Math 401: Homework 4

Instructions: Only turn in solutions to the questions marked in blue. However, please look at the other question

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1. (5 points) In 3-D find the free-space Green's function for

$$\Delta G - \kappa^2 G = \frac{\delta(r)}{4\pi r^2}\,, \quad \text{with } r^2 = x^2+y^2+z^2\,; \qquad G \to 0 \quad \text{as } r\to\infty\,,$$

where $\kappa > 0$ is a constant. (Hint: recall in 3-D that if G is radially symmetric, then $\Delta G = G_{rr} + 2r^{-1}G_r$.)

2. (5 points) By using the method of images to first find the relevant Green's function, find an integral representation for the solution to each of the following:

   (a) (5 points) In 2-D:

$$\begin{aligned} \Delta u - \kappa^2 u = F(x,y)\,, &\quad -\infty < x < \infty\,,\ \ y > 0\,, \\ u(x,0) = h(x)\,, &\qquad u \to 0 \quad \text{as } x^2+y^2 \to \infty\,, \end{aligned}$$

   where $\kappa > 0$ is a constant.

   (b) (0 points) In 3-D:

$$\begin{aligned} \Delta u = F(x,y,z)\,, &\quad -\infty < x < \infty\,,\ \ -\infty < y < \infty\,,\ \ z > 0\,, \\ u(x,y,0) = h(x,y)\,, &\qquad u \to 0 \quad \text{as } x^2+y^2+z^2 \to \infty\,. \end{aligned}$$

3. (5 points) By using the method of images to first find the relevant Green's function, find an integral representation for the solution to

$$\begin{aligned} \Delta u \equiv u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = F(r,\theta)\,, &\quad 0 \le r \le 1\,,\ \ 0 \le \theta \le \pi\,, \\ u(1,\theta) = 0\,, \quad u \text{ bounded as } r \to 0\,; &\qquad u(r,0) = 0\,,\ \ u(r,\pi) = 0\,. \end{aligned}$$

Here (r,θ) are the usual polar coordinates and the domain is the half unit-disk.

4. (5 points) Consider Laplace's equation $\Delta u = 0$ in a sphere of radius a with boundary condition $u(a,\theta,\phi) = F(\theta,\phi)$ on the boundary of the sphere. Here, the spherical coordinates are defined in the usual way by

$$x = r\cos\theta\sin\phi\,, \qquad y = r\sin\theta\sin\phi\,, \qquad z = r\cos\phi\,.$$

We assume that u is 2π periodic in θ and is bounded at the two poles $\phi = 0, \pi$ and at the origin as $r \to 0$. By first finding the relevant Green's function by using the method of images, show that

$$u(r,\theta,\phi) = \frac{a}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{(a^2-r^2)F(\hat\theta,\hat\phi)}{[a^2+r^2-2ar\cos\beta]^{3/2}}\sin\hat\theta\, d\hat\phi\, d\hat\theta\,,$$

where β is defined by $\cos\beta = \cos\phi\cos\hat\phi + \sin\phi\sin\hat\phi\cos(\theta-\hat\theta)$.

5. (5 points) Let $\mathbf{x} \in \mathbb{R}^3$ and suppose that $F(\mathbf{x})$ has compact support in $\mathbb{R}^3$. Let $u(\mathbf{x})$ satisfy

$$\Delta u = F(\mathbf{x}) \,, \quad \mathbf{x} \in \mathbb{R}^3 \,, \qquad \text{with} \quad \lim_{|\mathbf{x}| \to \infty} |\mathbf{x}| u \quad \text{finite} \,.$$

(a) (2 points) Show that

$$u(\mathbf{x}) = -\frac{1}{4\pi} \int_{\Omega_f} \frac{F(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \, d\mathbf{x}' \,.$$

Here the 3-D integral $\int_{\Omega_f}$ is taken over the support of F, i.e. the region where $F \not\equiv 0$.

(b) (3 points) From your formula in (a) above, let $|\mathbf{x}| \to \infty$ and show that u has the limitng behavior

$$u(\mathbf{x}) \sim \frac{C}{|\mathbf{x}|} + \frac{\mathbf{p} \cdot \mathbf{x}}{|\mathbf{x}|^3} + \ldots \,,$$

where you are to find the scalar C (called the capacitance) and the vector $\mathbf{p}$ (called the dipole moment) in terms of F.