{1}{2}\frac{t}{1-t^2} + \frac{A}{1-t^2}$$

$$\hat{v} = \frac{1}{4}\log(1-t^2) + \frac{A}{2}\left[\log(1-t) - \log(1+t)\right] +$$

CAN CHOOSE $A = 1/2$ OR $A = -1/2$

THUS WE HAVE (IF $A = 1/2$) $\rightarrow \hat{v} = \frac{1}{2}\log(1-t)$ NO SINGULARITY AT $t = -1$

(IF $A = -1/2$) $\rightarrow \hat{v} = \frac{1}{2}\log(1+t)$ NO SINGULARITY AT $t = 1$

THEN

$$\hat{v} = \begin{cases} \frac{1}{2}\log(1-t) + B & 0 < t < x \\ \frac{1}{2}\log(1+t) + D & x < t < 1 \end{cases}$$

NOW THE CONTINUITY AND JUMP CONDITIONS ARE

$$\hat{v}(x^+) - \hat{v}(x^-) = 0 \quad \rightarrow \quad \frac{1}{2}\log(1+x) + D = \frac{1}{2}\log(1-x) + B$$

$$\hat{v}'(x^+) - \hat{v}'(x^-) = \frac{1}{1-x^2} \qquad \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \frac{1}{1-x^2} \quad \text{SATISFIED}$$

THUS WE HAVE

$$\hat{v} = \begin{cases} \frac{1}{2}\log(1-t) + \frac{1}{2}\log(1+x) + C & 0 < t < x \\ \frac{1}{2}\log(1+t) + \frac{1}{2}\log(1-x) + C & x < t < 1 \end{cases}$$

C ARBITRARY.

THEN $(\hat{V}, LU) = (U, \hat{L}\hat{V}) = U(X) + C_1 (U, V_H)$ $\quad C_1 = -1/2$

SINCE WE CAN ADD ANY MULTIPLE OF $V_H(t) = 1$, THEN

$$U(X) = B + \int_0^1 \hat{V}(t; X) f(t)\, dt$$

ASSUMING $\int_0^1 f(t)\, dt = 0$, THEN

$$U(X) = B + \int_0^X \frac{1}{2} \log[(1-t)(1+X)] f(t)\, dt + \int_X^1 \frac{1}{2} \log[(1+t)(1-X)] f(t)\, dt$$

PROBLEM 5 FIND A SOLVABILITY CONDITION FOR

$$LU = U'' + K^2 U = F(X) \qquad 0 < X < 1$$

$$U(0) = U(1) \qquad U'(0) = -U'(1)$$

TO HAVE A SOLUTION.

NOW $(V, LU) = (U'V - UV')\Big|_0^1 + (U, LV)$

NOW $(V, LU) = (U'(1)V(1) - U(1)V'(1)) - (U'(0)V(0) - U(0)V'(0)) + (U, LV)$

$$(V, LU) = (U'(1)(V(1) + V(0)) - U(1)(V'(1) - V'(0))) + (U, LV)$$

CHOOSE $V(1) = -V(0) \qquad V'(1) = V'(0)$

THE HOMOGENEOUS ADJOINT PROBLEM IS

$$LV = 0 \qquad 0 < t < 1$$

$$V(1) = -V(0) \qquad V'(1) = V'(0)$$

NOW THE BOUNDARY CONDITIONS SUGGEST WE LOOK FOR A FUNCTION THAT IS ODD ABOUT $t = 1/2$.

THUS $V = A \sin(K(t - 1/2)) + B \cos(K(t - 1/2))$

$V(1) = -V(0) \rightarrow A \sin(K/2) + B\cos(K/2) = A\sin(K/2) - B\cos(K/2)$

$V'(1) = V'(0) \rightarrow AK\cos(K/2) - BK\sin(K/2) = AK\cos(K/2) + BK\sin(K/2)$

THUS $B\cos(K/2) = 0$ AND $BK\sin(K/2) = 0 \rightarrow B = 0$ FOR ALL K. NOTE:

$V_H(t) = \sin[K(t - 1/2)]$ SOLVES HOMOGENEOUS PROBLEM.

THEN $(V_H, LU) = 0 \rightarrow \int_0^1 \sin[K(t - 1/2)] f(t)\, dt = 0$ IS the SOLVABILITY condition FOR ALL K.

PROBLEM 4 CONSIDER $Lu \equiv u'' + 2u' + u = f(x)$ ON $0 < x < 1$

WITH $u(0) = 0$, $u(1) = 1$.

(i) FIND AN INTEGRAL REPRESENTATION BY FINDING G FOR L^*.

BY INTEGRATION BY PARTS

$$(v, Lu) = \left[vu' + u(-v' + 2v)\right]\Big|_0^1 + (u, L^* v) \quad \text{WHERE} \quad L^* v \equiv v'' - 2v' + v.$$

NOW $u(0) = 0$ AND $u(1) = 1$ YIELDS

$$(v, Lu) = vu'\Big|_0^1 + (-v' + 2v)\Big|_{t=1} + (u, L^* v).$$

TO OBTAIN AN EXPLICITLY DETERMINED BOUNDARY TERM TAKE $v = 0$ AT $t = 0, 1$, AND $L^* v = \delta(t - x)$. THEN

$$u(x) = \int_0^1 v(t; x) f(t)\, dt + v_t(1; x) \qquad (+).$$

HERE

$$L^* v \equiv v'' + 2v' + v = \delta(t - x)$$

$$v = 0 \quad \text{AT} \quad t = 0, 1.$$

THE HOMOGENEOUS SOLUTION HAS $v = \text{SPAN}\{e^t, te^t\}$.

TO SATISFY THE BC,

$$v = \begin{cases} c_1 te^t, & 0 < t < x \\ c_2 (t-1) e^t, & x < t < 1. \end{cases}$$

NOW BY CONTINUITY ACROSS $t = x$ WE HAVE

$$v = \begin{cases} c\, te^t (x-1) e^x, & 0 < t < x \\ c\, (t-1) e^t\, x e^x, & x < t < 1, \end{cases}$$

WHERE c IS FOUND FROM

$$v'(x^+) - v'(x^-) = 1.$$

NOW

$$v'(x^+) = c\, xe^x \left[e^t + (t-1)e^t\right]\Big|_{t=x}$$

$$v'(x^-) = c\, (x-1)e^x \left[te^t + e^t\right]\Big|_{t=x}$$

SUBTRACTING $v'(x^+) - v'(x^-) = c\left[xe^{2x} - (x-1)e^{2x}\right] = 1 \Rightarrow c = 1/e^{2x}$.

THUS

$$v(t; x) = \begin{cases} t(x-1)\, e^{t-x}, & 0 < t < x \\ (t-1)\, x\, e^{t-x}, & x < t < 1. \end{cases} \qquad (++)$$

MOREOVER $v_t(1; x) = xe^{-x}\left[(t-1)e^{t}\right]'\Big|_{t=1} = xe^{1-x}$.

WE CONCLUDE FROM (+) THAT

$$u(x) = xe^{1-x} + \int_0^1 v(t;x)\, f(t)\, dt$$

WHERE v IS DEFINED IN (++).

(ii) NOW MULTIPLY BY $p(x)$ TO MAKE THE PROBLEM SELF-ADJOINT.

WE HAVE $pu'' + 2pu' + pu = p\,f(x)$.

WE WANT $(pu')' + pu = f(x)\,p$

AND SO NEED $p' = 2p$ OR $p = e^{2x}$.

IN THIS WAY WE HAVE THE SELF-ADJOINT PROBLEM

$$\mathcal{L}u \equiv [e^{2x}u']' + e^{2x}u = \tilde{f}(x), \quad 0 < x < 1 \quad \text{WHERE } \tilde{f}(x) = e^{2x}f(x)$$

AND $u(0) = 0, \quad u(1) = 1.$

USING LAGRANGE'S IDENTITY

$$(v, \mathcal{L}u) = e^{2t}[u'v - uv']\Big|_0^1 + (u, \mathcal{L}v)$$

WE PUT $u(0) = 0$, $u(1) = 1$ AND CHOOSE $v = 0$ AT $t = 0, 1$.

THEN WITH

$$\mathcal{L}v = \delta(t-x) \quad \text{ON } 0 < t < 1$$

$$v = 0 \quad \text{AT } t = 0, 1$$

WE HAVE

$$(v, \mathcal{L}u) = -e^2\, v'(1) + (u, \delta(t-x)),$$

SO THAT

$$u(x) = v'\big|_{t=1} e^2 + (v, \tilde{f}).$$

IN THIS WAY,

$$u(x) = v'\big|_{t=1} e^2 + \int_0^1 v(t;x)\, e^{2t} f(t)\, dt. \quad (+++)$$

NOW THE GREEN'S FUNCTION v SATISFIES $[e^{2t}v']' + e^{2t}v = 0$ FOR $t \neq x$.

THIS YIELDS THAT $v'' + 2v' + v = 0$ $\quad v = \text{SPAN}\{e^{-t}, te^{-t}\}$.

IMPOSING $v = 0$ AT $t = 0, 1$ YIELDS

$$v = \begin{cases} c_1\, te^{-t}, & 0 < t < x \\ c_2\, (t-1)e^{-t}, & x < t < 1. \end{cases}$$

NOW IMPOSING THAT ν CONTINUOUS ACROSS $t = x$ YIELDS

$$\nu = \begin{cases} c\, t e^{-t} (x-1) e^{-x}, & 0 < t < x \\ c\, x e^{-x} (t-1) e^{-t}, & x < t < 1 \end{cases}$$

WHERE C IS TO BE FOUND FROM THE JUMP CONDITION $\nu'(x^+) - \nu'(x^-) = \frac{1}{p(x)} = e^{-2x}$.

SOME ALGEBRA YIELDS THAT $C = 1$.

THUS
$$\nu = \begin{cases} t e^{-t} (x-1) e^{-x}, & 0 < t < x \\ x e^{-x} (t-1) e^{-t}, & x < t < 1 \end{cases}$$

AND $\nu' \big|_{t=1} = x e^{-x-1}$. THIS YIELDS FROM (+++) THAT

$$u(x) = x e^{-x+1} + \int_0^1 \nu(t; x)\, e^{2t} f(t)\, dt.$$

THIS IS THE SAME AS FOUND IN PART (i) SINCE WE OBSERVE THAT $v(t; x) = e^{2t} \nu(t; x)$.