MATH 400 HW # 4

PROBLEM 1 IN 3-D FIND THE FREE-SPACE GREEN'S FUNCTION FOR

$$\Delta G - k^2 G = \frac{\delta(r)}{4\pi r^2} \quad \text{WITH} \quad r^2 = x^2 + y^2 + z^2 \quad \text{AND} \quad G \to 0 \text{ AS } r \to \infty,$$

WHERE $k > 0$.
(HINT: RECALL IN 3-D THAT IF G IS RADIALLY SYMMETRIC $\Delta G = G_{rr} + \frac{2}{r} G_r$).

PROBLEM 2 USING THE METHOD OF IMAGES TO FIND THE GREEN'S FUNCTION, FIND AN INTEGRAL REPRESENTATION FOR THE SOLUTION TO EACH OF THE FOLLOWING:

(i) 2-D $\quad \Delta u - k^2 u = F(x,y) \quad$ IN $-\infty < x < \infty,\ y > 0$ WITH $k > 0$

$u(x,0) = h(x)$ AND $u \to 0$ AS $x^2 + y^2 \to \infty$.

(ii) 3-D $\quad \Delta u = F(x,y,z) \quad$ IN $-\infty < x < \infty,\ -\infty < y < \infty,\ z > 0$

$u(x,y,0) = h(x,y)$ AND $u \to 0$ AS $x^2 + y^2 \to \infty$.

PROBLEM 3 USING THE METHOD OF IMAGES TO FIND THE GREEN'S FUNCTION, FIND AN INTEGRAL REPRESENTATION FOR THE SOLUTION TO

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = F(r,\varphi) \quad \text{IN} \quad 0 \le r \le 1,\ 0 \le \varphi \le \pi$$

WITH $\quad u = 0$ ON $r = 1$; u BOUNDED AS $r \to 0$

$u = 0$ ON $\varphi = 0, \pi$.

PROBLEM 4 CONSIDER $\Delta u = 0$ IN A SPHERE OF RADIUS a WITH

$$u(a,\varphi,\phi) = F(\varphi,\phi)$$

HERE IN SPHERICAL COORDINATES (r,φ,ϕ) ARE DEFINED BY

$x = r\cos\varphi \sin\phi$

$y = r\sin\varphi \sin\phi$

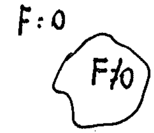
$z = r\cos\phi$.

BY FINDING THE GREEN'S FUNCTION USING THE METHOD OF IMAGES, SHOW THAT

$$u(r,\varphi,\phi) = \int_0^{2\pi}\int_0^{\pi} \frac{a(a^2 - r^2)\, F(\hat\varphi,\hat\phi)\sin\hat\phi}{4\pi\left[a^2 + r^2 - 2ar\cos\beta\right]^{3/2}}\, d\hat\phi\, d\hat\varphi$$

WHERE $\quad \cos\beta = \cos\phi\cos\hat\phi + \sin\phi\sin\hat\phi\cos(\varphi - \hat\varphi)$

PROBLEM 5 SUPPOSE THAT $\underline{x} \in \mathbb{R}^3$ AND THAT $F(\underline{x})$ HAS COMPACT SUPPORT IN $\mathbb{R}^3$. LET $u(\underline{x})$ SATISFY



$$\Delta u = F(\underline{x}) \quad \text{IN } \mathbb{R}^3 \quad \text{WITH} \quad u = O\left(\frac{1}{|\underline{x}|}\right) \quad \text{AS } |\underline{x}| \to \infty.$$

(i) SHOW THAT

$$u(\underline{x}) = -\frac{1}{4\pi} \int_{\Omega_F} \frac{F(\underline{x}')\, d\underline{x}'}{|\underline{x}' - \underline{x}|} \qquad (1)$$

WHERE THE 3-D INTEGRAL $\int_{\Omega_F}$ IS TAKEN OVER THE SUPPORT OF F, i.e. THE REGION WHERE $F \not\equiv 0$

(ii) SHOW FROM (1) THAT FOR $|\underline{x}| \to \infty$ WE HAVE THE ASYMPTOTIC BEHAVIOR

$$u(\underline{x}) \sim \frac{C}{|\underline{x}|} + \frac{\underline{p} \cdot \underline{x}}{|\underline{x}|^3} + \cdots$$

WHERE C (THE CAPACITANCE) AND $\underline{p}$ (THE DIPOLE VECTOR) ARE TO BE FOUND.

PROBLEM 1 FIND THE FREE-SPACE GREEN'S FUNCTION FOR

$$\Delta V - K^2 V = \frac{\delta(r)}{4\pi r^2} \quad \text{IN 3-D.}$$

WITH $V \to 0$ AS $r \to \infty$.

SOLUTION FOR $r \neq 0$, $\Delta V - K^2 V = 0$. LET $\Delta V = V_{rr} + \frac{2}{r} V_r$.

HENCE, $V_{rr} + \frac{2}{r} V_r - K^2 V = 0$.

LET $V(r) = f(r)/r$. THEN $V_r = f'/r - f/r^2$, $V_{rr} = f''/r - 2f'/r^2 + 2f/r^3$.

THUS, $$\frac{f''}{r} - \frac{2f'}{r^2} + \frac{2f}{r^3} + \frac{2}{r}\left(\frac{f'}{r} - \frac{f}{r^2}\right) - \frac{K^2 f}{r} = 0.$$

THIS YIELDS THAT $f'' - K^2 f = 0$ SO $f = e^{-Kr}$ WITH $f \to 0$ AS $r \to \infty$.

HENCE, $V = A e^{-Kr}/r$.

NOW INTEGRATE OVER A SMALL SPHERE OF RADIUS $r = \varepsilon$ AND LET $\varepsilon \to 0^+$.

THEN, $$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \Delta V \, d\underline{x} - \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} K^2 V \, d\underline{x} = \int_{\Omega_\varepsilon} \delta(\underline{x}) \, d\underline{x} = 1.$$

THEN BY DIV. THEOREM,

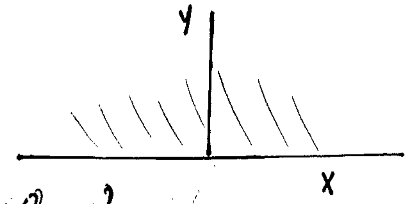
$$1 = \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \left.\frac{\partial V}{\partial r}\right|_{r=\varepsilon} \varepsilon^2 \sin\phi \, d\phi \, d\theta - \underbrace{\lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \int_0^{\varepsilon} K^2 \frac{A e^{-Kr}}{r} r^2 \sin\phi \, d\phi \, d\theta}_{\to 0 \text{ AS } \varepsilon \to 0.}$$

HENCE, WITH $V = A e^{-Kr}/r$, $\partial V/\partial r = -\frac{A}{r^2} e^{-Kr} - \frac{K A e^{-Kr}}{r}$,

SO $$1 = \lim_{\varepsilon \to 0} 4\pi \left(-\frac{A}{\varepsilon^2} e^{-K\varepsilon} - \frac{KA}{\varepsilon} e^{-K\varepsilon} \right) \varepsilon^2 = -4\pi A.$$

SO $A = -1/4\pi \longrightarrow V = -\frac{1}{4\pi r} e^{-Kr}$.

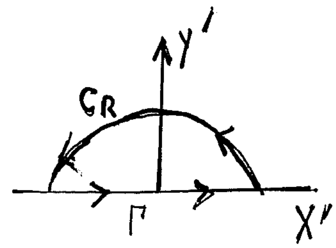
PROBLEM 2(i) USING THE METHOD OF IMAGES SOLVE



$$\Delta u - k^2 u = F(x,y) \quad \text{IN} \quad -\infty < x < \infty, \; y > 0$$

$$u(x,0) = h(x), \quad -\infty < x < \infty; \quad u \to 0 \quad \text{AS} \quad x^2 + y^2 \to \infty$$

SOLUTION WE WRITE



$$\Delta' u - k^2 u = F(x', y') \quad \text{IN} \quad -\infty < x' < \infty, \; y' > 0$$

$$u(x', 0) = h(x'), \quad u \to 0 \quad \text{AS} \quad x'^2 + y'^2 \to \infty.$$

NOW INTEGRATE OVER SEMI-CIRCLE AND LET $R \to \infty$ WITH $x'^2 + y'^2 = R^2$.

$$\lim_{R\to\infty} \left[\iint_\Omega (\Delta' u - k^2 u)\, G \, d\underline{x}' - \int_\Omega u [\Delta' G - k^2 G]\, d\underline{x}' \right] = \lim_{R\to\infty} \int_{C_R} (G \partial_n u - u \partial_n G)\, ds$$

$$+ \int_{-\infty}^{\infty} (G \partial_n u - u \partial_n G) \Big|_{y'=0} dx'$$

ASSUMING THAT $u \to 0$ AND $G \to 0$ AS $x'^2 + y'^2 \to \infty$,

WE OBTAIN

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} F(x', y')\, G \, dx' \, dy' - u(\underline{x}) = - \int_{-\infty}^{\infty} u \, \partial_n G \Big|_{y'=0} dx'.$$

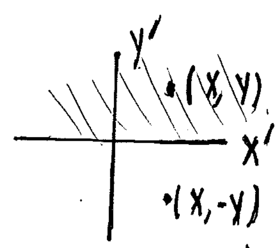
NOW $\partial_n G \Big|_{y'=0} = - \dfrac{\partial G}{\partial y'} \Big|_{y'=0}$. HENCE,

$$u(\underline{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x', y')\, G(\underline{x}', \underline{x})\, d\underline{x}' - \int_{-\infty}^{\infty} h(x') \frac{\partial G}{\partial y'} \Big|_{y'=0} dx',$$

WHERE

$$\Delta' G - k^2 G = \delta(\underline{x}' - \underline{x}) \quad \text{IN} \quad -\infty < x' < \infty, \; y' > 0$$

$$G = 0 \quad \text{ON} \quad y' = 0.$$



BY THE METHOD OF IMAGES, $G = -\dfrac{1}{2\pi} K_0(k|\underline{x}' - \underline{x}|) + \dfrac{1}{2\pi} K_0(k|\underline{x}' - \underline{x}_I|)$

WHERE $K_0(z)$ IS MODIFIED BESSEL FUNCTION

AND WHERE $\underline{x}_I = (x, -y)$, $\underline{x} = (x, y)$; $\underline{x}_I$ IS THE IMAGE CHARGE.

PROBLEM 2(ii) USING THE METHOD OF IMAGES, FIND THE SOLUTION TO

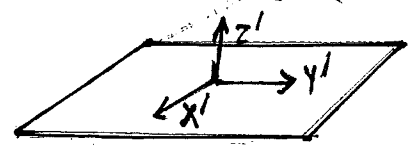
$$\Delta u = F(x,y,z) \quad \text{FOR } z>0,\ -\infty<x<\infty,\ -\infty<y<\infty$$

$$u(x,y,0) = h(x,y), \quad u\to 0 \text{ AS } r^2 = x^2+y^2+z^2 \to \infty.$$

SOLUTION

WE WRITE $\Delta' u = F(\underline{x}')$ FOR $z'>0$ WITH $\underline{x}' = (x',y',z')$

$u = h(x',y')$ ON $z'=0$



NOW USING GREEN'S SECOND IDENTITY, INTEGRATING OVER A SEMI-SPHERE OF RADIUS R AND LETTING $R\to\infty$. WITH $u\to 0$ AS $R\to\infty$ WE OBTAIN,

$$\int_\Omega (G\Delta' u - u\Delta' G)\,d\underline{x}' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(G\left[-\frac{\partial u}{\partial z'}\Big|_{z'=0}\right] - u\left[-\frac{\partial G}{\partial z'}\Big|_{z'=0}\right]\right) dx'dy'.$$

BY GREEN'S SECOND IDENTITY, WHERE $\Omega = \{(x',y',z') \mid z'\geq 0\}$.

NOW PUT G TO SATISFY $\left\{\begin{array}{l} \Delta' G = \delta(\underline{x}'-\underline{x}) \quad \text{IN } -\infty<x',y'<\infty,\ z'\geq 0 \\ G = 0 \quad \text{ON } z'=0. \end{array}\right.$

THEN (*) $$u(\underline{x}) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x',y')\frac{\partial G}{\partial z'}\Big|_{z'=0} dx'dy' + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} FG\,d\underline{x}'$$

NOW BY THE METHOD OF IMAGES

$$G = -\frac{1}{4\pi r} + \frac{1}{4\pi r_I} \quad \text{WITH} \quad r = \sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$$

$$\text{AND} \quad r_I = \sqrt{(x'-x)^2+(y'-y)^2+(z'+z)^2}.$$

NOW $$\frac{\partial G}{\partial z'}\Big|_{z'=0} = \frac{1}{4\pi r^2}\frac{\partial r}{\partial z'}\Big|_{z'=0} - \frac{1}{4\pi r_I^2}\frac{\partial r_I}{\partial z'}\Big|_{z'=0} = \frac{1}{2}\left(\frac{1}{4\pi r^3} 2r\frac{\partial r}{\partial z'}\Big|_{z'=0} - \frac{1}{4\pi r_I^3}2r_I\frac{\partial r_I}{\partial z'}\Big|_{z'=0}\right)$$

$$= \frac{1}{4\pi r^3}\frac{2(z'-z)}{2}\Big|_{z'=0} - \frac{1}{4\pi r_I^3}\frac{2(z'+z)}{2}\Big|_{z'=0} = \frac{-z}{2\pi r^3}.$$

HENCE IN (*) $\partial G/\partial z'|_{z'=0} = -\frac{z}{\;}\left[(x'-x)^2+(y'-y)^2+z^2\right]^{-3/2}$

PROBLEM 3

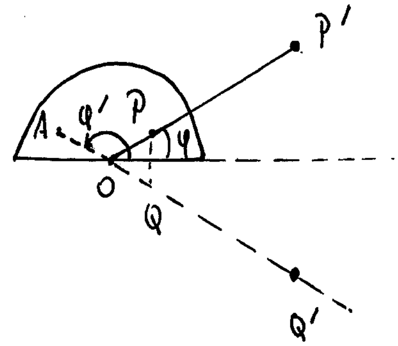
(i) WE FIRST FIND THE GREEN'S FUNCTION SATISFYING

$$\Delta G = \frac{1}{r'}\,\delta(r'-r)\,\delta(\varphi'-\varphi) \quad \text{IN} \quad 0\le r'\le 1,\ 0\le\varphi'\le\pi$$

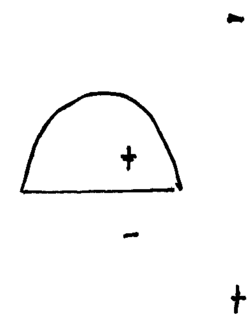
WITH A FIXED POINT (r,φ) IN THE UPPER 1/2 DISK. THIS WOULD GIVE

$$G(r',\varphi';r,\varphi).$$

THE PICTURE IS:



$$\Omega \equiv \{(r',\varphi') \mid 0\le r'\le 1,\ 0\le\varphi'\le\pi\}.$$



- P IS FIXED POINT AT (r,φ) WHERE DELTA SINGULARITY IS.
- P' IS INVERSE POINT TO P AT $(1/r,\varphi)$
- Q IS REFLECTION OF P IN x'-AXIS AT $(r,-\varphi)$
- Q' IS REFLECTION OF P' IN x'-AXIS AT $(1/r,-\varphi)$

CONSIDER AN ARBITRARY POINT A AT (r',φ') IN Ω.

LET $r_1 = \text{dist}(A,P)$ $\quad r_2 = \text{dist}(A,P')$, $\quad r_3 = \text{dist}(A,Q)$, $\quad r_4 = \text{dist}(A,Q')$

- NOW WHEN POINT A IS ON $r'=1$ WE HAVE $r_2 = \frac{1}{r} r_1$ AND $r_4 = \frac{1}{r} r_3$. WE WANT $G=0$
- ALSO WHEN A IS ON x'-AXIS THEN $r_1 = r_3$ AND $r_2 = r_4$. WE WANT $G=0$.

IN THIS WAY WE ALSO NEED $G \sim \frac{1}{2\pi}\log r_1$ NEAR POINT P.

THE G FUNCTION IS THEN

$$G = \frac{1}{2\pi}\left[\log(r_1) - \log(r\,r_2) - \log r_3 + \log(r\,r_4)\right] = \frac{1}{2\pi}\log\left(\frac{r_1 r_4}{r_2 r_3}\right)$$

THUS THERE ARE 3 IMAGE CHARGES. THE POINTS ARE

$$r_1 = \left[(r\cos\varphi - r'\cos\varphi')^2 + (r\sin\varphi - r'\sin\varphi')^2\right]^{1/2}$$

$$r_2 = \left[(r'\cos\varphi' - \tfrac{1}{r}\cos\varphi)^2 + (r'\sin\varphi' - \tfrac{1}{r}\sin\varphi)^2\right]^{1/2}$$

$$r_3 = \left[(r'\cos\varphi' - r\cos\varphi)^2 + (r'\sin\varphi' + r\sin\varphi)^2\right]^{1/2}$$

$$r_4 = \left[(r'\cos\varphi' - \tfrac{1}{r}\cos\varphi)^2 + (r'\sin\varphi' + \tfrac{1}{r}\sin\varphi)^2\right]^{1/2}$$

FINALLY, BY USING GREEN'S IDENTITY

$$u(r,\varphi) = \int_0^1\int_0^\pi G(r',\varphi';r,\varphi)\,F(r',\varphi')\,r'\,dr'\,d\varphi'.$$

PROBLEM 4 USING THE METHOD OF IMAGES FIND THE SOLUTION TO

$$\Delta u = 0 \quad \text{IN A SPHERE OF RADIUS } a$$

WITH $u = f(\theta, \phi)$ ON $r = a$.

SOLUTION WE FIRST WRITE

$$\Delta' u = 0, \quad \text{IN } |\underline{x}'| \leq a$$

WITH $u = f(\hat{\theta}, \hat{\phi})$ ON $|\underline{x}'| = a$.

WE DEFINE $\Omega = \{\underline{x}' \mid |\underline{x}'| \leq a\}$. WE OBTAIN

$$\int_\Omega [G\Delta' u - u\Delta' G]\, d\underline{x}' = \int_{\partial\Omega} [G\partial_n u - u\partial_n G]\, ds.$$

NOW CHOOSE G TO SATISFY

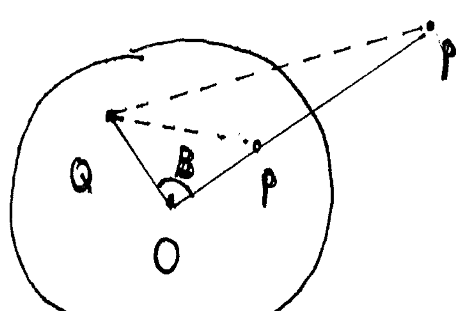
$$\begin{cases} \Delta' G = \delta(\underline{x}' - \underline{x}) & \text{IN } |\underline{x}'| \leq a \\ G = 0 & \text{ON } |\underline{x}'| = a. \end{cases}$$

THIS GIVES $-\int_\Omega u\, \delta(\underline{x}' - \underline{x})\, d\underline{x}' = -\int_{\partial\Omega} u\, \partial_n G\, ds.$

THIS YIELDS THAT $-u(\underline{x}) = -\int_0^{2\pi}\int_0^{\pi} f(\hat{\theta}, \hat{\phi}) \left.\frac{\partial G}{\partial \rho}\right|_{\rho = a} a^2 \sin\hat{\phi}\, d\hat{\phi}\, d\hat{\theta}.$

$$(*) \quad \Big\{ \; u(\underline{x}) = \int_0^{2\pi}\int_0^{\pi} f(\hat{\theta}, \hat{\phi}) \left.\frac{\partial G}{\partial \rho}\right|_{\rho = a} a^2 \sin\hat{\phi}\, d\hat{\phi}\, d\hat{\theta}.$$

NOW WE BEGIN WITH



- LET P BE THE FIXED POINT WHERE DELTA FUNCTION IS LOCATED
  $P = \{\underline{x} \mid$ WITH $\underline{x} = (x, y, z)$ AND $|\underline{x}| = r\}$
- LET Q BE ANY POINT IN THE DISK
  $Q = \{\underline{x}' \mid$ WITH $|\underline{x}'| = \rho$ AND $\underline{x}' = (x', y', z')$
- ALSO O IS CENTER OF SPHERE

NOW $\quad \operatorname{dist}(O,Q) = \rho \qquad \operatorname{dist}(P,Q) = r_1 \qquad \operatorname{dist}(Q,P') = r_2.$

$\qquad\quad \operatorname{dist}(O,P) = r \qquad \operatorname{dist}(O,P') = R$

NOTICE THAT OQP LIES ON A PLANE IN 3-DIMENSIONAL SPACE WITH B BEING THE ANGLE BETWEEN OQ AND OP. ALSO THE POINT P' LIES ON THE LINE OP AND WE CHOOSE R SUCH THAT

$$R = a^2/r.$$

<u>KEY FACT</u> ON $\rho = a$, I.E. WHEN Q IS ON THE BOUNDARY OF THE SPHERE, THEN $r_2 = r_1\, a/r$.

<u>PROOF</u> BY LAW OF COSINES,

$$(\operatorname{dist}(QP'))^2 = (\operatorname{dist}(OP'))^2 + (\operatorname{dist}(OQ))^2 - 2\operatorname{dist}(OQ)\operatorname{dist}(OP')\cos B$$

SO THAT $\quad r_2^2 = R^2 + \rho^2 - 2R\rho\cos B.$

BUT $R = a^2/r$, SO $\quad \left\{ r_2^2 = a^4/r^2 + \rho^2 - \dfrac{2a^2\rho}{r}\cos B. \right.$

NOW ON $\rho = a$ (Q ON BOUNDARY), THEN

$$r_2^2 = \frac{a^4}{r^2} + a^2 - \frac{2a^3}{r}\cos B = \frac{a^2}{r^2}\left[a^2 + r^2 - 2ar\cos B\right] = \frac{a^2 r_1^2}{r^2}.$$

HENCE $\quad r_2 = r_1\, a/r. \;\blacksquare$

NOW IN 3-DIMENSIONS, $\quad G^F = -\dfrac{1}{4\pi r_1}.$

WE THEN TAKE $\quad H = \dfrac{C}{4\pi r_2}$, FOR SOME CONSTANT C.

THEN $$G = -\frac{1}{4\pi r_1} + \frac{C}{4\pi r_2}.$$

WE NEED $G=0$ ON $r_2 = r_1\, a/r$. HENCE, $C = a/r$.

WE OBTAIN $$\left\{ G = -\frac{1}{4\pi r_1} + \frac{a}{r}\frac{1}{4\pi r_2}, \right.$$

WHERE $$r_2^2 = \frac{a^4}{r^2} + \rho^2 - \frac{2a^2\rho}{r}\cos B$$

AND $$r_1^2 = \rho^2 + r^2 - 2r\rho\cos B.$$

THIS DEFINES THE GREEN'S FUNCTION. IN THIS FORMULA ρ, B ARE THE TWO "VARIABLES", WHERE a AND r ARE FIXED VALUES.

FINALLY, WE MUST CALCULATE $\left.\frac{\partial G}{\partial \rho}\right|_{\rho=a}$ AND SUBSTITUTE INTO $(*)$

AND IDENTIFY THE ANGLE B WITH SPHERICAL COORDINATES.

$$\frac{\partial G}{\partial \rho} = \frac{1}{4\pi r_1^2}\partial_\rho r_1 - \frac{a}{4\pi r}\frac{1}{r_2^2}\partial_\rho r_2 = \frac{1}{4\pi}\left[\frac{r_1\partial_\rho r_1}{r_1^3} - \frac{a}{r}\frac{r_2\partial_\rho r_2}{r_2^3}\right]$$

NOW $$2r_1\partial_\rho r_1 = 2[\rho - r\cos B] \longrightarrow \left. r_1\partial_\rho r_1\right|_{\rho=a} = a - r\cos B$$

$$2r_2\partial_\rho r_2 = 2\left[\rho - \frac{a^2}{r}\cos B\right] \Longrightarrow \left. r_2\partial_\rho r_2\right|_{\rho=a} = a - \frac{a^2}{r}\cos B.$$

NOW ON $\rho = a$ THEN,

$$r_1^2 = a^2 + r^2 - 2ar\cos B, \quad r_2 = r_1\, a/r$$

THIS GIVES:

$$\left.\frac{\partial G}{\partial \rho}\right|_{\rho=a} = \frac{1}{4\pi r_1^3}\left[(a - r\cos B) - \left(\frac{a}{r}\right)\frac{r_1^3}{r_2^3}\left(a - \frac{a^2}{r}\cos B\right)\right]$$

BUT $r_1 = \left(\frac{r}{a}\right) r_2$, SO $r_1/r_2 = r/a$ ON $\rho = a$.

THUS, $$\left.\frac{\partial G}{\partial \rho}\right|_{\rho=a} = \frac{1}{4\pi r_1^3}\left[(a - r\cos B) - \frac{r^2}{a^2}\left(a - \frac{a^2}{r}\cos B\right)\right] = \frac{1}{4\pi r_1^3}\left[a - r^2/a\right],$$

WHICH YIELDS $$\left.\frac{\partial G}{\partial \rho}\right|_{\rho=a} = \frac{1}{4\pi a r_1^3}\left(a^2 - r^2\right).$$

RECALLING (*) FOR $u(\underline{x})$ AND $r_1 = \left(a^2 + r^2 - 2ar\cos B\right)^{1/2}$

WE OBTAIN

$$u(\underline{x}) = \int_0^{2\pi}\int_0^{\pi} \frac{a f(\hat{\varphi}, \hat{\phi})\,(a^2 - r^2)\sin\hat{\phi}}{4\pi\left[a^2 + r^2 - 2ar\cos B\right]^{3/2}}\, d\hat{\phi}\, d\hat{\varphi}.$$

FINALLY, WE MUST IDENTIFY B IN TERMS OF SPHERICAL COORDINATES.

NOW LET $\underline{x} = r\left(\cos\varphi\sin\phi, \sin\varphi\sin\phi, \cos\phi\right)$

$\underline{x}' = \rho\left(\cos\hat{\varphi}\sin\hat{\phi}, \sin\hat{\varphi}\sin\hat{\phi}, \cos\hat{\phi}\right)$

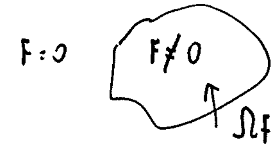
THEN $\underline{x}\cdot\underline{x}' = |\underline{x}|\,|\underline{x}'|\cos B$ WHERE B IS ANGLE BETWEEN $\underline{x}$ AND $\underline{x}'$.

HENCE, $$\cos B = \cos\varphi\cos\hat{\varphi}\,\underline{\sin\phi\sin\hat{\phi}} + \sin\varphi\,\underline{\sin\phi}\sin\hat{\varphi}\,\underline{\sin\hat{\phi}} + \cos\phi\cos\hat{\phi}$$

HENCE $$\cos B = \left(\cos\varphi\cos\hat{\varphi} + \sin\varphi\sin\hat{\varphi}\right)\sin\phi\sin\hat{\phi} + \cos\phi\cos\hat{\phi}.$$

THIS YIELDS, $$\cos B = \cos(\varphi - \hat{\varphi})\sin\phi\sin\hat{\phi} + \cos\phi\cos\hat{\phi}.$$

PROBLEM 5 CONSIDER IN 3-D THAT $u(\underline{x})$ SATISFIES



$$\Delta u = F(\underline{x}) \quad \text{IN } \mathbb{R}^3$$

WHERE $F(\underline{x})$ HAS COMPACT SUPPORT WITH $F(\underline{x}) \neq 0$ FOR $\underline{x} \in \Omega_f$ AND $F(\underline{x}) \equiv 0$ FOR $\underline{x} \notin \Omega_f$.

(i) NOW IMPOSING $u = O(1/|\underline{x}|)$ AS $|\underline{x}| \to \infty$ WE USE GREEN'S IDENTITY WITH THE FREE-SPACE G-FUNCTION $G_f \equiv -\dfrac{1}{4\pi|\underline{x}'-\underline{x}|}$. LET Ω_R BE A BALL OF RADIUS R.

WE INTEGRATE

$$\lim_{R\to\infty} \int_{\Omega_R} \left(G_f \Delta' u - u \Delta' G_f \right) d\underline{x}' = \lim_{R\to\infty} \int_{\partial\Omega_R} \left(G_f \frac{\partial u}{\partial r} - u \frac{\partial G_f}{\partial r} \right)\Bigg|_{r=|\underline{x}|=R} ds$$

NOW SINCE $ds = R^2 \sin\phi \, d\phi \, d\varphi$ WHILE $G_f \dfrac{\partial u}{\partial r}\Big|_{r=R} = O(1/R^3)$ AND $u \dfrac{\partial G_f}{\partial r} = O(1/R^3)$. THUS THE INTEGRAL OVER $\partial\Omega_R$ VANISHES AS $r \to \infty$. SINCE $\Delta' G_f = \delta(\underline{x}'-\underline{x})$ WE OBTAIN,

$$\int_{\mathbb{R}^3} G_f \Delta' u \, d\underline{x}' = \int_{\mathbb{R}^3} u \Delta' G_f \, d\underline{x}' = u(\underline{x}).$$

NOW $\Delta' u = F(\underline{x}')$ AND SO WE CONCLUDE THAT, GIVEN THAT F IS NON-ZERO ONLY IN Ω_f,

$$\left. u(\underline{x}) = \iiint_{\mathbb{R}^3} G_f F(\underline{x}') \, d\underline{x}' = -\frac{1}{4\pi} \int_{\Omega_f} \frac{F(\underline{x}')}{|\underline{x}'-\underline{x}|} d\underline{x}' \right\} \quad (*)$$

NOTICE THAT IN 3-D, THE IMPROPER INTEGRAL IS WELL-DEFINED SINCE $d\underline{x}' = r^2 \sin\phi \, d\phi \, d\varphi$.

(ii) NOW WE CALCULATE

$$|\underline{x}'-\underline{x}|^2 = (\underline{x}'-\underline{x}) \cdot (\underline{x}'-\underline{x}) = |\underline{x}'|^2 + |\underline{x}|^2 - 2\underline{x}' \cdot \underline{x}$$

$$|\underline{x}-\underline{x}'|^2 = |\underline{x}|^2 \left[1 - \frac{2\underline{x}' \cdot \underline{x}}{|\underline{x}|^2} + \frac{1}{|\underline{x}|^2} \right]$$

NOW

$$\frac{1}{|\underline{x}-\underline{x}'|} = \frac{1}{|\underline{x}|} \left(1 - \frac{2\underline{x}' \cdot \underline{x}}{|\underline{x}|^2} + O\left(\frac{1}{|\underline{x}|^2}\right) \right)^{-1/2}$$

RECALL BY TAYLOR SERIES THAT $(1-z)^{-1/2} \cong 1 + \dfrac{z}{2} + \dots$ AS $z \to 0$. THUS,

$$\frac{1}{|\underline{x}-\underline{x}'|} \cong \frac{1}{|\underline{x}|} \left(1 + \frac{\underline{x}' \cdot \underline{x}}{|\underline{x}|^2} + \dots \right) = \frac{1}{|\underline{x}|} + \frac{\underline{x}' \cdot \underline{x}}{|\underline{x}|^3} + \dots \quad \text{AS } |\underline{x}| \to \infty.$$

WE SUBSTITUTE THIS INTO $(*)$ TO GET $u(\underline{x}) = -\dfrac{1}{4\pi|\underline{x}|} \displaystyle\int_{\Omega_f} F(\underline{x}') \, d\underline{x}' - \dfrac{\underline{x} \cdot}{4\pi|\underline{x}|^3} \displaystyle\int_{\Omega_f} \underline{x}' F(\underline{x}') \, d\underline{x}' + \dots$

WE CONCLUDE FOR $|\underline{x}| \to \infty$ THAT

$$u(\underline{x}) \sim \frac{C}{|\underline{x}|} + \frac{\underline{P} \cdot \underline{x}}{|\underline{x}|^3} + \dots \quad \text{AS } |\underline{x}| \to \infty \quad \text{WHERE} \quad C = -\frac{1}{4\pi} \int_{\Omega_f} F(\underline{x}') \, d\underline{x}'$$

$$\underline{P} = -\frac{1}{4\pi} \int_{\Omega_f} \underline{x}' F(\underline{x}') \, d\underline{x}'.$$