EIGENVALUE PROBLEMS

CONSIDER EITHER THE WAVE EQUATION WITH p, q, r REAL-VALUED

$$r(\underline{x})\, u_{tt} = \nabla \cdot [p(\underline{x}) \nabla u] - q(\underline{x})\, u \qquad (1)$$

OR HEAT EQUATION

$$r(\underline{x})\, u_t = \nabla \cdot [p(\underline{x}) \nabla u] - q(\underline{x})\, u \qquad (2)$$

IN A BOUNDED DOMAIN Ω IN $\mathbb{R}^N$. ASSUME $r(\underline{x}) > 0$, $p(\underline{x}) > 0$ IN $\bar{\Omega}$. NOW IN (1) IF WE PUT $u(\underline{x},t) = e^{-i\omega t} \phi(\underline{x})$ THEN WE GET

$$\nabla \cdot [p(\underline{x}) \nabla \phi] - q(\underline{x})\, \phi = -\lambda\, r(\underline{x})\, \phi \quad \text{IN } \Omega \qquad (3)$$

WHERE $\lambda = \omega^2$. IF WE PUT $u = e^{-\lambda t} \phi(\underline{x})$ INTO (2) WE GET (3). HERE $\bar{\Omega} = \Omega \cup \partial\Omega$.

OUR GOAL NOW IS TO FIND ALL VALUES OF λ (CALLED EIGENVALUES) FOR WHICH

$$L\phi \equiv \nabla \cdot [p(\underline{x}) \nabla \phi] - q(\underline{x})\, \phi = -\lambda\, r(\underline{x})\, \phi \quad \text{IN } \Omega \qquad (4a)$$

WITH HOMOGENEOUS BOUNDARY CONDITIONS, HAS NONTRIVIAL SOLUTIONS (CALLED EIGENFUNCTIONS). THE TYPES OF BOUNDARY CONDITIONS ARE EITHER:

$$\phi = 0 \text{ ON } \partial\Omega \text{ (DIRICHLET)}; \quad \partial_n \phi = 0 \text{ ON } \partial\Omega \text{ (NEUMANN)}; \quad \partial_n \phi + b\phi = 0 \text{ ON } \partial\Omega \text{ (ROBIN)} \qquad (4b)$$

WE REFER TO (4a, b) AS THE BASIC EIGENVALUE PROBLEM. TECHNICALLY, WE ARE LOOKING FOR C^2 SOLUTIONS.

WE NOW DERIVE A USEFUL LEMMA BY GREEN'S IDENTITIES.

LEMMA $$\int_\Omega (v\,Lu - u\,Lv)\, d\underline{x} = \int_{\partial\Omega} p(\underline{x}) \left[v\, \partial_n u - u\, \partial_n v \right] ds. \qquad (5)$$

PROOF RECALL $\nabla \cdot [f \underline{F}] = \nabla f \cdot \underline{F} + f\, \nabla \cdot \underline{F}$.

THUS $\nabla \cdot [v\, p(\underline{x}) \nabla u] = \nabla v \cdot p \nabla u + v \nabla \cdot [p \nabla u]$

AND $\nabla \cdot [u\, p(\underline{x}) \nabla v] = \nabla u \cdot p \nabla v + u \nabla \cdot [p \nabla v]$

NOW SUBTRACT: $v \nabla \cdot [p \nabla u] - u \nabla \cdot [p \nabla v] = \nabla \cdot [v p \nabla u] - \nabla \cdot [u p \nabla v]$.

NOW ADDING AND SUBTRACTING $-quv$ WE GET

$$v[\nabla \cdot (p \nabla u) - q u] - u[\nabla \cdot (p \nabla v) - q v] = \nabla \cdot [v p \nabla u] - \nabla \cdot [u p \nabla v]$$

THUS USING DIV. THEOREM AND RECALLING L DEFINED IN (4a):

$$\int_\Omega (v\,Lu - u\,Lv)\, d\underline{x} = \int_\Omega \left(\nabla \cdot [v p \nabla u] - \nabla \cdot (u p \nabla v) \right) ds = \int_{\partial\Omega} \left(v p \frac{\partial u}{\partial n} - u p \frac{\partial v}{\partial n} \right) ds. \quad \square$$

REMARK! WE CAN ALSO ADD/SUBTRACT A TERM IN (5) TO GET

$$\int_\Omega (v\,Lu - u\,Lv)\, d\underline{x} = \int_{\partial\Omega} p(\underline{x}) \left[v\,(\partial_n u + b u) - u\,(\partial_n v + b v) \right] ds. \qquad (6).$$

PROPOSITION 1

(i) IF λ IS AN EIGENVALUE OF (4a) WITH EITHER OF THE BOUNDARY CONDITIONS IN (4b) THEN λ IS REAL-VALUED.

(ii) IF λ_i, λ_j ARE TWO DISTINCT EIGENVALUES WITH $\lambda_i \neq \lambda_j$, THEN $\int_\Omega r\,\phi_i \phi_j \, d\underline{x} = 0$ SO THAT ϕ_i, ϕ_j ARE ORTHOGONAL WRT WEIGHT FUNCTION $r(\underline{x})$.

PROOF THE PROOF IS SIMILAR TO THAT FOR STURM-LIOUVILLE PROBLEMS.

(i) WE HAVE $L\phi = -\lambda r \phi$ AND $L\bar{\phi} = -\bar{\lambda} r \bar{\phi}$.

WE USE (5) TO GET $\int_\Omega (\bar{\phi} L\phi - \phi L \bar{\phi})\, d\underline{x} = \int_{\partial\Omega} p[\bar{\phi}\, \partial_n \phi - \phi\, \partial_n \bar{\phi}]\, ds.$

NOW IF EITHER $\phi = 0$, OR $\partial_n \phi = 0$, OR $\partial_n \phi + b\phi = 0$ ON $\partial\Omega$ (WHERE WE USE (6))

WE OBTAIN $\int_\Omega \bar{\phi} L \phi \, d\underline{x} = \int_\Omega \phi L \bar{\phi}\, d\underline{x} \longrightarrow -\lambda \int_\Omega r \phi \bar{\phi}\, d\underline{x} = -\bar{\lambda} \int_\Omega r\phi\bar{\phi}\, d\underline{x}$.

THUS $(\lambda - \bar{\lambda}) \int_\Omega r|\phi|^2 d\underline{x} = 0$. HENCE SINCE $\phi \equiv 0 \longrightarrow \lambda = \bar{\lambda}. \to \lambda$ REAL.

(ii) $L\phi_j = -\lambda_j r \phi_j$ AND $L\phi_i = -\lambda_i r \phi_i$. WE GET IN SAME WAY THAT

$$\int_\Omega \phi_i L \phi_j \, d\underline{x} = \int_\Omega \phi_j L \phi_i \, d\underline{x} \longrightarrow (\lambda_i - \lambda_j) \int_\Omega r \phi_i \phi_j d\underline{x} = 0.$$

THUS SINCE $\lambda_i \neq \lambda_j \rightarrow \int_\Omega r\phi_i\phi_j\, d\underline{x} = 0$. "ORTHOGONALITY". $\square$

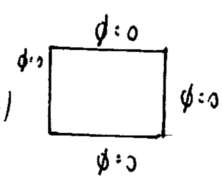
WE CAN THEN NORMALIZE EIGENFUNCTIONS SO THAT

$$(\phi_i, \phi_i) = 1 \qquad \text{WHERE} \quad (\phi_i, \phi_i) \equiv \int_\Omega r\,\phi_i^2 \, d\underline{x}. \quad (7)$$

REMARK 2 EIGENVALUES CAN, IN CERTAIN INSTANCES, BE DEGENERATE IN THAT FOR A SPECIFIED EIGENVALUE THERE MAY BE MORE THAN ONE INDEPENDENT EIGENFUNCTION. THESE EIGENFUNCTIONS CAN BE MADE ORTHOGONAL VIA A GRAM-SCHMIDT PROCESS.

TO ILLUSTRATE THIS CONSIDER THE 2-D PROBLEM

$$\left.\begin{aligned} &\Delta\phi + \lambda\phi = 0 \quad \text{IN} \quad 0 < x < \pi,\ 0 < y < \pi \\ &\phi = 0 \quad \text{ON ALL SIDES.} \end{aligned}\right\} (8)$$



THEN IT IS READILY VERIFIED THAT

$$\phi = \sin[nx]\sin[my] \quad \Big\} (8)$$

IS AN EIGENFUNCTION WITH EIGENVALUE $\lambda = m^2 + n^2$ WITH $m, n = 1, 2, 3, \ldots$.

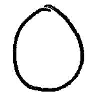
SUPPOSE THAT $\lambda = 5$. THEN EITHER $m = 1, n = 2$ OR $m = 2, n = 1$.

HENCE FOR $\lambda = 5$. $\phi = \text{SPAN}\,\{\sin(x)\sin(2y), \sin(2x)\sin(y)\}$ DEGENERATE.

ANOTHER WELL-KNOWN PROBLEM IS

$$\Delta\phi + \lambda\phi = 0 \quad \text{IN} \quad 0 < r < a$$

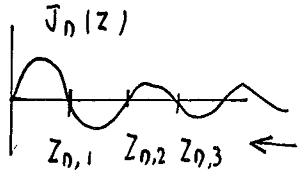
$$\phi = 0 \quad \text{ON} \quad r = a$$



IF WE SEPARATE VARIABLES, WITH $n = 0, 1, 2, \ldots$

$$(9) \left\{ \begin{array}{l} \phi = J_0(\sqrt{\lambda_{0,m}}\, r) \quad \text{WITH} \quad J_0(z_{0,m}) = 0, \quad \lambda_{0,m} = z_{0,m}^2/a^2, \quad m = 1, 2, \ldots \quad \text{(RADIALLY SYMMETRIC)} \\ \phi = J_n(\sqrt{\lambda_{n,m}}\, r)\left[A_n \cos n\varphi + B_n \sin(n\varphi)\right], \quad J_0(z_{n,m}) = 0, \quad \lambda_{n,m} = z_{n,m}^2/a^2 \end{array} \right.$$

HERE FOR <u>FIXED</u> $n \geq 1$ WE HAVE



WE CONCLUDE THAT THE EIGENVALUES FOR THE NON-RADIALLY SYMMETRIC EIGENFUNCTIONS ARE DEGENERATE IN THAT THERE ARE 2 SUCH EIGENFUNCTIONS: $J_n(\sqrt{\lambda_{n,m}}\, r)\cos(n\varphi)$ AND $J_n(\sqrt{\lambda_{n,m}}\, r)\sin(n\varphi)$. DEGENERACY HERE IS CAUSED BY SYMMETRY OF THE DOMAIN.

OVERALL, THE EIGENVALUES ARE $\lambda_{n,m} = \dfrac{z_{n,m}^2}{a^2}$ WITH $n = 0, 1, 2, \ldots$, $m = 1, 2, 3 \ldots$

THE LOWEST EIGENVALUE IS $z_{0,1} \cong 2.4..$

$$\left.\text{AND} \quad \lambda_{0,1} \cong 5.76/a^2 \right\} \quad (10)$$

NEXT, WE DERIVE AN EQUATION THAT CAN BE USED TO OBTAIN THE SIGN OF THE EIGENVALUE. WE WRITE

$$\lambda r(\underline{x})\phi = q(\underline{x})\phi - \nabla\cdot[p(\underline{x})\nabla\phi].$$

MULTIPLYING BY ϕ AND USING $\nabla\cdot[\phi p \nabla\phi] = p\nabla\phi\cdot\nabla\phi + \phi\nabla\cdot[p\nabla\phi]$

WE GET

$$\lambda r(\underline{x})\phi^2 = q(\underline{x})\phi^2 - \phi\nabla\cdot[p\nabla\phi] = q(\underline{x})\phi^2 + p|\nabla\phi|^2 - \nabla\cdot[\phi p\nabla\phi].$$

INTEGRATING OVER Ω:

$$\lambda\int_\Omega r(x)\phi^2\,d\underline{x} = \int_\Omega [p|\nabla\phi|^2 + q\phi^2]\,d\underline{x} - \int_\Omega \nabla\cdot[\phi p\nabla\phi]\,d\underline{s}.$$

USING THE DIVERGENCE THEOREM:

$$\lambda\int_\Omega r(\underline{x})\phi^2\,d\underline{x} = \int_\Omega [p|\nabla\phi|^2 + q\phi^2]\,d\underline{x} - \int_{\partial\Omega}\phi p \frac{\partial\phi}{\partial n}\,ds. \quad (11)$$

NOW IF EITHER $\phi = 0$ OR $\partial\phi/\partial n = 0$ ON $\partial\Omega$ WE HAVE

$$\lambda\int_\Omega r(\underline{x})\phi^2\,d\underline{x} = \int_\Omega [p|\nabla\phi|^2 + q\phi^2]\,d\underline{x} \quad (11a).$$

IF, ON THE OTHER HAND $\partial_n\phi + b\phi = 0$ ON $\partial\Omega$, (11) GIVES

$$\lambda\int_\Omega r(\underline{x})\phi^2\,d\underline{x} = \int_\Omega [p|\nabla\phi|^2 + q\phi^2]\,d\underline{x} + b\int_{\partial\Omega} p\phi^2\,ds. \quad (11b)$$

WE CONCLUDE THAT IF $r, p > 0$, $q \geq 0$ IN Ω THEN FROM (11a) WE HAVE $\lambda \geq 0$ FOR EITHER $\phi = 0$ OR $\partial_n \phi = 0$ ON $\partial\Omega$. WE CAN, HOWEVER, SAY MORE.

- SUPPOSE $q = 0$ IN (11a). THEN WE CLAIM $\lambda > 0$ IF $\phi = 0$ ON $\partial\Omega$. SUPPOSE, INSTEAD $\lambda = 0$. THEN $\nabla\phi = 0$ IN (11a) IN Ω AND THIS MEANS $\phi = C$ IN Ω. BUT $\phi = 0$ ON $\partial\Omega$ GIVES $C = 0$. BUT $\phi \equiv 0$ IN $\bar{\Omega}$ IS NOT AN EIGENFUNCTION. HENCE $\lambda > 0$ (STRICT INEQUALITY).
- SUPPOSE $q = 0$ IN (11a) AND $\partial_n \phi = 0$ ON $\partial\Omega$. THEN $\lambda = 0$ IS POSSIBLE FOR $\phi = C \neq 0$.

THE SAME COMMENT APPLIES TO (11b) FOR A ROBIN CONDITION.

THE UPSHOT IS THAT FOR $r, p > 0$ AND $q \geq 0$ WE HAVE $\lambda > 0$ FOR $\phi = 0$ ON $\partial\Omega$ AND $\partial_n \phi + b\phi = 0$ ON $\partial\Omega$. ONLY IN THE CASE WHERE $q = 0$ AND $\partial_n \phi = 0$ ON $\partial\Omega$ IS $\lambda = 0$ AN EIGENVALUE.

NEXT, WE WILL ESTABLISH A VARIATIONAL PRINCIPLE TO CHARACTERIZE THE EIGENPAIRS OF

$$\left.\begin{aligned} L\phi = \nabla\cdot[p(\underline{x})\nabla\phi] - q(\underline{x})\phi = -\lambda r(\underline{x})\phi \quad \text{IN } \Omega \\ \text{WITH} \qquad \phi = 0 \quad \text{ON } \partial\Omega. \end{aligned}\right\} \quad (12)$$

WE WILL FIRST CONSIDER THIS DIRICHLET PROBLEM AND THEN DISCUSS THE MODIFICATION NEEDED FOR THE NEUMANN $\partial_n \phi = 0$ ON $\partial\Omega$ AND ROBIN CONDITION $\partial_n \phi + b\phi = 0$. LAPLACIAN EIGENVALUES OBTAINED BY SETTING $p = 1$, $q = 0$, $r = 1$ IN (12).

MOTIVATED BY (11a) WE DEFINE THE <u>RAYLEIGH QUOTIENT</u> R FOR ANY C^2 FUNCTION $u(\underline{x})$ BY

$$R \equiv \frac{E(u)}{(u, u)} \qquad (13a)$$

WHERE THE ENERGY INTEGRAL $E(u) \equiv -\int_\Omega u\, Lu \, d\underline{x} = \int_\Omega [p(\underline{x})|\nabla u|^2 + q(\underline{x}) u^2]\, d\underline{x}$ (13b)

AND $(u, u) \equiv \int_\Omega u^2 r \, d\underline{x}$. (13c).

NOW FROM (11a) IF WE TAKE $u = \phi$, WHERE ϕ IS AN EIGENFUNCTION OF (12) THEN $R \equiv \lambda$, THE CORRESPONDING EIGENVALUE. WE NOW ASK WHETHER (13) CAN BE USED IN A VARIATIONAL PRINCIPLE TO CHARACTERIZE THE EXISTENCE OF EIGENPAIRS.

<u>THEOREM 1</u> (MINIMUM EIGENVALUE : VARIATIONAL PRINCIPLE). "<u>DIRICHLET BC</u>"

LET $H_0 = \{ w \mid w \in C^2(\Omega),\ w = 0 \text{ on } \partial\Omega,\ w \not\equiv 0 \}$ BE A SET OF TRIAL FUNCTIONS. SUPPOSE THAT THERE EXISTS A FUNCTION $u \in H_0$ SUCH THAT u MINIMIZES

$$R \equiv \frac{E(w)}{(w,w)} \quad \text{OVER } \underline{\text{ALL}} \text{ TRIAL FUNCTIONS } w \in H_0.$$

THAT IS

$$m = \frac{E(u)}{(u,u)} = \min_{w \in H_0} \frac{E(w)}{(w,w)}. \qquad (14)$$

THEN m IS THE FIRST (OR LOWEST EIGENVALUE) OF (12). THAT IS $m = \lambda_1$, AND u IS THE CORRESPONDING EIGENFUNCTION ϕ_1.

<u>PROOF</u> SUPPOSE u IS THE MINIMIZER OF THE RAYLEIGH QUOTIENT AND m IS THE RAYLEIGH QUOTIENT OF u, i.e.

$$m = \frac{\int_\Omega [\, p|\nabla u|^2 + q u^2 \,]\, d\underline{x}}{\int_\Omega r u^2 \, d\underline{x}} \equiv \frac{E(u)}{(u,u)}$$

PICK ANY FUNCTION $v \in H_0$. LET $w = u + \sigma v \in H_0$. AND DEFINE $f(\sigma)$ BY

$$f(\sigma) \equiv \frac{\int_\Omega [\, p|\nabla(u+\sigma v)|^2 + q(u+\sigma v)^2 \,]\, d\underline{x}}{\int_\Omega r (u+\sigma v)^2 \, d\underline{x}} \equiv \frac{\tilde{E}(\sigma)}{G(\sigma)}.$$

NOW IF u MINIMIZES RAYLEIGH QUOTIENT, THEN WE MUST HAVE $f'(0) = 0$.

FROM QUOTIENT RULE

$$f'(0) = \left(\frac{\tilde{E}'(\sigma)}{G(\sigma)} - \frac{E(\sigma)\,\tilde{G}'(\sigma)}{(G(\sigma))^2} \right)\Bigg|_{\sigma=0} = 0. \qquad (15)$$

WE CALCULATE :

$$G(0) = (u,u), \qquad G'(0) = 2\int_\Omega r u v \, d\underline{x}$$

$$\hat{E}(0) = \int_\Omega (p|\nabla u|^2 + q u^2)\, d\underline{x}, \qquad \hat{E}'(0) = \int_\Omega (2p\nabla u \cdot \nabla v + 2 q u v)\, d\underline{x}.$$

NOW WE SUBSTITUTE INTO (15) TO OBTAIN

$$f'(0) = \frac{2\int_\Omega (p \nabla u \cdot \nabla v + q u v)\, d\underline{x}}{(u,u)} - \frac{2\hat{E}(0)\int_\Omega r u v \, d\underline{x}}{(u,u)^2}$$

$$f'(0) = \frac{2}{(u,u)}\left[\int_\Omega (p \nabla u \cdot \nabla v + q u v)\, d\underline{x} - \frac{\tilde{E}(0)}{(u,u)} \int_\Omega r u v \, d\underline{x} \right]. \qquad (16)$$

NOW WE WILL SIMPLIFY THE FIRST TERM IN (16).

$$\nabla \cdot [p v \nabla u] = p \nabla u \cdot \nabla v + v \nabla \cdot [p \nabla u].$$

SO $$\int_\Omega (p \nabla u \cdot \nabla v + q u v)\, d\underline{x} = \int_\Omega v[-\nabla \cdot (p \nabla u) + q u]\, d\underline{x} + \int_\Omega \nabla \cdot (p v \nabla u)\, d\underline{x}$$

NOW USE DIV. THEOREM:

$$\int_\Omega (p \nabla u \cdot \nabla v + q u v)\, d\underline{x} = \int_\Omega [-\nabla \cdot (p \nabla u) + q u] v \, d\underline{x} + \int_{\partial\Omega} p v \frac{\partial u}{\partial n}\, ds \qquad (17)$$

SINCE $v \in H_0$, THEN $v = 0$ ON $\partial\Omega$ AND THE LAST TERM VANISHES. SUBSTITUTE INTO (16) WE OBTAIN

$$f'(0) = \frac{2}{(u,u)}\left[\int_\Omega [-\nabla \cdot (p \nabla u) + q u] v \, d\underline{x} - \frac{\hat{E}(0)}{(u,u)} \int_\Omega r u v \, d\underline{x} \right]$$

NOW $\dfrac{\hat{E}(0)}{(u,u)} \equiv \dfrac{E(u)}{(u,u)}$. COMBINE ALL UNDER ONE INTEGRAL SIGN TO GET

$$f'(0) = \frac{2}{(u,u)}\left[\int_\Omega \left[-\nabla \cdot (p \nabla u) + q u - \frac{E(u)}{(u,u)} r u \right] v \, d\underline{x}. \qquad (18).\right.$$

WE NOW NEED A LEMMA FOR THE NEXT STEP.

<u>LEMMA 1</u> SUPPOSE THAT $\int_\Omega f(\underline{x})\, v(\underline{x})\, d\underline{x} = 0$ FOR ALL C^2 FUNCTIONS $v(\underline{x})$ WHERE $f(\underline{x})$ IS CONTINUOUS IN Ω. THEN $f(\underline{x}) \equiv 0$ IN Ω.

<u>PROOF</u> ASSUME TO THE CONTRARY THAT $f(\underline{x}_0) \neq 0$ FOR SOME $\underline{x}_0$ IN Ω. THEN, WLOG, $f(\underline{x}_0) > 0$ AND BY CONTINUITY $f(\underline{x}) > 0$ IN SOME NEIGHBOURHOOD $|\underline{x} - \underline{x}_0| < \sigma$ WITH $\sigma > 0$ IN Ω. LET $v(\underline{x})$ BE A FUNCTION WITH $v > 0$ FOR $|\underline{x} - \underline{x}_0| < \sigma$ WITH $v = v_n = 0$ ON $|\underline{x} - \underline{x}_0| = \sigma$ AND $v \equiv 0$ OUTSIDE $|\underline{x} - \underline{x}_0| \geq \sigma$ BUT IN Ω.

THEN $\int_\Omega f(\underline{x})\, v(\underline{x})\, d\underline{x} > 0 \rightarrow$ CONTRADICTION.

USING THIS LEMMA ON (18) WE OBTAIN THAT

$$\nabla \cdot (p \nabla u) - q u + \frac{E(u)}{(u,u)} r u = 0 \quad \text{IN } \Omega$$

WITH $\quad u = 0 \quad$ ON $\partial\Omega$.

THUS $\quad m = \dfrac{E(u)}{(u,u)}$ IS AN EIGENVALUE OF L WITH EIGENFUNCTION $\phi = u$.

IT REMAINS TO SHOW THAT m IS THE SMALLEST EIGENVALUE. SUPPOSE ψ_i IS ANOTHER EIGENFUNCTION OF (12) WITH EIGENVALUE λ_i. SINCE $\psi_i \in H_0$ WE HAVE

$$m = \frac{E(u)}{(u,u)} \leq \frac{E(\psi_i)}{(\psi_i,\psi_i)} = -\frac{\int \psi_i L \psi_i \, d\underline{x}}{\int \psi_i^2 r \, dx} = \frac{\int + \psi_i (\lambda r \psi_i) \, d\underline{x}}{\int \psi_i^2 r \, dx} = \lambda_i . \qquad \square .$$

THEREFORE, FOR THE DIRICHLET PROBLEM (12), THE <u>MINIMUM</u> EIGENVALUE IS CHARACTERIZED BY THE SMALLEST VALUE $\displaystyle\min_{u \in H_0} \frac{E(u)}{(u,u)}$.

NEXT, WE DISCUSS HOW TO MODIFY THE THEORY FOR THE CASE OF A ROBIN BOUNDARY CONDITION

$$\frac{\partial \phi}{\partial n} + b \phi = 0 \quad \text{ON } \partial\Omega.$$

FROM (11b) WE NOW DEFINE THE RAYLEIGH QUOTIENT AS

$$R_b = \frac{E_b(u)}{(u,u)} \quad \text{WHERE} \quad E_b(u) = \int_\Omega (p |\nabla u|^2 + q u^2) \, d\underline{x} + \int_{\partial\Omega} b p u^2 \, ds. \tag{19}$$

<u>THEOREM 2</u> (MINIMUM EIGENVALUE: ROBIN BOUNDARY CONDITION)

LET $H = \{ w \mid w \in C^2(\Omega), \, w \neq 0 \}$ BE A SET OF TRIAL FUNCTIONS. SUPPOSE $\exists$ A FUNCTION $u \in H$ SUCH THAT u MINIMIZES R_b OVER <u>ALL</u> $w \in H$, THAT IS

$$m_b = \frac{E_b(u)}{(u,u)} = \min_{w \in H} \frac{E_b(w)}{(w,w)} . \tag{20}$$

THEN m_b IS THE LOWEST EIGENVALUE OF $L\phi = -\lambda r \phi$ WITH $\partial\phi/\partial n + b\phi = 0$ ON $\partial\Omega$, AND u IS THE CORRESPONDING FIRST EIGENFUNCTION.

REMARK 3 (KEY POINT) NOTICE THAT FOR THE DIRICHLET CASE WITH $\phi = 0$ ON $\partial\Omega$, THE ADMISSIBLE TRIAL FUNCTIONS W MUST VANISH ON $\partial\Omega$, I.e. $W \in H_0$. FOR THE ROBIN PROBLEM NO BOUNDARY CONDITIONS FOR W ON $\partial\Omega$ ARE NEEDED.

THUS $$H_0 \subset H . \qquad (21).$$

PROOF WE CAN REPEAT ESSENTIALLY VERBATIM THE PROOF OF THEOREM 1 TO GET FOR THE MINIMIZER U AND ANY $V \in H$ THAT

$$f'(0) = \frac{2}{(u,u)}\left[\int_\Omega (p\nabla u\cdot\nabla v + q\,u\,v)\,d\underline{x} - \frac{E_b(u)}{(u,u)}\int_\Omega r\,u\,v\,d\underline{x} + \int_{\partial\Omega} b\,p\,u\,v\,ds\right]$$

NOW USING (17) WE OBTAIN

$$f'(0) = \frac{2}{(u,u)}\left[\int_\Omega \left[-\nabla\cdot(p\nabla u) + q\,u - \frac{E_b(u)}{(u,u)}\, r\,u\right] v\,d\underline{x} + \int_{\partial\Omega} p\left(\frac{\partial u}{\partial n} + b\,u\right) v\,ds\right]$$

NOW A NECESSARY CONDITION FOR U TO BE A MINIMIZER IS THAT $f'(0) = 0$. I.e. THAT

$$\int_\Omega \left[-\nabla\cdot(p\nabla u) + q\,u - \frac{E_b(u)}{(u,u)}\, r\,u\right] v\,d\underline{x} + \int_{\partial\Omega} p\left(\frac{\partial u}{\partial n} + b\,u\right) v\,ds = 0 . \qquad (22)$$

FOR ALL $v \in H$.

NOW IN (22) LET V BE ANY FUNCTION IN H WITH $V = 0$ ON $\partial\Omega$. THEN THE SECOND TERM IN (22) VANISHES AND FROM THE FIRST TERM AND LEMMA 1 WE CONCLUDE THAT

$$\nabla\cdot[p\nabla u] - q\,u + \frac{E_b(u)}{(u,u)}\, r\,u = 0 \quad \text{IN } \Omega .$$

THEN WE ARE LEFT WITH $\int_{\partial\Omega} p\left(\partial u/\partial n + u\,b\right) v\,ds = 0$ AS A CONDITION. SINCE THIS MUST HOLD FOR ALL $V \in H$ (WITHOUT V SATISFYING ANY CONDITION ON $\partial\Omega$) FROM A LEMMA 1 ARGUMENT WE MUST HAVE THAT $\partial u/\partial n + b\,u = 0$ ON $\partial\Omega$ WHEN U MINIMIZES R_b.

HENCE U SATISFIES

$$L u = -m_b\, r\, u \quad \text{IN } \Omega$$
$$\partial_n u + b\,u = 0 \quad \text{ON } \partial\Omega$$

WITH $m_b = E_b(u)/(u,u)$, AND m_b SMALLEST EIGENVALUE OF $L\phi = -\lambda r\phi$ IN Ω, $\partial_n\phi + b\phi = 0$ ON $\partial\Omega$

REMARK 4: FROM OUR ANALYSIS OF (22) IT FOLLOWS THAT THE MINIMIZER u MUST AUTOMATICALLY SATISFY THE CONDITION $\partial u/\partial n + bu = 0$ ON $\partial\Omega$.

THERE IS NO NEED TO IMPOSE THIS CONDITION IN THE SPACE OF TRIAL FUNCTIONS FOR V. IN THIS SENSE $\partial u/\partial n + bu = 0$ ON $\partial\Omega$ IS A "NATURAL BOUNDARY CONDITION" FOR THE VARIATIONAL PROBLEM

$$\min_{w \in H} \frac{E_b(w)}{(w,w)} . \qquad E_b(w) = \int_\Omega (p|\nabla w|^2 + q w^2)\, d\underline{x} + \int_{\partial\Omega} b p w^2 \, dS.$$

THE MINIMIZER u WILL AUTOMATICALLY SATISFY THE CONDITION $\partial u/\partial n + bu = 0$ ON $\partial\Omega$.

NOW WE GIVE A TRIVIAL, BUT STILL SLICK, PROOF OF A SIMPLE INEQUALITY.

<u>THEOREM</u> LET λ_1 BE THE MINIMUM EIGENVALUE OF $L\phi = -\lambda r \phi$ FOR THE DIRICHLET PROBLEM, $\phi = 0$ ON $\partial\Omega$. LET μ_1 BE THE MINIMUM EIGENVALUE OF $L\phi = -\lambda r\phi$ FOR THE NEUMANN PROBLEM, $\partial_n \phi = 0$ ON $\partial\Omega$.

THEN $$\mu_1 \le \lambda_1 .$$

<u>PROOF</u> WHEN $b = 0$, WE HAVE THAT $E_b(w) = \int_\Omega (p|\nabla w|^2 + q w^2)\, d\underline{x} = E(w)$.

THE VARIATIONAL PRINCIPLE IS

$$\lambda_1 = \min_{w \in H_0} \frac{E(w)}{(w,w)} \quad \text{AND} \quad \mu_1 = \min_{w \in H} \frac{E(w)}{(w,w)} .$$

SINCE $H_0 \subset H$ THEN H IS A LARGER SET OF FUNCTIONS THAN H_0 AND CONSEQUENTLY $\mu_1 \le \lambda_1$. ▯

THE RAYLEIGH QUOTIENT IS USEFUL FOR OBTAINING GOOD UPPER BOUNDS ON THE LOWEST EIGENVALUE λ_1 OF EITHER

$$L\phi = -\lambda r \phi \quad \text{IN } \Omega ; \quad \phi = 0 \qquad (23a)$$

OR

$$L\phi = -\lambda r \phi \quad \text{IN } \Omega ; \quad \partial_n \phi + b\phi = 0. \qquad (23b)$$

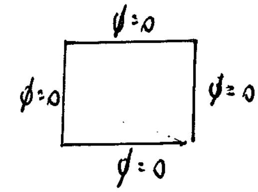
IN THE FIRST CASE, JUST PICK ANY "REASONABLE" TRIAL FUNCTION ϕ^* WITH $\phi^* = 0$ ON $\partial\Omega$. THEN $\lambda_1 \le E(\phi^*)/(\phi^*, \phi^*)$.

FOR (23b), JUST PICK ANY TRIAL FUNCTION ϕ^* (NOT NECESSARILY SATISFYING $\partial_n \phi^* + b\phi^* = 0$ ON $\partial\Omega$), AND THEN

$$\lambda_1 \le \frac{E_b(\phi^*)}{(\phi^*, \phi^*)} .$$

EXAMPLE 1 USING THE TRIAL FUNCTION $w^* = xy(\pi - x)(\pi - y)$ FIND AN UPPER BOUND ON THE LOWEST EIGENVALUE OF

$$\Delta \phi + \lambda \phi = 0 \quad \text{IN} \quad 0 < x < \pi, \; 0 < y < \pi$$
$$\phi = 0 \quad \text{ON} \quad \partial\Omega.$$



SOLUTION $\quad \lambda_1 = \min\limits_{w \in H_0} \dfrac{E(w)}{(w,w)} \leq \dfrac{E(w^*)}{(w^*, w^*)}$ SINCE $w^* \in H_0$ (VANISHES ON $\partial\Omega$).

NOW $p = 1$, $q = 0$, $r = 1$ SO THAT

$$\frac{E(w^*)}{(w^*, w^*)} = \frac{\int_0^\pi \int_0^\pi |\nabla w^*|^2 \, dx\, dy}{\int_0^\pi \int_0^\pi w^{*2} \, dx\, dy}.$$

NOW $\displaystyle\int_0^\pi \int_0^\pi w^{*2} \, dx\, dy = \left[\int_0^\pi y^2 (\pi - y)^2 \, dy \right]^2 = \left(\pi^5/30 \right)^2.$

NOW $\quad w_x^* = y(\pi - y)[\pi - 2x] \qquad w_y^* = x(\pi - x)[\pi - 2y].$

WE CALCULATE $\quad \dfrac{E(w^*)}{(w^*, w^*)} = \dfrac{900}{\pi^{10}} \displaystyle\int_0^\pi \int_0^\pi \left[x^2(\pi - x)^2(\pi - 2y)^2 + y^2(\pi - y)^2(\pi - 2x)^2 \right] dx\,dy$

WE OBTAIN, AFTER TEDIOUS EVALUATION, THAT $\quad \dfrac{E(w^*)}{(w^*, w^*)} = 2.03.$

THUS $\quad \lambda_1 \leq 2.03.$

HOWEVER, ON PAGE 2 EQUATION (8) WE KNOW THAT $\phi_1 = \sin(\;x)\sin(y)$ AND $\lambda_1 = 2$. THE BOUND 2.03 IS VERY CLOSE.

EXAMPLE 2 USING TRIAL FUNCTIONS FIND AN UPPER BOUND ON THE LOWEST EIGENVALUE OF

$$\Delta \phi + \lambda \phi = 0 \quad \text{IN } \Omega$$
$$\phi = 0 \quad \text{ON } \partial\Omega$$

WHEN Ω IS THE UNIT DISK.

SOLUTION ON PAGE 3 IN EQUATION (9) WE SHOWED THE LOWEST EIGENVALUE IS

$\lambda_1 = z_1^2$ WHERE $J_0(z_1) = 0$ AND z_1 SMALLEST ROOT OF $J_0(z) = 0$ GIVEN BY $z_1 \doteq 2.4$. SO $\lambda_1 \doteq 5.76$ AND $\phi_1 = J_0(\sqrt{\lambda_1}\, r)$.

NOW CHOOSE TWO TRIAL FUNCTIONS

(i) $\; w^* = 1 - r^2 \qquad$ (ii) $\; w^* = \cos(\pi r/2)$.

BOTH SATISFY $w^* \in H_0$ SINCE $w^* = 0$ ON $r=1$ IN BOTH CASES. THUS $\lambda_1 \le \frac{E(w^*)}{(w^*,w^*)} = \frac{\int_\Omega |\nabla w^*|^2 dx}{\int_\Omega w^{*2} dx}$ (11)

FOR (i) WE USE POLAR COORDINATES AND CALCULATE

$$\lambda_1 \le \frac{\int_0^1 \int_0^{2\pi} r (w_r^*)^2 \, dr \, d\varphi}{\int_0^1 \int_0^{2\pi} (w^*)^2 \, r \, dr \, d\varphi} = \frac{\int_0^1 r (w_r^*)^2 dr}{\int_0^1 r (w^*)^2 dr} = \frac{\int_0^1 r [4r^2] dr}{\int_0^1 r (1-r^2)^2 dr} = 6.$$

THUS FOR (i) WE HAVE $\lambda_1 \le 6$.

NOW FOR (ii) WE HAVE $w_r^* = -\pi/2 \sin(\pi r/2)$.

THEN $$\lambda_1 \le \frac{\int_0^1 \frac{\pi^2}{4} r \sin^2\left(\frac{\pi r}{2}\right) dr}{\int_0^1 r \cos^2(\pi r/2) \, dr} = \frac{\pi^2}{4}\left[\frac{1/4 + 1/\pi^2}{1/4 - 1/\pi^2}\right] = 5.803.$$

THIS GIVES $\lambda_1 \le 5.8303$, WHICH IS A BETTER UPPER BOUND SINCE $\lambda_1 = 5.76\ldots$ (EXACT).

EXAMPLE 3 FIND AN UPPER BOUND ON THE LOWEST EIGENVALUE OF

$$[(1+x)\phi']' + \lambda\phi = 0 \quad \text{WITH} \quad \phi(0) = \phi(1) = 0.$$

SOLUTION RAYLEIGH'S QUOTIENT WORKS IN 1-D SINCE WE CAN MAKE A 2-D EIGENVALUE PROBLEM HAVE SOLUTIONS INDEPENDENT OF SECOND DIMENSION, OR WE CAN DO A 1-D THEORY DIRECTLY. THE MINIMUM EIGENVALUE λ_1 IS

$$\lambda_1 = \min_{w \in H_0} \frac{E(w)}{(w,w)} \quad \text{WHERE} \quad E(w) = \int_0^1 (1+x) w'^2 dx, \quad (w,w) = \int_0^1 w^2 dx.$$

LET $w^* = x(1-x)$, THEN $$\lambda_1 \le \frac{E(w^*)}{(w^*,w^*)} = \frac{\int_0^1 (1+x)(1-2x)^2 dx}{\int_0^1 (x-x^2)^2 dx} = \frac{\int_0^1 (1+x)(1-4x+4x^2) dx}{\int_0^1 x^2(1-2x+x^2) dx}.$$

WE CALCULATE $\lambda_1 \le 15$.

NEXT, WE WILL DERIVE A VARIATIONAL PRINCIPLE TO CHARACTERIZE THE HIGHER EIGENVALUES. WE WILL FIRST CONSIDER THE DIRICHLET PROBLEM

$$\left.\begin{aligned} L\phi = \nabla\cdot[p(x)\nabla\phi] - q(x)\phi &= -\lambda r(x)\phi \quad \text{IN } \Omega \quad \text{WITH } p, r > 0 \text{ IN } \bar\Omega \\ \phi &= 0 \quad \text{ON } \partial\Omega \end{aligned}\right\} \quad (24)$$

THEOREM 3 (MINIMUM PRINCIPLE FOR n^{th} EIGENVALUE: DIRICHLET BC).

FIX AN INTEGER $n \geq 2$. LET $\phi_1, \ldots, \phi_{n-1}$ BE THE FIRST $n-1$ EIGENFUNCTIONS OF (24). WLOG, ASSUME THEY ARE NORMALIZED AS $(\phi_i, \phi_i) = \int_\Omega r \phi_i^2 \, d\underline{x} = 1$. DEFINE

$$H_{n-1} \equiv \{ w \mid w \in C^2(\Omega),\ w = 0 \text{ ON } \partial\Omega,\ w \not\equiv 0,\ (w, \phi_i) = 0,\ i = 1, \ldots, n-1 \}.$$

SUPPOSE $\exists$ A FUNCTION $u_n \in H_{n-1}$ WHICH MINIMIZES THE RAYLEIGH QUOTIENT R OVER ALL $w \in H_{n-1}$; THAT IS SUPPOSE

$$m_n = \frac{E(u_n)}{(u_n, u_n)} = \min_{w \in H_{n-1}} \frac{E(w)}{(w, w)} \qquad E(w) \equiv \int_\Omega (p|\nabla w|^2 + q w^2)\, d\underline{x}$$

$$(w, w) = \int_\Omega r w^2 \, d\underline{x}.$$

THEN m_n IS THE n^{th} EIGENVALUE OF (24), i.e. $\lambda_n = m_n$ AND $u_n = \phi_n$, THE nTH EIGENFUNCTION.

PROOF WE PROCEED AS IN THE PROOF OF THEOREM 1. PICK ANY $v \in H_{n-1}$ AND DEFINE

$$f(\sigma) = \frac{\int_\Omega [\, p\,|\nabla(u_n + \sigma v)|^2 + q(u_n + \sigma v)^2]\, d\underline{x}}{\int_\Omega r (u_n + \sigma v)^2 \, d\underline{x}} = \frac{\hat{E}(\sigma)}{G(\sigma)}.$$

SETTING $f'(\sigma) = 0$ WE OBTAIN AS IN EQUATION (18) THAT WE MUST HAVE

$$\int_\Omega [-\nabla\cdot(p\nabla u_n) + q u_n - m_n r u_n]\, v \, d\underline{x} = 0 \qquad \forall\ v \in H_{n-1} \qquad (25)$$

TO PROVE THAT THIS IMPLIES THAT

$$-\nabla\cdot[p\nabla u_n] + q u_n - m_n r u_n = 0$$

WE MUST PROVE THAT (25) IS TRUE $\forall\ v \in H_0$ (NOT JUST FOR THOSE TRIAL FUNCTIONS THAT ARE ORTHOGONAL TO THE FIRST $n-1$ EIGENFUNCTIONS).

NOW LET $h(\underline{x}) \in H_0 \equiv \{ w \mid w \in C^2(\Omega),\ w = 0 \text{ ON } \partial\Omega,\ w \not\equiv 0 \}$. DEFINE

$$v = h(\underline{x}) - \sum_{k=1}^{n-1} c_k \phi_k(\underline{x}) \quad \text{WITH} \quad c_k = (h, \phi_k) \quad \text{WITH} \quad (\phi_k, \phi_k) = 1 \text{ NORMALIZED.}$$

WE CALCULATE

$$(v, \phi_j) = (h, \phi_j) - \sum_{k=1}^{n-1} c_k (\phi_k, \phi_j) = (h, \phi_j) - c_j(\phi_j, \phi_j) = (h, \phi_j) - (h, \phi_j) = 0$$

FOR $j = 1, \ldots, n-1$ BY ORTHOGONALITY.

THUS $v \in H_{n-1}$. AS SUCH $f'(0) = 0 \longrightarrow \int_\Omega [-\nabla\cdot(p\nabla u_n) + q u_n - m_n r u_n]\, v\, d\underline{x} = 0$ FOR ALL $v \in H_{n-1}$, IMPLIES THAT THIS MUST HOLD FOR $v = h(x) - \sum_{k=1}^{n-1} c_k \phi_k(\underline{x})$.

THUS $\int_\Omega [-\nabla\cdot(p\nabla u_n)+q u_n - m_n r u_n]\, h(x)\, d\underline{x} = \sum_{k=1}^{n-1} c_k \int_\Omega [-\nabla\cdot(p\nabla u_n)+q u_n - m_n r u_n]\, \phi_k\, d\underline{x}.$ (26).

ON (26) WE USE GREEN'S IDENTITY: SINCE $u_n \in H_{n-1}$ AND $\phi_k = 0$ ON $\partial\Omega$ WE GET

$$\begin{aligned}\int_\Omega [-\nabla\cdot(p\nabla u_n) + q u_n - m_n r u_n]\phi_k\, d\underline{x} &= \int_\Omega [-L u_n \phi_k]\, d\underline{x} - \int_\Omega m_n r u_n \phi_k\, d\underline{x}\\ &= \int_\Omega -L\phi_k\, u_n\, d\underline{x} - \int_\Omega m_n r u_n \phi_k\, d\underline{x}\\ &= \int_\Omega \lambda_k r \phi_k u_n\, d\underline{x} - \int_\Omega m_n r u_n \phi_k\, dx\\ &= (\lambda_k - m_n)\int_\Omega r\phi_k u_n\, dx = 0 \quad \text{FOR } k=1,\dots,n-1\end{aligned}$$

SINCE $u_n \in H_{n-1}$. AS A RESULT, THE RIGHT SIDE OF (26) VANISHES AND

$$\int_\Omega [-\nabla\cdot(p\nabla u_n) + q u_n - m_n r u_n]\, h\, d\underline{x} = 0 \quad \forall\, h \in H_0.$$

AS A RESULT, BY LEMMA I WE HAVE THAT

$$\left.\begin{aligned}\nabla\cdot[p\nabla u_n] - q u_n &= m_n r u_n \quad \text{IN } \Omega.\\ u_n &= 0 \quad \text{ON } \partial\Omega\end{aligned}\right\} \quad (27)$$

THEREFORE, u_n IS AN EIGENFUNCTION WITH EIGENVALUE m_n. FROM $u_n \in H_{n-1}$, IT FOLLOWS THAT THIS EIGENFUNCTION IS ORTHOGONAL TO $\phi_1,\dots,\phi_{n-1}$. THE SMALLEST RAYLEIGH QUOTIENT MUST BE $\lambda_n = m_n$, $\phi_n = u_n$ IN THIS SUBSPACE.

<u>REMARK 5</u> SINCE $H_{n-1} \subset H_{n-2} \subset \dots H_1 \subset H_0$ IT FOLLOWS THAT $m_n \geq \lambda_{n-1} \geq \lambda_{n-2} \geq \dots \geq \lambda_1$. IN THIS WAY WE OBTAIN THE ORDERING PRINCIPLE

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_{n-1} \leq \lambda_n \leq \dots \quad (28).$$

<u>REMARK 6</u> THIS VARIATIONAL PRINCIPLE IS USEFUL FOR PROVING THE EXISTENCE OF A COUNTABLY INFINITE SET OF EIGENVALUES (IN TERMS OF A VARIATIONAL PROBLEM), HOWEVER IT HAS A DISADVANTAGE FOR THE HIGHER EIGENVALUES THAT IT IS INTRACTABLE TO ANALYTICALLY CALCULATE THE TRIAL SET H_{n-1} SINCE $\phi_1,\dots,\phi_{n-1}$ ARE IN GENERAL NOT EXPLICITLY AVAILABLE.

FOR THE CASE OF ROBIN BOUNDARY CONDITIONS $\partial_n\phi + b\phi = 0$ ON $\partial\Omega$ WE NEED ONLY A SLIGHT MODIFICATION OF THEOREM 3.

THEOREM 4 (MINIMUM PRINCIPLE FOR n^{th} EIGENVALUE : ROBIN BC)

FIX AN INTEGER $n \geq 2$. LET $\phi_1, \ldots, \phi_{n-1}$ BE THE FIRST $n-1$ EIGENFUNCTIONS OF (24) WITH $\partial_n \phi + b\phi = 0$ ON $\partial\Omega$. WLOG, ASSUME $(\phi_i, \phi_i) = 1 \ \forall i = 1, \ldots, n-1$.

DEFINE
$$H_{n-1}^* = \{ w \mid w \in C^2(\Omega),\ w \not\equiv 0 \text{ IN } \Omega,\ (w, \phi_i) = 0,\ i = 1, \ldots, n-1 \}.$$

SUPPOSE $\exists$ A FUNCTION $u_n \in H_{n-1}^*$ WHICH MINIMIZES THE RAYLEIGH QUOTIENT R_b OVER ALL $w \in H_{n-1}^*$; THAT IS SUPPOSE
$$m_n^* = \frac{E_b(u_n)}{(u_n, u_n)} = \min_{w \in H_{n-1}^*} \frac{E_b(w)}{(w,w)}, \qquad E_b(w) \equiv \int_\Omega (p|\nabla w|^2 + q w^2)\, d\underline{x} + b \int_{\partial\Omega} p w^2 \, ds.$$
$$(w,w) \equiv \int_\Omega r w^2 \, d\underline{x}$$

THEN m_n^*, u_n IS THE n^{th} EIGENPAIR OF
$$L\phi = -\lambda r \phi \text{ IN } \Omega; \quad \partial_n \phi + b\phi = 0 \text{ ON } \partial\Omega. \ \square \qquad \left(\text{i.e. } \begin{array}{l} \lambda_n = m_n^* \\ \phi_n = u_n \end{array} \right)$$

THE PROOF IS THE SAME AS IN THEOREM 3 WITH THE MODIFICATION NEEDED IN THEOREM 2. WE LEAVE THIS FOR THE READER. THE KEY POINT IS THAT NO BC IS IMPOSED IN H_{n-1}^* AND THAT THE ENERGY E_b DEPENDS ON b.

REMARK 7 SET $b = 0$. WE CONCLUDE SINCE $H_{n-1} \subset H_{n-1}^*$ THAT $\mu_n \leq \lambda_n$ WHERE μ_n IS THE nTH EIGENVALUE OF $L\phi = -\mu r \phi$ WITH $\partial_n \phi = 0$ ON $\partial\Omega$ AND λ_n IS THE nTH EIGENVALUE OF $L\phi = -\lambda r \phi$ WITH $\phi = 0$ ON $\partial\Omega$.

OUR VARIATIONAL PRINCIPLES CHARACTERIZE THE EXISTENCE OF A COUNTABLY INFINITE SET OF EIGENVALUES $\lambda_1, \lambda_2, \lambda_3, \ldots$ WITH $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \ldots$

THERE IS A FURTHER RESULT, WHICH IS STATED WITHOUT PROOF, BASED ON THE MAXIMUM PRINCIPLE

THEOREM 5 THE EIGENFUNCTION ϕ_1 FOR THE DIRICHLET PROBLEM $\phi_1 = 0$ ON $\partial\Omega$ FOR THE LOWEST EIGENVALUE λ_1 IS OF ONE SIGN, i.e. $\phi_1 > 0$ IN Ω. THE CORRESPONDING EIGENVALUE λ_1 IS NON-DEGENERATE (i.e. A SIMPLE EIGENVALUE).

THEREFORE FOR THE DIRICHLET PROBLEM WE HAVE $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \ldots$

BEFORE GIVING THE NEXT RESULT WE CALCULATE THE RAYLEIGH QUOTIENT

$R \equiv \dfrac{E(w)}{(w,w)}$ WHEN $w = \sum_{j=1}^{n} c_j \phi_j$ WHERE $\phi_1, .., \phi_n$ ARE NORMALIZED EIGENFUNCTIONS OF $L\phi = -\lambda r \phi$. WE HAVE $(w,w) = \sum_{j=1}^{n}\sum_{k=1}^{n} c_j c_k \int_\Omega r \phi_j \phi_k \, d\underline{x} = \sum_{j=1}^{n} c_j^2$. SINCE $(\phi_j, \phi_k) = 1$ IF $j = k$ AND 0 OTHERWISE. SUPPOSE THAT EITHER $\phi_j = 0$ OR $\partial_n \phi_j = 0$ ON $\partial\Omega$. THEN, WE HAVE:

$$R = \frac{E(w)}{(w,w)} = \frac{\int_\Omega - w L w \, d\underline{x}}{(w,w)} = -\frac{\sum_{j=1}^{n}\sum_{k=1}^{n} c_j c_k \int_\Omega \phi_j L \phi_k \, d\underline{x}}{\sum_{j=1}^{n} c_j^2}.$$

BUT $L\phi_k = -r\lambda_k \phi_k$ SO THAT

$$R = + \frac{\sum_{j=1}^{n}\sum_{k=1}^{n} c_j c_k \int_\Omega r \lambda_k \phi_j \phi_k \, d\underline{x}}{\sum_{j=1}^{n} c_j^2} = + \frac{\sum_{j=1}^{n} \lambda_j c_j^2}{\sum_{j=1}^{n} c_j^2}.$$

THEREFORE IF EITHER $\phi_j = 0$ OR $\partial_n \phi_j = 0$ ON $\partial\Omega$, WE CONCLUDE THAT

$$R = \frac{\sum_{j=1}^{n} \lambda_j c_j^2}{\sum_{j=1}^{n} c_j^2}.$$

WE CONCLUDE THAT SINCE $\lambda_n \geq \lambda_j$ FOR $j = 1, .., n$ WE HAVE

$$R \leq \lambda_n \sum_{j=1}^{n} c_j^2 \Big/ \sum_{j=1}^{n} c_j^2 = \lambda_n \implies R \leq \lambda_n. \qquad (29)$$

THIS OBSERVATION LEADS TO THE NEXT RESULT. THIS RESULT IS KEY FOR GETTING GOOD BOUNDS.

THEOREM 6 (COURANT MAX-MIN PRINCIPLE).

LET $m_1(\underline{x}), .., m_{n-1}(\underline{x})$ BE AN ARBITRARY COLLECTION OF FUNCTIONS AND LET $H \equiv \{ w \mid w \in C^2(\Omega),\ w = 0 \text{ ON } \partial\Omega,\ (w, m_j) = 0 \text{ FOR } j = 1, .., n-1 \}$.

WE THEN CLAIM THAT

$$\lambda_n = \max_{m_1, .., m_{n-1}} \left[\min_{w \in H} \frac{E(w)}{(w,w)} \right] \qquad (30)$$

WHERE λ_n IS THE n^{th} EIGENVALUE OF $L\phi = -\lambda r \phi$ IN Ω WITH $\phi = 0$ ON $\partial\Omega$.

FOR THE CORRESPONDING NEUMANN PROBLEM $L\phi = -\lambda r\phi$ IN Ω WITH $\partial_n \phi = 0$ ON $\partial\Omega$ WE SIMPLY REPLACE $W = 0$ ON $\partial\Omega$ WITH $\partial_n W = 0$ ON $\partial\Omega$ IN THE DEFINITION OF $\mathcal{H}$.

REMARK 8 IN THE MINIMIZATION STEP IN (30), $m_1, .., m_{n-1}$ IS ANY SPECIFIC COLLECTION. THE MAXIMIZATION STEP IS TO VARY ALL $m_1, .., m_{n-1}$. IN (30) WE DO NOT NEED TO KNOW THE EIGENFUNCTIONS $\phi_1, .., \phi_{n-1}$.

PROOF LET $\{ m_1(\underline{x}), .., m_{n-1}(\underline{x}) \}$ BE AN ARBITRARY SET OF FUNCTIONS IN $C^2(\Omega)$. DEFINE $W^* = \sum_{j=1}^{n} c_j \phi_j(\underline{x})$ WHERE $\phi_1, .., \phi_n$ ARE FIRST n EIGENFUNCTIONS OF $L\phi = -\lambda r\phi$ WITH EITHER $\phi = 0$ OR $\partial_n \phi = 0$ ON $\partial\Omega$. THEN $W^* \in \mathcal{H}$, AND SO IS ADMISSIBLE, WHEN

$$(W^*, m_j) = 0 \quad \text{FOR} \quad j = 1, .., n-1.$$

THUS $c_1, .., c_n$ IS THE SYSTEM $\sum_{j=1}^{n} c_j (m_k, \phi_j) = 0$, $k = 1, .., n-1$ WHICH IS AN UNDERDETERMINED SYSTEM FOR $k = 1, .., n-1$, i.e. AN $(n-1) \times n$ MATRIX PROBLEM. THUS $\exists$ A SOLUTION $c_1, .., c_n$ WITH AT LEAST ONE c_j ARBITRARY. WE CHOSE THIS CONSTANT TO SATISFY $(W^*, W^*) = \sum_{j=1}^{n} c_j^2 = 1$.

THEREFORE W^* CAN BE MADE ADMISSIBLE AND SO BY (29),

$$\frac{E(W^*)}{(W^*, W^*)} = \sum_{j=1}^{n} c_j^2 \lambda_j \leq \lambda_n \sum_{j=1}^{n} c_j^2 = \lambda_n .$$

HENCE

$$\frac{E(W^*)}{(W^*, W^*)} \leq \lambda_n .$$

SINCE W^* IS A <u>SPECIFIC</u> ADMISSIBLE FUNCTION, THEN

$$\min_{W} \frac{E(W)}{(W, W)} \leq \frac{E(W^*)}{(W^*, W^*)} \leq \lambda_n . \qquad (31)$$

FINALLY, SINCE $\{ m_1, .., m_{n-1} \}$ IS AN ARBITRARY SET, THEN (31) MUST HOLD FOR THAT SET THAT MAXIMIZES THE MIN IN (31). THUS, WE CONCLUDE THAT

$$\max_{m_1, \ldots, m_{n-1}} \left[\min_{W \in \mathcal{H}} \frac{E(W)}{(W, W)} \right] \leq \lambda_n . \qquad (32)$$

TO CONCLUDE THE PROOF, WE MUST SHOW THAT WE CAN ACHIEVE EQUALITY IN (+).

THIS IS CLEAR IF WE TAKE $\{m_1, .., m_{n-1}\} = \{\phi_1, .., \phi_{n-1}\}$ AND $w = \phi_n$.

THUS $\displaystyle \max_{m_1,..,m_{n-1}} \left[\min_{w \in H} \frac{E(w)}{(w,w)} \right] = \lambda_n$, WHICH CONCLUDES THE PROOF.

<u>REMARK 9</u> IN THE COURANT MAX, MIN PRINCIPLE WE MUST TAKE EITHER $w = 0$ OR $\partial_n w = 0$ ON $\partial\Omega$ IN H. THIS IS BECAUSE WE USE $E(w) = -\int_\Omega w L w \, d\underline{x}$ TO CALCULATE (29).

WE NOW USE THE COURANT MAX-MIN PRINCIPLE TO DETERMINE BOUNDS ON THE EIGENVALUES FOR EITHER THE DIRICHLET OR NEUMANN PROBLEM.

<u>BOUNDS ON THE COEFFICIENTS</u>

CONSIDER
$$L\phi = \nabla\cdot[p\nabla\phi] - q\phi = -\lambda r \phi \quad \text{IN } \Omega$$
WITH EITHER $\phi = 0$ OR $\partial_n \phi = 0$ ON $\partial\Omega$.

THEN
$$\lambda_n = \max_{m_1,..,m_{n-1}} \left[\min_{w \in H} \frac{E(w)}{(w,w)} \right], \quad H = \{ w \mid w \text{ IS } C^2(\Omega),\ w=0 \text{ OR } \partial_n w = 0 \text{ ON } \partial\Omega,\ (w, m_j) = 0,\ j = 1,..,n-1 \}$$
$$E(w) = \int_\Omega (p|\nabla w|^2 + q w^2)\, d\underline{x}, \quad (w,w) = \int_\Omega r w^2 \, d\underline{x}.$$

NOW SUPPOSE THAT IN Ω,
$$0 < p_{MIN} \le p \le p_{MAX}, \quad 0 \le q_{MIN} \le q \le q_{MAX}, \quad 0 < r_{MIN} \le r \le r_{MAX},$$

AND DEFINE
$$E_{MAX}(w) \equiv \int_\Omega [p_{MAX} |\nabla w|^2 + q_{MAX} w^2]\, d\underline{x}, \quad \|w\|^2_{MAX} \equiv \int_\Omega r_{MAX} w^2 \, d\underline{x}$$
$$E_{MIN}(w) \equiv \int_\Omega [p_{MIN} |\nabla w|^2 + q_{MIN} w^2]\, d\underline{x}, \quad \|w\|^2_{MIN} = \int_\Omega r_{MIN} w^2 \, d\underline{x}$$

THEN FOR ANY FUNCTION $w \in H$ WE HAVE
$$\frac{E_{MIN}(w)}{\|w\|^2_{MAX}} \le \frac{E(w)}{\|w\|^2} \le \frac{E_{MAX}(w)}{\|w\|^2_{MIN}} \qquad (33)$$

THUS, IF λ_{nMAX} IS THE n^{th} EIGENVALUE FOR
$$\nabla\cdot[p_{MAX}\nabla\phi] - q_{MAX}\phi = -\lambda_{nMAX}\, r_{MIN}\, \phi \quad \text{IN } \Omega \qquad (34)$$
WITH $\phi = 0$ OR $\partial_n \phi = 0$ ON $\partial\Omega$

AND IF λ_{nMIN} IS THE n^{th} EIGENVALUE FOR

$$\left.\begin{array}{l}\nabla\cdot[p_{MIN}\nabla\phi]-q_{MIN}\,\phi=-\lambda_{0MIN}\,r_{MAX}\,\phi \quad \text{IN } \Omega \\ \text{WITH} \quad \phi=0 \text{ OR } \partial_n\phi=0 \text{ ON } \partial\Omega\end{array}\right\} \quad (35)$$

WE HAVE FROM THE MAX-MIN PRINCIPLE THAT

$$\lambda_{0MIN}\le\lambda_0\le\lambda_{0MAX}. \qquad (36)$$

NOW SINCE (34), (35) HAVE CONSTANT COEFFICIENTS, WE MAY BE ABLE TO CALCULATE λ_{0MIN} AND λ_{0MAX} EXPLICITLY.

<u>REMARK 10</u> IN DERIVING (36) FROM (33) WE MUST HAVE THAT THE ADMISSIBILITY CRITERIA IS THE SAME FOR ALL 3 PROBLEMS. THIS IS NOT IMMEDIATELY CLEAR SINCE THE ORTHOGONALITY RELATION $(W, m_j)=0$ DEPENDS ON r.

FOR MIDDLE OF (33) WE NEED $\int_\Omega r\,W\,m_j\,d\underline{x}=0$, WHEREAS FOR THE LEFT AND RIGHT SIDES OF (33) WE NEED $\int_\Omega r_{MAX}\,W\,m_j\,d\underline{x}=0$ AND $\int_\Omega r_{MIN}\,W\,m_j\,d\underline{x}=0$. HOWEVER, SINCE m_j ARE <u>ARBITRARY</u> WE CAN REPLACE $m_j=\frac{r}{r_{MAX}}\hat{m}_j$ AND $m_j=\frac{r}{r_{MIN}}\hat{m}_j$ SO THAT ADMISSIBILITY CRITERIA IS THE SAME IN EACH CASE.

<u>EXAMPLE 4</u> FIND BOUNDS ON THE FIRST EIGENVALUE FOR

$$\phi''-x\phi=-\lambda\phi \quad \text{ON } 0\le x\le\pi$$
$$\text{WITH} \quad \phi=0 \text{ AT } x=0,\pi.$$

<u>SOLUTION</u>

$$\lambda_0=\min_{u\in H}\frac{\int_0^\pi\left((u')^2+x\,u^2\right)dx}{\int_0^\pi u^2\,dx}, \quad H=\{u \mid u\in C^2(\Omega),\ u=0 \text{ ON } \partial\Omega\}$$

NOW $p=1$, $q=x$ SO $q_{MAX}=\pi$, $q_{MIN}=0$.
$r=1$

THUS $\lambda_{0MIN}\le\lambda_0\le\lambda_{0MAX}$

WHERE λ_{0MAX} IS LOWEST EIGENVALUE OF $\phi''-\pi\phi=-\lambda_{0MAX}\phi$, $0<x<\pi$, $\phi(0)=\phi(\pi)=0$

AND λ_{0MIN} IS LOWEST EIGENVALUE OF $\phi''=-\lambda_{0MIN}\phi$, $0<x<\pi$, $\phi(0)=\phi(\pi)=0$.

WE CALCULATE $\lambda_{0MIN}=1$, $\lambda_{0MAX}=1+\pi$. (FOR λ_{0MAX} PROBLEM $\phi=\sin\left(\sqrt{\lambda_{0MAX}-\pi}\,x\right)$ SO $\sqrt{\lambda_{0MAX}-\pi}\,\pi=n\pi$ ARE EIGENVALUES. SET $n=1$, $\lambda_{0MAX}=1+\pi$).

BY COURANT MAX-MIN WE GET $1\le\lambda_0\le1+\pi$.

NOW WE CAN GET A BETTER UPPER BOUND BY USING A TRIAL FUNCTION.

TAKE $u = \sin x$, THEN

$$\lambda_0 \le \frac{\int_0^\pi (u'^2 + x u^2)\,dx}{\int_0^\pi u^2\,dx} = \frac{\int_0^\pi (\cos^2 x + x \sin^2 x)\,dx}{\int_0^\pi \sin^2 x\,dx} = 1 + \frac{\pi}{2}.$$

THUS, $\quad 1 \le \lambda_0 \le 1 + \pi/2.$

<u>EXAMPLE 5</u> BY BOUNDING THE COEFFICIENTS FIND SIMPLE UPPER AND LOWER BOUNDS FOR THE LOWEST EIGENVALUE IN THE UNIT DISK FOR THE 2-D PROBLEM:

$$\nabla \cdot [\, p(r) \nabla \phi \,] + \lambda \phi = 0 \quad \text{IN} \quad 0 < r < 1, \; 0 \le \varphi \le 2\pi$$

$$\phi = 0 \quad \text{ON} \quad \partial\Omega$$

WITH $\qquad p = 1 + r^2.$

<u>SOLUTION</u> IN UNIT DISK $\qquad 1 \le p_{MIN} \le p \le p_{MAX} = 2.$

NOW FOR $p_{MIN} = 1 \longrightarrow \lambda_{0MIN} = z_0^2$, $\quad z_0 \cong 2.4$ SATISFIES $J_0(z_0) = 0$ (LOWEST ROOT).

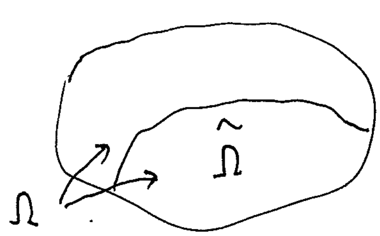
NOW FOR $p_{MAX} = 2 \longrightarrow \quad \phi_{rr} + \frac{1}{r}\phi_r + \frac{\lambda_{0MAX}}{2}\phi = 0 \longrightarrow \frac{\lambda_{0MAX}}{2} = z_0^2.$

WE CONCLUDE THAT $\quad z_0^2 < \lambda_0 < 2 z_0^2$, WHERE $z_0 \cong 2.4$ IS SMALLEST ROOT OF $J_0(z) = 0$.

NOW WE WILL <u>CONSIDER BOUNDS ON THE GEOMETRY</u>. THE ANALYSIS BELOW IS VALID, AS WE WILL SEE, FOR THE DIRICHLET PROBLEM

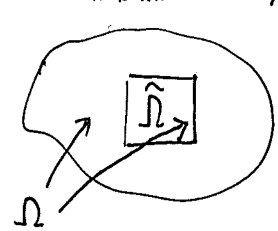
$$L\phi \equiv \nabla \cdot [\, p \nabla \phi \,] - q\,\phi = -\lambda r \phi \quad \text{IN } \Omega \;; \quad \phi = 0 \text{ ON } \partial\Omega.$$

LET λ_n BE THE n^{th} EIGENVALUE FOR Ω. NOW LET $\tilde{\Omega} \subset \Omega$ WHERE PART OF THE BOUNDARY OF Ω AND $\hat{\Omega}$ MAY COINCIDE



Ω IS THE WHOLE REGION SHOWN WITH $\hat{\Omega}$ BEING THE SUBREGION AS SHOWN.

OR



WHERE THEY DO NOT SHARE A BOUNDARY.

CONSIDER $\dfrac{E(W)}{\|W\|^2}$ FOR THE MAX-MIN VARIATIONAL PROBLEM IN Ω.

WE MUST HAVE THAT W SATISFIES

(i) $W = 0$ ON $\partial\Omega$

(ii) $\int_\Omega W r m_j \, d\underline{x} = 0$ FOR $j = 1, .., n-1$.

THEN W IS ADMISSIBLE FOR THE MAX-MIN PROBLEM IN Ω. SUPPOSE, THAT IN ADDITION TO SATISFYING (i) AND (ii), W ALSO SATISFIES

(iii) $W = 0$ ON $\partial\tilde{\Omega}$

(iv) $W \equiv 0$ IN THE REGION EXTERIOR TO $\hat{\Omega}$ BUT INSIDE Ω.

THEN IF W SATISFIES (i)-(iv) IT IS CLEAR THAT W IS ADMISSIBLE FOR THE VARIATIONAL PROBLEM IN $\hat{\Omega}$; i.e. $0 = \int_\Omega r W m_j \, d\underline{x} = \int_{\hat{\Omega}} r W m_j \, d\underline{x}$ SINCE $W = 0$ IN $\Omega \setminus \hat{\Omega}$.

HOWEVER, THE SET OF FUNCTIONS, LABELLED BY $H_{\tilde{\Omega}}$, SATISFYING (i)-(iv) ARE MORE RESTRICTIVE THAN THE SET H_Ω OF FUNCTIONS W SATISFYING ONLY (i) AND (ii). THUS

$$H_{\hat{\Omega}} \subset H_\Omega$$

AS A CONSEQUENCE, THE MAX-MIN FOR THE VARIATIONAL PROBLEM IN Ω IS $\leq$ THAT FOR THE VARIATIONAL PROBLEM IN $\hat{\Omega}$, i.e. FOR A FIXED $\{m_1, .., m_{n-1}\}$

$$\min_{W \in H_\Omega} \frac{E(W)}{(W,W)} \leq \min_{W \in H_{\tilde{\Omega}}} \frac{E(W)}{(W,W)}.$$

$\rightarrow$ TAKING $\max\limits_{m_1,..,m_{n-1}}$, WE OBTAIN THAT THE n^{th} EIGENVALUE λ_n FOR Ω AND $\hat{\lambda}_n$ FOR $\tilde{\Omega}$ SATISFY

$$\lambda_n \leq \tilde{\lambda}_n \tag{37}$$

WE CONCLUDE FOR THE DIRICHLET EIGENVALUE THAT IF THE AREA (2-D) OR VOLUME (N-D) DECREASES THE nTH EIGENVALUE CANNOT DECREASE. THIS IS <u>DOMAIN MONOTONICITY PRINCIPLE</u>.

<u>EXAMPLE 6</u> <u>(IMPORTANT)</u>

CONSIDER OUR DIRICHLET EIGENVALUE PROBLEM

$$\left.\begin{aligned} &\nabla \cdot [p(\underline{x}) \nabla \phi] - q(\underline{x}) \phi = -\lambda r(\underline{x}) \phi \quad \text{IN } \Omega \subset \mathbb{R}^N \\ &\text{WITH} \quad \phi = 0 \quad \text{ON } \partial\Omega \end{aligned}\right\} \tag{38}$$

SUPPOSE, AS USUAL THAT $p(\underline{x}) > 0$, $r(\underline{x}) > 0$ ON $\bar{\Omega}$.

<u>CLAIM</u>: THE n^{th} EIGENVALUE $\lambda_n \to \infty$ AS $n \to \infty$.

<u>PROOF</u> BY REPLACING $p \to p_{MIN}$, $q \to q_{MIN}$, $r \to r_{MAX}$, WHICH BOUNDS THE COEFFICIENTS WE OBTAIN

$$\lambda_{n\,MIN} \leq \lambda_n \qquad \forall\, n = 1, 2, \ldots \qquad (39)$$

HERE $\lambda_{n\,min}$ IS THE n^{th} EIGENVALUE OF THE CONSTANT COEFFICIENT PROBLEM

$$p_{MIN}\,\Delta\phi - q_{MIN}\,\phi = -\lambda\, r_{MAX}\,\phi \quad \text{IN } \Omega$$
$$\phi = 0 \quad \text{ON } \partial\Omega$$

NOW WE ENCLOSE Ω IN AN N-DIMENSIONAL CUBE OF SIDE LENGTH L. WE DENOTE $\square$ BY THIS DOMAIN. SINCE VOLUME $\uparrow$, THEN BY (37), WE HAVE

$$\lambda_{n\,MIN}(\square) \leq \lambda_{n\,MIN}$$

WHICH FROM (39) YIELDS

$$\lambda_{n\,MIN}(\square) \leq \lambda_{n\,MIN} \leq \lambda_n \qquad (40)$$

WHERE $\square$ IS THE CUBE $\square = \left\{ \underline{x} \;\middle|\; -\frac{L}{2} < x_i < \frac{L}{2} \text{ FOR } i = 1, \ldots N \right\}$

AND $\lambda_{n\,MIN}(\square)$ IS THE n^{th} EIGENVALUE OF

$$\left.\begin{aligned} p_{MIN}\,\Delta\phi - q_{MIN}\,\phi &= -\lambda\, r_{MAX}\,\phi \quad \text{IN } \square \\ \phi &= 0 \quad \text{ON } \partial\Omega \end{aligned}\right\} \quad (41)$$

SINCE (41) IS A CONSTANT COEFFICIENT PROBLEM IN A SEPARABLE GEOMETRY WE CAN FIND λ EXPLICITLY FOR (41). THE EIGENFUNCTION IS IN $\mathbb{R}^N$,

$$\phi = \sin\left[j_1 \pi \frac{(x_1 + L/2)}{L} \right] \cdots \sin\left[j_N \pi \frac{(x_N + L/2)}{L} \right], \quad \text{FOR } j_m = 1, 2, \ldots \text{ FOR EACH } m = 1, \ldots, N.$$

WE SUBSTITUTE AND CALCULATE:

$$\left[-p_{MIN} \frac{\pi^2}{L^2} \sum_{m=1}^{N} j_m^2 - q_{MIN} + \lambda\, r_{MAX} \right] \phi = 0.$$

THUS THE EIGENVALUES ARE $\lambda = \frac{1}{r_{MAX}} \left[p_{MIN} \frac{\pi^2}{L^2} \sum_{m=1}^{N} j_m^2 + q_{MIN} \right]$

SINCE $j_m = 1, 2, \ldots$ IS UNBOUNDED, THE n^{th} EIGENVALUE OF (41), λ_n, TENDS TO ∞ AS $n \to \infty$,

i.e. $\lambda_n \to \infty$ AS $n \to \infty$.

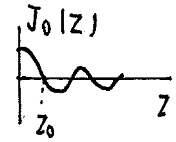
EXAMPLE 7 CONSIDER THE LOWEST EIGENVALUE OF THE LAPLACIAN IN A DISK OF RADIUS a, SO THAT

$$\Delta\phi + \lambda\phi = 0 \quad \text{IN } 0 < r < a,\ 0 < \varphi < 2\pi$$
$$\phi(a,\varphi) = 0, \quad \phi \text{ IS } 2\pi \text{ PERIODIC IN } \varphi.$$

FIND BOUNDS ON λ_0 BY INSCRIBING THE DISK IN A SQUARE AND BY CONSIDERING LARGEST INSCRIBED SQUARE.

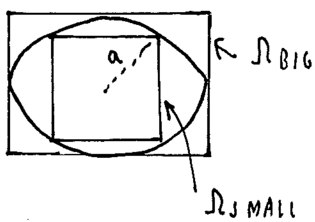
SOLUTION AS SHOWN PREVIOUSLY THE SMALLEST EIGENVALUE IS EXACTLY

$$\lambda_1 = z_0^2/a^2 \quad \text{WHERE } z_0 \cong 2.4 \text{ IS LOWEST ROOT OF } J_0(z) = 0.$$



THUS $\lambda_0 \cong 5.76/a^2$.

NOW CONSIDER AN INSCRIBED Ω_{SMALL} AND CIRCUMSCRIBED Ω_{BIG} SQUARE FOR Ω, AS SHOWN.



- NOW Ω_{BIG} IS $-a < x < a,\ -a < y < a$.
  IF WE PUT $\phi = 0$ ON $\partial\Omega_{BIG}$ WE HAVE LOWEST EIGENVALUE:
  $$\lambda_{BIG} = \frac{\pi^2}{(2a)^2} + \frac{\pi^2}{(2a)^2} = \frac{\pi^2}{2a^2}$$
- NOW Ω_{SMALL} IS $-a/\sqrt{2} < x < a/\sqrt{2},\ -a/\sqrt{2} < y < a/\sqrt{2}$.
  IF WE PUT $\phi = 0$ ON $\partial\Omega_{SMALL}$ WE HAVE LOWEST EIGENVALUE
  $$\lambda_{SMALL} = \frac{\pi^2}{(\sqrt{2}a)^2} + \frac{\pi^2}{(\sqrt{2}a)^2} = \frac{\pi^2}{2a^2} + \frac{\pi^2}{2a^2} = \frac{\pi^2}{a^2}$$

REMARK: IF $0 < x < L$, $0 < y < H$ THE LOWEST LAPLACIAN EIGENVALUE WITH DIRICHLET B.C. IS
$$\lambda_1 = \frac{\pi^2}{L^2} + \frac{\pi^2}{H^2}.$$

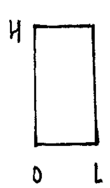
WE CONCLUDE BY DOMAIN MONOTONICITY RESULT

$$\lambda_{BIG} < \lambda_1 < \lambda_{SMALL} \longrightarrow \frac{\pi^2}{2a^2} < \lambda_1 < \frac{\pi^2}{a^2} \Rightarrow \frac{4.93}{a^2} < \lambda_1 < \frac{9.87}{a^2}.$$

NOW WE WILL SHOW THROUGH A COUNTEREXAMPLE THAT THE DOMAIN MONOTONICITY PRINCIPLE DOES NOT HOLD FOR NEUMANN BOUNDARY CONDITIONS, i.e. $\partial_n\phi = 0$ ON $\partial\Omega$. IN OTHER WORDS IF $\hat{\Omega} \subset \Omega$ IT DOES NOT IMPLY THAT $\hat{\lambda}_n > \lambda_n$ WHERE $\hat{\lambda}_n$ AND λ_n n^{th} EIGENVALUE OF $\hat{\Omega}$ AND Ω. THE COUNTEREXAMPLE RELIES ON AN OBSERVATION: FOR A 2-D RECTANGLE $0 < x < L$, $0 < y < H$, THE EIGENPAIRS OF THE LAPLACIAN $\Delta\phi + \lambda\phi = 0$ IN Ω WITH $\partial_n\phi = 0$ ON $\partial\Omega$ ARE

$$\lambda = \frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{H^2}, \quad m,n = 0,1,2,\ldots \quad \phi = \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi y}{H}\right).$$

NOW IF $H > L$

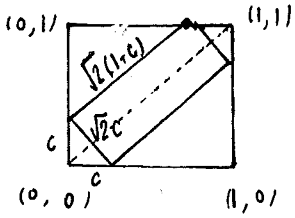


THE SECOND EIGENVALUE IS TO SET $m = 0$, $n = 1$ SO
$$\lambda_1 = \pi^2/H^2 \quad \text{(DEPENDS ON THE LONGEST SIDE)}.$$

THE LOWEST EIGENVALUE IS CLEARLY ZERO.

THE IDEA IS TO CONSTRUCT TWO DOMAINS Ω AND $\hat{\Omega}$, BOTH RECTANGULAR, WITH $\hat{\Omega} \subset \Omega$ BUT WHERE THE LONGEST SIDE OF $\hat{\Omega}$ IS LONGER THAN THAT OF Ω.

TO DO SO, LET Ω BE THE UNIT SQUARE AND FOR $0<c<1$ TAKE $\hat{\Omega}$ AS SHOWN:



- NOW LET Ω BE UNIT SQUARE $\rightarrow$ SECOND EIGENVALUE IS $\lambda_1 = \pi^2$.
- NOW LET $\hat{\Omega}$ BE RECTANGLE OF LENGTH $\sqrt{2}c$ AND WIDTH $\sqrt{2}(1-c)$

NOW SUPPOSE $c < 1 - 1/\sqrt{2}$ $\rightarrow$ LONG SIDE OF RECTANGLE LARGER THAN SIDE LENGTH OF SQUARE. CHOOSE $c < 1/2$ SO THAT WIDTH > LENGTH OF RECTANGLE. THE EIGENVALUES OF RECTANGLE ARE

$$\hat{\lambda} = \frac{\pi^2}{(\sqrt{2}c)^2} n^2 + \frac{\pi^2 m^2}{(\sqrt{2}(1-c))^2}, \qquad m, n = 0, 1, 2, \ldots$$

NOW IF $c < 1/2$, SECOND EIGENVALUE IS FROM $m=1$, $n=0$ SO

$$\hat{\lambda}_1 = \frac{\pi^2}{2(1-c)^2}.$$

NOTICE IF $c < -1/\sqrt{2} + 1 \Rightarrow c < \frac{1}{2}$ AND WE CONCLUDE THAT $2(1-c)^2 > 1$.

THUS IF $c < 1 - 1/\sqrt{2} \Rightarrow \hat{\lambda}_1 < \lambda_1$ YET $\hat{\Omega} \subset \Omega$.

EIGENVALUE $\downarrow$ AS DOMAIN $\downarrow$. WE CONCLUDE THAT DOMAIN MONOTONICITY PRINCIPLE FAILS FOR THE NEUMANN PROBLEM.

NEXT, WE WILL PROVE THE COMPLETENESS RELATION FOR EIGENFUNCTIONS OF

$$\left.\begin{aligned} L\phi &\equiv \nabla\cdot[p\nabla\phi] - q\phi = -\lambda r\phi \quad \text{IN } \Omega \\ \phi &= 0 \quad \text{ON } \partial\Omega \end{aligned}\right\} \quad (41)$$

<u>THEOREM 7.</u> LET ϕ_j FOR $j \geq 1$ BE THE EIGENFUNCTIONS OF (41) NORMALIZED BY $(\phi_j, \phi_j) \equiv \int_\Omega \phi_j^2 r\, dx = 1$. CONSIDER ANY FUNCTION $u(\underset{\sim}{x}) \in C^2(\Omega)$, WITH $u=0$ ON $\partial\Omega$. THEN, WE HAVE THAT THE RESIDUAL $R_n(\underset{\sim}{x}) \equiv u(\underset{\sim}{x}) - \sum_{j=1}^{n-1} c_j \phi_j(\underset{\sim}{x})$ WITH $c_j = (\phi_j, u)$

SATISFIES $\| R_n(\underset{\sim}{x}) \|^2 \equiv \int_\Omega R_n^2\, r\, d\underset{\sim}{x} \rightarrow 0$ AS $n \rightarrow \infty$. IN THIS SENSE, ANY REASONABLE FUNCTION $u(\underset{\sim}{x})$ CAN BE WRITTEN AS $\sum_{j=1}^{\infty} c_j \phi_j(\underset{\sim}{x})$ WITH CONVERGENCE IN THE MEAN. THUS $\{\phi_1, \phi_2, \ldots\}$ FORM A COMPLETE SET.

PROOF:

NOTICE THAT $(R_n, \phi_j) = (u - \sum_{k=1}^{n-1} c_k \phi_k, \phi_j) = (u, \phi_j) - \sum_{k=1}^{n-1} c_k (\phi_k, \phi_j) = (u, \phi_j) - c_j = 0, \quad j = 1, .., n-1.$

THUS $R_n(x)$ IS AN ADMISSIBLE FUNCTION FOR THE VARIATIONAL PRINCIPLE IN THEOREM 3 ON PAGE 12 FOR λ_n, SINCE WITH $u = 0$ ON $\partial\Omega$, WE HAVE $R_n = 0$ ON $\partial\Omega$.

WE CONCLUDE THAT $\lambda_n \leq \dfrac{E(R_n)}{\|R_n\|^2}$ WITH $E(u) = \int_\Omega (p|\nabla u|^2 + q u^2) \, d\underline{x}$

THIS GIVES

$$\|R_n\|^2 \leq \frac{E(R_n)}{\lambda_n}. \qquad (42)$$

WE KNOW FROM OUR EXAMPLE 6 ON DOMAIN MONOTONICITY THAT $\lambda_n \to \infty$ AS $n \to \infty$ (SEE PAGE 21). WE SIMPLY NEED TO SHOW THAT $E(R_n)$ IS BOUNDED.

SINCE $R_n = 0$ ON $\partial\Omega$, WE HAVE

$$E(R_n) = -\int_\Omega R_n L R_n \, d\underline{x} = -\int_\Omega \left[u(\underline{x}) - \sum_{i=1}^{n-1} c_i \phi_i(\underline{x}) \right] \left[Lu - \sum_{j=1}^{n-1} c_j L\phi_j \right] d\underline{x}$$

$$E(R_n) = -\int_\Omega u L u \, d\underline{x} + 2\sum_{i=1}^{n-1} c_i \int_\Omega \phi_i L u \, d\underline{x} - \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} c_i c_j \int_\Omega \phi_i L\phi_j \, d\underline{x}.$$

NOW SINCE $\phi_i = 0$ AND $u = 0$ ON $\partial\Omega$, WE HAVE BY GREEN'S IDENTITY $\int_\Omega \phi_i L u \, d\underline{x} = \int_\Omega u L\phi_i \, d\underline{x} = -\lambda_i \int_\Omega r \phi_i u \, d\underline{x}$

THUS $E(R_n) = E(u) - 2\sum_{i=1}^{n-1} c_i \lambda_i \int_\Omega r \phi_i u \, d\underline{x} + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} c_i c_j \int_\Omega \lambda_j \phi_i \phi_j r \, d\underline{x}$

BUT $\int_\Omega r \phi_i u \, d\underline{x} = c_i$ AND $\int_\Omega \phi_i \phi_j r \, d\underline{x} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$.

THUS, $E(R_n) = E(u) - 2\sum_{i=1}^{n-1} c_i^2 \lambda_i + \sum_{i=1}^{n-1} c_i^2 \lambda_i = E(u) - \sum_{i=1}^{n-1} c_i^2 \lambda_i < E(u).$

AS SUCH WE HAVE $\|R_n\|^2 \leq \dfrac{E(R_n)}{\lambda_n} \leq \dfrac{E(u)}{\lambda_n} \to 0$ AS $n \to \infty$ ∎

NOW A SIMILAR COMPLETENESS RESULT HOLDS FOR NEUMANN EIGENFUNCTIONS AND ROBIN EIGENFUNCTIONS. WE HOWEVER NEED A DIFFERENT APPROACH OTHER THAN THE DOMAIN MONOTONICITY APPROACH TO PROVE THAT $\lambda_n \to \infty$ AS $n \to \infty$ FOR THESE EIGENFUNCTIONS.

RAYLEIGH-RITZ METHOD

WE WANT TO APPROXIMATE THE LOWEST EIGENVALUE λ_1 OF

$$\left.\begin{aligned} \nabla\cdot[p\nabla u] - q u = -\lambda r u \quad \text{IN } \Omega \\ u = 0 \quad \text{ON } \partial\Omega \end{aligned}\right\} \quad (43)$$

WITH $p, r > 0$ IN Ω, $q \geq 0$ IN Ω, BY THE RAYLEIGH QUOTIENT

$$\lambda_1 = \min_{w \in H} \frac{E(w)}{(w,w)} \qquad \left.\begin{aligned} E(w) &= \int_\Omega (p|\nabla w|^2 + q w^2)\, d\underline{x} \\ (w,w) &= \int_\Omega r w^2\, d\underline{x} \end{aligned}\right\} \quad (44)$$

$$H = \{ w \mid w = 0 \text{ ON } \partial\Omega,\ w \not\equiv 0 \}$$

WE CHOOSE TRIAL FUNCTIONS $v_1(\underline{x}), \ldots, v_m(\underline{x})$ WITH $v_j \in H$ FOR $j = 1, \ldots, m$ AND FOR CONSTANTS $c_1, \ldots, c_m$ DEFINE THE TRIAL FUNCTION w^* BY

$$w^* = \sum_{j=1}^{m} c_j v_j .$$ ASSUME $v_j \not\equiv 0$ FOR EACH $j = 1, \ldots, m$; $v_1, \ldots, v_m$ INDEPENDENT

WE HAVE $\lambda_1 \leq \dfrac{E(w^*)}{(w^*, w^*)}$. WE WILL DETERMINE $c_1, \ldots, c_m$ TO MINIMIZE $\dfrac{E(w^*)}{(w^*, w^*)}$.

WE NOW CALCULATE $E(w^*)$. WE WRITE $\underline{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$, $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$

SO THAT $w^* = \underline{c}^T \underline{v}$. RECALLING $\underline{a}\cdot\underline{b} = \underline{a}^T\underline{b} = \underline{b}^T\underline{a}$ WE GET

$$p\, \nabla w^* \cdot \nabla w^* = p\, \underline{c}^T \nabla\underline{v} \cdot \underline{c}^T \nabla \underline{v} = p\, \underline{c}^T \nabla\underline{v} \cdot \nabla\underline{v}^T \underline{c} .$$

IN A SIMILAR WAY TO CALCULATE $q\, w^{*2} = q\, (\underline{c}^T\underline{v})(\underline{c}^T\underline{v}) = q\, \underline{c}^T (\underline{v}\,\underline{v}^T)\underline{c}$.

THIS YIELDS THAT

$$\begin{aligned} E(w^*) &= \underline{c}^T A\, \underline{c} \\ (w^*, w^*) &= \underline{c}^T B\, \underline{c} \end{aligned} \qquad \text{WHERE} \qquad \left.\begin{aligned} A &\equiv \int_\Omega (p\, \nabla\underline{v}\cdot\nabla\underline{v}^T + q\, \underline{v}\,\underline{v}^T)\, d\underline{x} \\ B &\equiv \int_\Omega r\, \underline{v}\,\underline{v}^T\, d\underline{x} \end{aligned}\right\} \quad (45)$$

BOTH A AND B ARE $m \times m$ SYMMETRIC MATRICES.

WE CONCLUDE THAT $\lambda_1 \leq \dfrac{\underline{c}^T A \underline{c}}{\underline{c}^T B \underline{c}}$ $\quad \forall\, c_1, \ldots, c_m$ WITH $\underline{c} \not\equiv \underline{0}$.

AS SUCH IF WE MINIMIZE OVER $\underline{c}$, $\quad \lambda_1 \leq \min\limits_{\substack{\underline{c} \neq \underline{0} \\ \underline{c} \in \mathbb{R}^m}} \dfrac{\underline{c}^T A \underline{c}}{\underline{c}^T B \underline{c}}$.

NOW WE CLAIM THAT A, B POSITIVE DEFINITE. TO SHOW THIS LET $\underline{y} \neq 0$ BE CONSTANT (BUT ARBITRARY) VECTOR. WE WILL CALCULATE $\underline{y}^T B \underline{y} > 0 \ \forall \underline{y} \neq 0$ WHICH PROVES B IS POSITIVE DEFINITE.

$$\underline{y}^T B \underline{y} = \int_\Omega r \, (\underline{y}^T \underline{v})(\underline{v}^T \underline{y}) \, d\underline{x} = \int_\Omega r \, \| \underline{y}^T \underline{v} \|^2 \, d\underline{x}, \quad \text{WHERE } \|\underline{a}\|^2 = \underline{a}^T \underline{a}.$$

NOW SINCE $\underline{y} \neq 0$, WE MUST HAVE $y_j \neq 0$ FOR AT LEAST SOME j, WHERE y_j IS j^{th} COMPONENT OF $\underline{y}$. THEN $\| \underline{y}^T \underline{v} \|^2 \geq y_j^2 \, v_j^2$ AND SO WITH v_j, j^{th} ELEMENT IN $\underline{v}$, WE HAVE:

$$\underline{y}^T B \underline{y} \geq y_j^2 \int_\Omega r \, v_j^2 \, d\underline{x} > 0 \quad \text{SINCE } r > 0 \text{ AND } v_j \not\equiv 0 \text{ IN } \Omega.$$

SIMILARLY

$$\underline{y}^T A \underline{y} = \int_\Omega \left(p \, \| y \nabla \underline{v}^T \|^2 + q \, \| y^T \underline{v} \|^2 \right) d\underline{x} \geq y_j^2 \int_\Omega q \, v_j^2 \, dx > 0$$

IF $q \geq 0$ AND $q \not\equiv 0$. $\rightarrow$ A IS POSITIVE DEFINITE.

OUR AIM IS TO MINIMIZE $\dfrac{\underline{c}^T A \underline{c}}{\underline{c}^T B \underline{c}}$ OVER $\underline{c}$.

THEN, WE OBTAIN THE LOWEST POSSIBLE UPPER BOUND FOR THIS TRIAL SET.

$$\left. \lambda_1 \leq \min_{\substack{\underline{c} \neq 0 \\ \underline{c} \in \mathbb{R}^m}} \left(\frac{\underline{c}^T A \underline{c}}{\underline{c}^T B \underline{c}} \right). \qquad R \equiv \frac{\underline{c}^T A \underline{c}}{\underline{c}^T B \underline{c}} \right\} \quad (46)$$

WE NEXT FIND A NECESSARY CONDITION FOR R TO HAVE A MINIMUM. WE LET $\underline{c}^*$ BE THE MINIMIZER IN (46). WE LET $\underline{c} = \underline{c}^* + \sigma \underline{V}$ FOR $\underline{V} \in \mathbb{R}^m$ AND $\sigma \ll 1$. A NECESSARY CONDITION FOR $\underline{c}^*$ TO BE A MINIMIZER IS THAT

$$R'(\sigma)\Big|_{\sigma=0} = 0 \quad \text{WHERE} \quad R(\sigma) \equiv \frac{\left(\underline{c}^{*T} + \sigma \underline{V}^T \right) A \left(\underline{c}^* + \sigma \underline{V} \right)}{\left(\underline{c}^{*T} + \sigma \underline{V}^T \right) B \left(\underline{c}^* + \sigma \underline{V} \right)}$$

WE CALCULATE USING $A = A^T$ AND SO $\underline{V}^T A \underline{c}^* = \underline{c}^{*T} A \underline{V}$,

$$R(\sigma) = \frac{\left[\underline{c}^{*T} A \underline{c}^* + 2\sigma \underline{V}^T A \underline{c}^* + O(\sigma^2) \right]}{\left[\underline{c}^{*T} B \underline{c}^* + 2\sigma \underline{V}^T B \underline{c}^* + O(\sigma^2) \right]}$$

NOW FOR $\sigma \ll 1$ WE USE $\dfrac{1}{a + \sigma b} = \dfrac{1}{a} \left(1 + \sigma b/a \right)^{-1} \simeq \dfrac{1}{a} - \dfrac{\sigma b}{a^2} + \cdots$

WE OBTAIN THAT

$$R(\sigma) = \frac{\underline{c}^{*T} A \underline{c}^{*}}{\underline{c}^{*T} B \underline{c}^{*}} + \frac{2\sigma}{\underline{c}^{*T} B \underline{c}^{*}} \left[\underline{V}^{T} \left(A \underline{c}^{*} - \frac{\underline{c}^{*T} A \underline{c}^{*}}{\underline{c}^{*T} B \underline{c}^{*}} B \underline{c}^{*} \right) \right] + O(\sigma^2).$$

NOW DEFINE $$R_{*} = \frac{\underline{c}^{*T} A \underline{c}^{*}}{\underline{c}^{*T} B \underline{c}^{*}} \equiv \min_{\substack{c \in \mathbb{R}^m \\ c \neq 0}} \frac{\underline{c}^{T} A \underline{c}}{\underline{c}^{T} B \underline{c}} .$$

WE CONCLUDE THAT A NECESSARY CONDITION FOR $\underline{c}^{*}$ TO BE A MINIMIZER IS THAT

$$\underline{V}^{T} \left[A \underline{c}^{*} - \lambda^{*} B \underline{c}^{*} \right] = 0, \quad \forall\, V \in \mathbb{R}^m ,$$

THUS λ^{*} IS A MATRIX EIGENVALUE OF

(47) $\quad (A - \lambda^{*} B)\, \underline{c}^{*} = 0 \quad$ (GENERALIZED MATRIX EIGENVALUE PROBLEM).

ANY EIGENVALUE λ^{*} OF (47) CORRESPONDS TO $R'(\sigma)\big|_{\sigma=0} = 0$. THE MINIMIZER OF R (AS OPPOSED TO A SIMPLE STATIONARY POINT OF R), SHOULD CORRESPOND TO THE LOWEST EIGENVALUE λ^{*}_{MIN} OF (47).

IN THIS WAY, $\quad \lambda_1 \leq \lambda^{*}_{MIN}$.

FOR EXAMPLE, IF $m = 2$ ALL WE HAVE TO DO IS TO CALCULATE THE LOWEST EIGENVALUE OF (47) AFTER FIRST BUILDING THE MATRICES A AND B USING EQUATION (45).