MATH 401 HW #7

PROBLEM 1 BY BOUNDING THE COEFFICIENT xy IN THE EIGENVALUE PROBLEM BELOW FIND UPPER AND LOWER BOUNDS FOR THE LOWEST EIGENVALUE OF

$$\Delta \phi - xy \Phi = -\lambda \Phi \quad \text{IN} \quad 0 < x < \pi, \ 0 < y < \pi$$

$$\Phi = 0 \quad \text{ON ALL SIDES OF THE SQUARE.}$$

PROBLEM 2 CONSIDER THE LOWEST EIGENVALUE λ_1 FOR THE LAPLACIAN IN THE UNIT SPHERE IN 3-D WITH DIRICHLET BC, WHERE

$$\left.\begin{array}{l} \Delta \phi + \lambda \phi = 0 \quad \text{IN} \quad 0 < r < 1 \\ \phi = 0 \quad \text{ON} \quad r = 1, \quad \phi \ \text{BOUNDED AS} \ r \to 0. \end{array}\right\} \quad (*)$$

(i) FORMULATE A 1-D VARIATIONAL PROBLEM FOR THE LOWEST EIGENVALUE λ_1 OF $(*)$

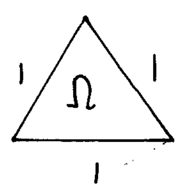
(ii) COMPUTE THE RAYLEIGH QUOTIENT WITH THE TRIAL FUNCTION $1-r$ TO GET AN UPPER BOUND FOR λ_1

(iii) REPEAT (ii) WITH THE TRIAL FUNCTION $\cos\left(\pi r/2\right)$

(iv) BY SETTING $\phi(r) = f(r)/r$ IN $(*)$ FIND THE LOWEST EIGENVALUE EXPLICITLY AND COMPARE WITH THE BOUNDS IN (ii) AND (iii).

PROBLEM 3 CONSIDER THE LOWEST EIGENVALUE λ_1 OF AN EQUILATERAL TRIANGLE Ω OF SIDELENGTH 1, FOR THE LAPLACIAN WITH DIRICHLET BOUNDARY CONDITIONS ON ALL SIDES, SO THAT

$$(*) \left\{\begin{array}{l} \Delta \phi + \lambda \phi = 0 \quad \text{IN} \quad \Omega \\ \phi = 0 \quad \text{ON} \quad \partial\Omega \end{array}\right.$$



$\Omega \in \mathbb{R}^2$

(i) BY INTRODUCING A COORDINATE SYSTEM, AND BY CALCULATING THE RADIUS OF THE LARGEST INSCRIBED DISK AND THE RADIUS OF THE SMALLEST DISK THAT CONTAINS Ω, FIND UPPER AND LOWER BOUNDS FOR λ_1.

(ii) AMAZINGLY ALL THE EIGENVALUES FOR $(*)$ ARE $\lambda_{mn} = \frac{16\pi^2}{9}(m^2 + n^2 + mn)$, $m, n = 1, 2, 3, \ldots$ FIND LOWEST EIGENVALUE AND COMPARE WITH RESULTS IN (i).

(iii) GIVE A SIMPLE TRIAL FUNCTION $w(x,y)$ AT MOST QUADRATIC IN (x,y) AND POSITIVE IN Ω THAT CAN BE USED IN RAYLEIGH QUOTIENT TO GET AN UPPER BOUND.

DO NOT EVALUATE THIS UPPER BOUND ANALYTICALLY.

PROBLEM 4 LET Ω BE THE ELLIPSE

$$x^2/a^2 + y^2/b^2 \le 1 \quad \text{WITH} \quad a < b$$

AND CONSIDER THE EIGENVALUE PROBLEM

$$(*) \left\{ \begin{array}{l} \Delta\phi + \lambda\phi = 0 \quad \text{IN } \Omega \\ \phi = 0 \quad \text{ON } \partial\Omega \end{array} \right.$$

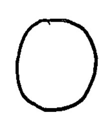
(i) FIND A SIMPLE UPPER BOUND ON THE LOWEST EIGENVALUE λ_1 OF $(*)$ BY USING THE LARGEST DISK THAT CAN BE INSCRIBED IN Ω.

(ii) BY USING THE TRIAL FUNCTION $W = 1 - x^2/a^2 - y^2/b^2$ DIRECTLY IN RAYLEIGH'S QUOTIENT FIND ANOTHER EXPLICIT UPPER BOUND ON THE LOWEST EIGENVALUE. (YOUR BOUND HERE WILL DEPEND ON a AND b.)

(HINT: IN PERFORMING THE INTEGRATIONS NEEDED YOU WILL FIND USE FOR $\int_0^{\pi/2} \sin^2\varphi \cos^2\varphi \, d\varphi = \frac{\pi}{16}$, $\int_0^{\pi/2} \cos^4\varphi \, d\varphi = \frac{3\pi}{16}$, $\int_0^{\pi/2} \cos^6\varphi \, d\varphi = \frac{5\pi}{32}$.)

PROBLEM 5 CONSIDER THE EIGENVALUE PROBLEM IN THE UNIT DISK

$$(*) \left\{ \begin{array}{l} \Delta\phi + \lambda\phi = 0 \quad \text{IN } 0 < r < 1 \\ \phi = 0 \text{ ON } r = 1, \quad \phi \text{ BOUNDED AS } r \to 0. \end{array} \right.$$



USING THE RAYLEIGH-RITZ METHOD WITH THE TWO TRIAL FUNCTIONS $W_1 = 1 - r^2$ AND $W_2 = (1 - r^2)^2$ SHOW THAT THE MINIMUM EIGENVALUE λ_1 OF $(*)$ IS SMALLER THAN THE MINIMUM EIGENVALUE μ_1 OF THE MATRIX PROBLEM $(A - \mu B)\underline{c} = \underline{0}$ WHERE

$$A = \begin{pmatrix} \int_0^1 4r^3 \, dr & \int_0^1 8r^2(1-r^2) r \, dr \\ \int_0^1 8r^2(1-r^2) r \, dr & \int_0^1 16 r^2 (1-r^2)^2 r \, dr \end{pmatrix}$$

AND

$$B = \begin{pmatrix} \int_0^1 (1-r^2)^2 r \, dr & \int_0^1 (1-r^2)^3 r \, dr \\ \int_0^1 (1-r^2)^3 r \, dr & \int_0^1 (1-r^2)^4 r \, dr \end{pmatrix}.$$

CALCULATE μ_1 ANALYTICALLY AND COMPARE WITH THE EXACT RESULT FOR λ_1.

MATH 401 HW # 7 SOLUTIONS

PROBLEM 1 FIND UPPER AND LOWER BOUNDS FOR THE EIGENVALUES OF

$$\Delta\phi - xy\,\phi = -\lambda\phi \quad \text{IN } D: 0<x<\pi,\ 0<y<\pi$$

$$\phi = 0 \text{ ON ALL SIDES}$$

$$\lambda_0 = \min_{w\in H} \frac{\int_D (|\nabla\phi|^2 + xy\,\phi^2)\,dx\,dy}{\int_D \phi^2\,dx\,dy}$$

NOW IF WE BOUND xy BY ITS MAX AND MIN VALUE OVER $0<x<\pi,\ 0<y<\pi$ IT IS CLEAR THAT

$$\lambda_{0_{MIN}} \le \lambda_0 \le \lambda_{0_{MAX}}$$

WHERE λ_{0MAX} IS THE FIRST EIGENVALUE OF

$$\Delta\phi - \pi^2\phi = -\lambda_{0MAX}\,\phi \quad \text{IN } D$$

$$\phi = 0 \text{ ON ALL sides}$$

AND λ_{0MIN} IS THE FIRST eigenvalue OF

$$\Delta\phi = -\lambda_{0MIN}\,\phi \quad \text{IN } D$$

$$\phi = 0 \text{ ON ALL sides}$$

RECALLING THAT THE EIGENVALUES OF $\Delta v = -\mu v$ IN D

$v = 0$ ON ALL SIDES $0\le x\le a \quad 0\le y\le b$

ARE $\mu = \dfrac{m^2\pi^2}{a^2} + \dfrac{n^2\pi^2}{b^2}$ $\qquad \mu_0 = \dfrac{\pi^2}{a^2} + \dfrac{\pi^2}{b^2}$

THUS $\lambda_{0MAX} = \pi^2 + 2$

$\lambda_{0MIN} = 2$

THUS $2 \le \lambda_0 \le 2 + \pi^2$

NOT AN IMPRESSIVE BOUND

NOW FOR THE OTHER EIGENVALUES WE HAVE

$$m^2 + n^2 \le \lambda_{m,n} \le 2 + m^2 + n^2$$

PROBLEM 2 $\Delta\phi + \lambda\phi = 0$ IN $0 \le r \le 1$

$\phi = 0$ ON $r = 1$, ϕ BOUNDED AS $r \to 0$.

(i) WE HAVE $\frac{1}{r^2}(r^2\phi_r)_r + \lambda\phi = 0$ FOR LOWEST EIGENVALUE.

NOW $(r^2\phi_r)_r + \lambda r^2 \phi = 0$

WE HAVE THAT $\lambda_1 \le \min_{\phi \in H} \dfrac{\int_0^1 (r^2 \phi'^2)\, dr}{\int_0^1 r^2 \phi^2\, dr}$, $H = \{\phi \mid \phi \text{ IS } C^2,\ \phi(1) = 0\}$

(ii) NOW TRY $\phi^* = 1 - r$ WE CALCULATE

$$\lambda_1 \le \frac{E(\phi^*)}{(\phi^*, \phi^*)} = \frac{\int_0^1 r^2 (\phi^{*\prime})^2\, dr}{\int_0^1 r^2 \phi^{*2}\, dr} = \frac{\int_0^1 r^2 (1)\, dr}{\int_0^1 r^2 (1-r)^2 dr} = \frac{1/3}{\int_0^1 (r^2 - 2r^3 + r^4)\, dr}$$

THIS YIELDS THAT $\lambda_1 \le \dfrac{1/3}{1/3 - 1/2 + 1/5} = \dfrac{1/3}{10/30 - 15/30 + 6/30} = \dfrac{30(1/3)}{10 - 15 + 6} = 10.$

WE CONCLUDE THAT $\lambda_1 \le 10$.

(iii) NOW $\phi^* = \cos(\pi r/2)$. WE CALCULATE

$$\lambda_1 \le \frac{\int_0^1 r^2 (\phi^{*\prime})^2\, dr}{\int_0^1 r^2 \phi^{*2}\, dr} = \frac{\int_0^1 r^2 \pi^2/4 \sin^2(\pi r/2)\, dr}{\int_0^1 r^2 \cos^2(\pi r/2)\, dr} \qquad (*)$$

WE CALCULATE WITH $\sin^2 x = (1 - \cos(2x))/2$, $\cos^2 x = (1 + \cos(2x))/2$, THAT

$$\frac{\pi^2}{4}\int_0^1 r^2 \sin^2\left(\frac{\pi r}{2}\right) dr = \frac{\pi^2}{4}\int_0^1 r^2 \left(\frac{1 - \cos(\pi r)}{2}\right) dr = \frac{\pi^2}{4}\left(\frac{1}{2}\right)\left[\frac{1}{3} - \int_0^1 r^2 \cos(\pi r)\, dr\right]$$

NOW USING $\int x^2 \cos(ax)\, dx = \dfrac{2x}{a^2}\cos(ax) + \left(\dfrac{x^2}{a} - \dfrac{2}{a^3}\right)\sin(ax)$, WE OBTAIN

$$\frac{\pi^2}{4}\int_0^1 r^2 \sin^2\left(\frac{\pi r}{2}\right) dr = \frac{\pi^2}{8}\left(\frac{1}{3} - \left(\frac{2}{\pi^2}(-1)\right)\right) = \frac{\pi^2}{24} + \frac{1}{4}. \qquad (**)$$

NOW WE CALCULATE $\int_0^1 r^2 \cos^2(\pi r/2)\, dr = \frac{1}{2}\int_0^1 r^2 (1 + \cos(\pi r))\, dr = \frac{1}{2}\left[\frac{1}{3} + \int_0^1 r^2 \cos(\pi r)\, dr\right]$

WE OBTAIN, $\int_0^1 r^2 \cos^2(\pi r/2)\, dr = \dfrac{1}{6} + \dfrac{1}{2}\left(\dfrac{2}{\pi^2}\right)(-1) = \dfrac{1}{6} - \dfrac{1}{\pi^2}$. $(***)$

WE SUBSTITUTE $(**)$ AND $(***)$ INTO $(*)$ TO OBTAIN $\lambda_1 \le \left(\dfrac{\pi^2}{4} + \dfrac{3}{2}\right) \Big/ \left(1 - 6/\pi^2\right) \cong 10.119$;

THIS GIVES A WORSE (SLIGHTLY) BOUND THAN IN (i).

(iii) PUT $\phi = f/r$. WE SUBSTITUTE INTO $\phi'' + 2/r\, \phi' + \lambda\phi = 0$ TO OBTAIN $f'' + \lambda f = 0$.

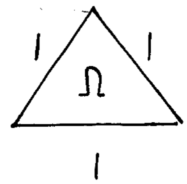
WE NEED $f(0) = 0$ SO THAT ϕ BOUNDED AT $r = 0$ AND $f(1) = 0 \to f(r) = \sin(\sqrt{\lambda} r) \mapsto \sin(\sqrt{\lambda}) = 0$.

THE LOWEST ROOT IS $\lambda = \lambda_1 = \pi^2$. THUS, THE EXACT EIGENVALUE IS $\lambda_1 = \pi^2$.

THE BOUND IN (i) GIVES $\lambda_1 \le 10$ (slightly better than (ii)).

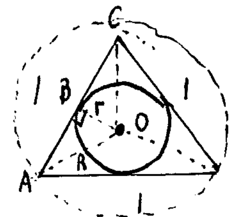
PROBLEM 3 CONSIDER LOWEST EIGENVALUE OF

$$(*)\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{IN } \Omega \\ \phi = 0 & \text{ON } \partial\Omega \end{cases}$$



WHERE Ω IS EQUILATERAL TRIANGLE.

(i) FIND BIGGEST INSCRIBED CIRCLE



BY SYMMETRY OAB IS A 30-60-90 TRIANGLE. THE AREA OF OCA IS $r/2$.

THUS SINCE THERE ARE 3 SUCH TRIANGLES $\longrightarrow 3\left(r/2\right) = \frac{\sqrt{3}}{4}$ (AREA OF 'Δ').

HENCE $r = \sqrt{3}/6$ FOR LARGEST INSCRIBED DISK.

NOW THE RADIUS OF THE SMALLEST CIRCUMSCRIBING SPHERE, LABELLED BY R IN THE FIGURE, IS FOUND BY PYTHAGORUS:

$$R^2 = r^2 + \left(\tfrac{1}{2}\right)^2 = r^2 + \frac{1}{4} = \frac{3}{36} + \frac{1}{4} = \frac{1}{12} + \frac{3}{12} = \frac{1}{3}$$

SO $R = 1/\sqrt{3} = \sqrt{3}/3$.

SINCE THE LOWEST EIGENVALUE FOR (*) IN A DISK OF RADIUS a IS $\frac{z_0^2}{a^2}$

WHERE $z_0 \doteq 2.4$ IS LOWEST ROOT OF $J_0(z) = 0$.

BY OUR BOUNDING THEOREMS THE LOWEST EIGENVALUE λ_1 FOR (*) IS

$$\lambda_{1\,\substack{BIG\\DISK}} \le \lambda_1 \le \lambda_{1\,\substack{SMALL\\DISK}} \longrightarrow \frac{z_0^2}{(1/\sqrt{3})^2} \le \lambda_1 \le \frac{z_0^2}{(\sqrt{3}/6)^2}$$

THUS $3z_0^2 \le \lambda_1 \le 12 z_0^2 \longrightarrow 3(5.76) \le \lambda_1 \le 12(5.76)$

(ii) ALL EIGENVALUES FOR 'Δ' ARE $\lambda_{m,n} = \frac{16\pi^2}{9}(m^2 + n^2 + mn)$.

THE LOWEST EIGENVALUE IS $\lambda_1 = \frac{16\pi^2}{9}(3) = 16\pi^2/3 \doteq 52.64$

WE CONCLUDE THAT $17.28 \le \lambda_1 \le 69.12$ WITH EXACT VALUE BEING $\lambda_1 \doteq 52.64$

(iii) WE LET THE BOUNDARY OF Ω BE $y = 0, 0 < x < 1$, $y = \sqrt{3}\,x$ FOR $0 \le x \le 1/2$, $y = \sqrt{3}(1-x)$ FOR $\frac{1}{2} \le x \le 1$

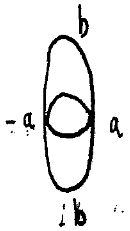
NOW DEFINE $w^* = y(\sqrt{3}\,x - y)(\sqrt{3}(1-x) - y)$. THEN $w^* > 0$ IN Ω. NOTICE $w^* = 0$ ON $\partial\Omega$.

NOW $$\lambda_1 \le \frac{\int_\Omega |\nabla w^*|^2\,dx\,dy}{\int_\Omega w^{*2}\,dx\,dy}$$

PROBLEM 4 FIND A SIMPLE UPPER BOUND ON THE LOWEST EIGENVALUE OF

$$\Delta\phi + \lambda\phi = 0 \text{ in } \Omega \qquad \text{WITH } \Omega \text{ THE ELLIPSE } x^2/a^2 + y^2/b^2 \le 1,$$
$$\phi = 0 \text{ on } \partial\Omega \qquad \text{AND } a < b.$$

(i)



NOW $\lambda_0(\Omega) \le \lambda_0(\Omega_{SMALL})$

WHERE $\lambda_0(\Omega_{SMALL})$ IS THE LOWEST EIGENVALUE OF

$$\Delta\phi + \lambda\phi = 0 \quad \text{IN } 0 < r < a,\ 0 \le \varphi \le 2\pi$$
$$\phi = 0 \text{ ON } r = a$$

FROM PROBLEM 1 WE HAVE $\lambda_0(\Omega_{SMALL}) \cong (2.4)^2/a^2 \cong 5.76/a^2$.

(ii) WE CAN DO A LITTLE BETTER WITH A RAYLEIGH QUOTIENT AND A SIMPLE TRIAL FUNCTION.

WE DEFINE $Lw \equiv \nabla\cdot(\nabla w)$. THEN $w\Delta w = -|\nabla w|^2 + \nabla\cdot(w\nabla w)$

WE HAVE
$$\int_\Omega |\nabla w|^2\, d\underline{x} = -\int_\Omega w\Delta w\, d\underline{x} + \int_{\partial\Omega} w\,\partial_n w\, ds.$$

NOW IF $w = 0$ ON $\partial\Omega$, THEN $\displaystyle\int_\Omega |\nabla w|^2 d\underline{x} = -\int_\Omega w\Delta w\, d\underline{x}$.

WE HAVE THAT
$$\lambda_0 = \min_{w\in H} \frac{\int_\Omega |\nabla w|^2 d\underline{x}}{\int_\Omega w^2 d\underline{x}} = \min_{w\in H}\left(-\frac{\int_\Omega w\Delta w\, d\underline{x}}{\int_\Omega w^2 d\underline{x}}\right)$$

WHERE $H = \{ w \mid w = 0 \text{ ON } \partial\Omega\}$. LET $w = 1 - x^2/a^2 - y^2/b^2$ SO THAT $w = 0$ ON $\partial\Omega$. WE CALCULATE $w_{xx} = -2/a^2$ $w_{yy} = -2/b^2$ SO THAT

$$w\Delta w = -\left(\frac{2}{a^2} + \frac{2}{b^2}\right)\left[1 - x^2/a^2 - y^2/b^2\right].$$

THUS
$$-\int_\Omega w\Delta w\, d\underline{x} = \left(\frac{2}{a^2}+\frac{2}{b^2}\right)\int_\Omega \left[1 - x^2/a^2 - y^2/b^2\right] d\underline{x}.$$
$$= \left(\frac{2}{a^2}+\frac{2}{b^2}\right)\left[\pi ab - \int_\Omega (x^2/a^2 + y^2/b^2)\,dx\,dy\right]$$
$$= \left(\frac{2}{a^2}+\frac{2}{b^2}\right)\left[\pi ab - 4\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} (x^2/a^2 + y^2/b^2)\,dx\,dy\right]$$

WE CALCULATE

$$\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) dx\,dy = \int_0^a \left[\frac{x^2}{a^2}\, b\sqrt{1-x^2/a^2} + \frac{1}{3b^2}\, b^3 \left(1 - x^2/a^2\right)^{3/2} \right] dx$$

$$= ab \int_0^{\pi/2} \sin^2\varphi \cos^2\varphi \, d\varphi + \frac{1}{3}\, ab \int_0^{\pi/2} \cos^4\varphi \, d\varphi.$$

NOW WE CALCULATE $\int_0^{\pi/2} \sin^2\varphi \cos^2\varphi \, d\varphi = \pi/16$, $\int_0^{\pi/2} \cos^4\varphi \, d\varphi = 3\pi/16$

THEN $\int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \left(x^2/a^2 + y^2/b^2 \right) dx\,dy = ab \dfrac{\pi}{16} + \dfrac{1}{3} ab \left(\dfrac{3\pi}{16} \right) = \dfrac{\pi ab}{8}$.

THUS $-\int_\Omega w \Delta w \, d\underline{x} = \left(\dfrac{2}{a^2} + \dfrac{2}{b^2} \right) \left[\pi ab - \dfrac{4\pi ab}{8} \right] = \left(\dfrac{2}{a^2} + \dfrac{2}{b^2} \right) \left(\dfrac{\pi ab}{2} \right)$

THIS YIELDS $-\int_\Omega w \Delta w \, d\underline{x} = \pi ab \left(1/a^2 + 1/b^2 \right) = \pi \left(b/a + a/b \right)$.

NOW WE CALCULATE $I_2 \equiv \int_\Omega w^2 \, d\underline{x} = 4 \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \left[1 - x^2/a^2 - y^2/b^2 \right]^2 dx\,dy$

SO $I_2 = 4 \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \left[(1-x^2/a^2)^2 - 2y^2/b^2 \left(1 - x^2/a^2\right) + y^4/b^4 \right] dx\,dy$

$$= 4 \int_0^a \left[(1-x^2/a^2)^2 y - \frac{2y^3}{3b^2} (1 - x^2/a^2) + \frac{1}{5} \frac{y^5}{b^4} \right] \Bigg|_{y = b\sqrt{1-x^2/a^2}} dx$$

$$I_2 = 4b \int_0^a (1-x^2/a^2)^{5/2} \left[1 - 2/3 + 1/5 \right] dx = 4b \left(\frac{1}{3} + \frac{1}{5} \right) \int_0^a (1 - x^2/a^2)^{5/2} dx = \frac{4b \cdot 8}{15}\, a \int_0^{\pi/2} \cos^6\varphi \, d\varphi$$

THUS $I_2 = \dfrac{32\,ab}{15} \int_0^{\pi/2} \cos^6\varphi \, d\varphi = \dfrac{32\,ab}{15} \left(\dfrac{5\pi}{32} \right) = \dfrac{\pi ab}{3}$, SINCE $\int_0^{\pi/2} \cos^6\varphi \, d\varphi = \dfrac{5\pi}{32}$

THUS $\int_\Omega w^2 \, d\underline{x} = \pi ab/3$ AND $-\int_\Omega w \Delta w \, d\underline{x} = \pi \left(a/b + b/a \right)$.

WE CONCLUDE THAT $\lambda_0 \leq \dfrac{-\int_\Omega w \Delta w \, d\underline{x}}{\int_\Omega w^2 \, d\underline{x}} = \dfrac{3}{ab} \left(\dfrac{a}{b} + \dfrac{b}{a} \right)$

WHICH YIELDS $\lambda_0 \leq \left(\dfrac{3}{b^2} + \dfrac{3}{a^2} \right)$. NOTICE IF $a = b$ THEN $\lambda_0 \leq 6/a^2$ (CLOSE TO $5.76/a^2$)

PROBLEM 5 CONSIDER LOWEST EIGENVALUE λ_1 OF THE 2-D PROBLEM:

$$(*)\left\{\begin{array}{l} \Delta\phi + \lambda\phi = 0 \quad \text{IN} \quad 0<r<1 \\ \phi = 0 \text{ ON } r=1,\ \phi \text{ BOUNDED AS } r\to 0. \end{array}\right. \qquad \Omega = \text{UNIT DISK IN 2-D}$$

USE RAYLEIGH RITZ-METHOD AS DESCRIBED IN NOTES WITH $w_1 = 1-r^2$, $w_2 = (1-r^2)^2$, TO APPROXIMATE THE LOWEST EIGENVALUE OF $(*)$. NOTICE $\Delta\phi = \frac{1}{r}(r\phi_r)_r \rightarrow (r\phi_r)_r + \lambda r\phi = 0$

SOLUTION

$$\lambda_1 = \min_{w\in H} \frac{\int_0^1 r(w')^2\,dr}{\int_0^1 r\, w^2\,dr} \qquad H = \{ w \mid w=0 \text{ on } r=1,\ w \text{ is } C^2(\Omega)\}.$$

WE TAKE $w = c_1 w_1 + c_2 w_2$ WITH $w_1 = 1-r^2$, $w_2 = (1-r^2)^2$ AS TRIAL FUNCTIONS.
NOW FROM THE DERIVATION IN THE NOTES, $\lambda_1 \le \mu_{MIN}$

WHERE μ_{MIN} IS THE SMALLEST EIGENVALUE OF THE 2×2 MATRIX EIGENVALUE PROBLEM

$$(A - \mu B)\underline{c} = \underline{0} \qquad A = \begin{pmatrix} \int_0^1 r(w_1')^2\,dr & \int_0^1 r\, w_1' w_2'\,dr \\ \int_0^1 r\, w_1' w_2'\,dr & \int_0^1 r(w_2')^2\,dr \end{pmatrix}$$

WHERE
$$B = \begin{pmatrix} \int_0^1 r\, w_1^2\,dr & \int_0^1 r\, w_1 w_2\,dr \\ \int_0^1 r\, w_1 w_2\,dr & \int_0^1 r\, w_2^2\,dr \end{pmatrix}$$

NOW $w_1' = -2r$, $w_2' = -4r(1-r^2)$

WE OBTAIN
$$A = \begin{pmatrix} \int_0^1 4r^3\,dr & \int_0^1 8r^3(1-r^2)\,dr \\ \int_0^1 8r^3(1-r^2)\,dr & \int_0^1 16r^3(1-r^2)^2\,dr \end{pmatrix}, \quad B = \begin{pmatrix} \int_0^1 r(1-r^2)^2\,dr & \int_0^1 r(1-r^2)^3\,dr \\ \int_0^1 r(1-r^2)^3\,dr & \int_0^1 r(1-r^2)^4\,dr \end{pmatrix}$$

NOW WE MUST CALCULATE THE ENTRIES.

$$\int_0^1 4r^3\,dr = r^4\Big|_0^1 = 1 \quad ; \quad \int_0^1 (8r^3 - 8r^5)\,dr = 2r^4\Big|_0^1 - \frac{4}{3}r^6\Big|_0^1 = \frac{6}{3} - \frac{4}{3} = \frac{2}{3}$$

$$\int_0^1 16r^3(1-r^2)^2\,dr = 16\int_0^1 r^3(1-2r^2+r^4)\,dr = 16\left[\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right] = 16\left(\frac{6-8+3}{24}\right) = \frac{2}{3}.$$

THUS $A = \begin{pmatrix} 1 & 2/3 \\ 2/3 & 2/3 \end{pmatrix}.$

NOW
$$\int_0^1 r(1-r^2)^2\,dr = \int_0^1 r(1-2r^2+r^4)\,dr = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.$$

$$\int_0^1 r(1-r^2)^3\,dr = \int_0^1 r(1-3r^2+3r^4-r^6)\,dr = \frac{1}{2} - \frac{3}{4} + \frac{3}{6} - \frac{1}{8} = \frac{1}{8}.$$

$$\int_0^1 r(1-r^2)^4\,dr = -\frac{1}{2}\int_1^0 t^4\,dt = \frac{1}{2}\int_0^1 t^4\,dt = \frac{1}{10}. \qquad \text{THUS } B = \begin{pmatrix} 1/6 & 1/8 \\ 1/8 & 1/10 \end{pmatrix}.$$

let $t = 1-r^2$, $dt = -2r\,dr$

NOW WE MUST FIND EIGENVALUES μ OF

$$(A - \mu B)\underline{c} = \begin{pmatrix} 1-\mu/6 & 2/3-\mu/8 \\ 2/3-\mu/8 & 2/3-\mu/10 \end{pmatrix} \underline{c} = 0.$$

HENCE μ SATISFIES: $$(1-\mu/6)(2/3-\mu/10) - (2/3-\mu/8)^2 = 0.$$

THIS YIELDS THAT $$\mu^2/60 - \mu\left(1/9 + 1/10\right) + \frac{2}{3} - \frac{4}{9} - \frac{\mu^2}{64} + \frac{4}{3}\left(\frac{1}{8}\right)\mu = 0.$$

$$\mu^2\left(\frac{1}{60} - \frac{1}{64}\right) - \mu\left(\frac{1}{9} + \frac{1}{10} - \frac{1}{6}\right)\mu + \frac{2}{9} = 0.$$

$$\frac{\mu^2}{4}\left(\frac{1}{15} - \frac{1}{16}\right) - \mu\left(\frac{10+9-15}{90}\right) + \frac{2}{9} = 0.$$

$$\frac{\mu^2}{4}\,\frac{1}{240} - \frac{2\mu}{45} + \frac{2}{9} = 0.$$

THUS $$\frac{\mu^2}{960} - \frac{2\mu}{45} + \frac{2}{9} = 0. \longrightarrow \frac{\mu^2}{320} - \frac{2\mu}{15} + \frac{2}{3} = 0.$$

THUS $$\mu_\pm = \frac{\left[\frac{2}{15} \pm \sqrt{\left(\frac{2}{15}\right)^2 - 4\left(\frac{2}{3}\right)\left(\frac{1}{320}\right)}\,\right]}{(1/160)}$$

THIS YIELDS $$\mu_\pm = 160\left[\frac{2}{15} \pm \sqrt{\left(\frac{2}{15}\right)^2 - \frac{1}{120}}\,\right]$$

WE CALCULATE $\mu_- = 5.7845..$ $\mu_+ = 36.88$

THUS $\mu_{MIN} = 5.7845.$

THE EXACT VALUE FOR λ_1 IS $\lambda_1 = z_0^2$, WHERE $J_0(z_0) = 0$, z_0 FIRST ROOT WHICH IS $z_0 \cong 2.4048$ SO $\lambda_1 = 5.780306$. NOTICE THAT μ_{MIN} IS SUPRISINGLY GOOD APPROXIMATION TO λ_1 AND ONLY 2 TRIAL FUNCTIONS WERE USED.