EQUATION

...T WE WILL CONSIDER THE WAVE EQUATION FOR MORE GENERAL SITUATIONS (1)
...F ARBITRARY INITIAL DATA AND FORCING.

WE WILL FIRST CALCULATE THE GREEN'S FUNCTION DUE TO A LOCALIZED SOURCE AT $\underline{x} = \underline{x}_0$ TURNED ON AT $t = 0^+$. IN VARIOUS DIMENSIONS WE WILL SOLVE

$$(1)\quad \begin{cases} \Delta G - \frac{1}{c^2} G_{tt} = -\frac{1}{c^2}\delta(\underline{x}-\underline{x}_0)\,\delta(t-0^+), & t > 0,\ \underline{x} \in \mathbb{R}^N \\ G = G_t = 0 \text{ AT } t = 0 \end{cases}$$

THIS MODELS A POINT "BLAST" LOCALIZED AT $\underline{x} = \underline{x}_0$ TURNED ON AT $t = 0^+$ IN AN OTHERWISE QUIESCENT MEDIUM.

<u>1-D PROBLEM</u>

$$G_{xx} - \frac{1}{c^2} G_{tt} = -\frac{1}{c^2}\delta(x-x_0)\,\delta(t-0^+)$$
$$G = G_t = 0 \text{ AT } t = 0.$$

WE TAKE LAPLACE TRANSFORM IN TIME: $\hat{G}(x,s) = \mathcal{L}[G(x,t)] = \int_0^\infty e^{-st} G(x,t)\,dt$.
SINCE $\mathcal{L}(G_{tt}) = s^2 \hat{G} - sG(x,0) - G_t(x,0) = s^2 \hat{G}$ WE OBTAIN USING $\mathcal{L}(G_{xx}) = \frac{d^2}{dx^2}\hat{G}$ THAT

$$\mathcal{L}(G_{xx}) - \frac{s^2}{c^2}\hat{G} = -\frac{1}{c^2}\mathcal{L}\left(\delta(x-x_0)\delta(t-0^+)\right) = -\frac{1}{c^2}\delta(x-x_0).$$

WHICH GIVES

$$\hat{G}_{xx} - \frac{s^2}{c^2}\hat{G} = -\frac{1}{c^2}\delta(x-x_0) \quad \text{IN } -\infty < x < \infty$$
$$\hat{G} \text{ BOUNDED AS } |x| \to \infty \text{ FOR } \mathrm{RE}(s) > 0.$$

WE HAVE BY CONTINUITY

$$\hat{G}(x,s) = \begin{cases} A e^{-\frac{s}{c}(x-x_0)}, & x > x_0 \\ A e^{\frac{s}{c}(x-x_0)}, & x < x_0 \end{cases}$$

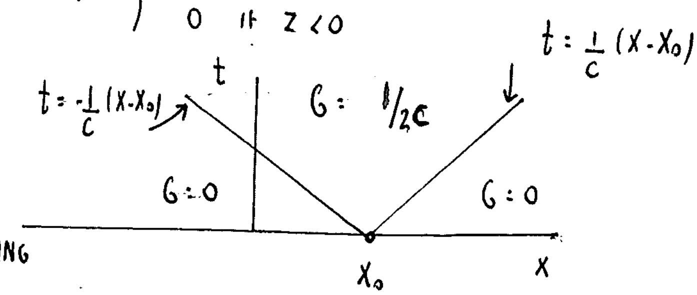
AND USING $\hat{G}_x\big|_{x_0^+} - \hat{G}_x\big|_{x_0^-} = -1/c^2 \longrightarrow -2As/c = -\frac{1}{c^2}$ SO $A = 1/2sc$

THIS YIELDS $\hat{G}(x,s) = \frac{1}{2sc} e^{-s/c\,|x-x_0|}$

FINALLY, WE INVERT THE TRANSFORM. RECALLING $\mathcal{L}^{-1}\left[\frac{e^{-rs}}{s}\right] = H(t-r)$, WHERE $H(z)$ IS THE HEAVY-SIDE FUNCTION: $H(z) \equiv \begin{cases} 1 & \text{IF } z \geq 0 \\ 0 & \text{IF } z < 0 \end{cases}$

WE OBTAIN

$$2) \quad G(x,t) = \frac{1}{2c} H\left[t - \frac{|x-x_0|}{c}\right].$$




WE CONCLUDE THAT $G$ IS NON-VANISHING ONLY INSIDE THE WEDGE.

2-D PROBLEM WE LET $r = |\underline{x} - \underline{x}_0|$ AND WRITE (1) AS

$$G_{rr} + \frac{1}{r} G_r - \frac{1}{c^2} G_{tt} = -\frac{\delta(r)}{2\pi r c^2}\,\delta(t - 0^+)$$

TAKING LAPLACE TRANSFORMS WE OBTAIN

$$\hat{G}_{rr} + \frac{1}{r}\hat{G}_r - \frac{s^2}{c^2}\hat{G} = -\frac{\delta(r)}{2\pi r c^2}.$$

THIS IS THE MODIFIED BESSEL EQUATION AND THE SINGULAR SOLUTION IS $\hat{G}(r,s) = A K_0(rs/c)$. RECALLING $K_0(z) \sim -\log z$ AS $z \to 0^+$ WE MUST HAVE $\hat{G} \sim -\frac{1}{2\pi c^2}\log r$ AS $r \to 0$ TO BALANCE DELTA SINGULARITY. HENCE $A = +\frac{1}{2\pi c^2}$ AND

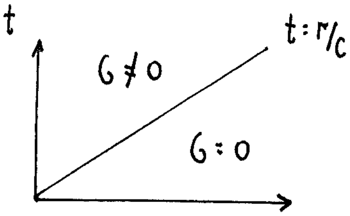
$$\hat{G}(r,s) = +\frac{1}{2\pi c^2} K_0\left(\frac{rs}{c}\right).$$

NOW WE HAVE FROM A HANDBOOK OF LAPLACE TRANSFORMS THAT

$$\mathcal{L}^{-1}\left[K_0(Bs)\right] = \frac{1}{\sqrt{t^2 - B^2}} H(t - B) \quad \text{FOR } B > 0.$$

SETTING $B = r/c$ WE CONCLUDE THAT

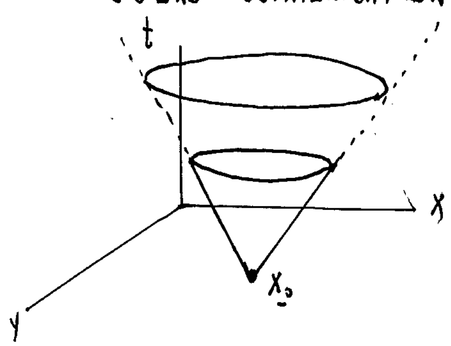
$$(3) \qquad G(r,t) = \frac{1}{2\pi c^2}\frac{H(t - r/c)}{\sqrt{t^2 - r^2/c^2}}$$




$$G = \frac{1}{2\pi c^2}\frac{H(t - |\underline{x} - \underline{x}_0|/c)}{\sqrt{t^2 - |\underline{x} - \underline{x}_0|^2/c^2}}$$

NOTICE THAT ONCE THE SIGNAL HAS PASSED, IN THE REGION $t > r/c$ THE SIGNALS DECAYS IN TIME LIKE $G \cong \frac{1}{2\pi\sqrt{t^2 - r^2/c^2}}\left(\frac{1}{c^2}\right)$

THEREFORE THE SIGNAL IS NOT CRISP AND THERE IS THIS ALGEBRAICALLY DECAYING TAIL THAT REMAINS BEHIND. IF WE LIVED IN A 2-D WORLD IT WOULD BE POOR SOUND COMMUNICATION. OUR PICTURE IN $\underline{x} = (x, y)$ AND $t$ SPACE IS




$G \neq 0$ IN THE CONE.

$G = 0$ OUTSIDE THE CONE.

**3-D PROBLEM** WE LET $r = |\underline{x} - \underline{x}_0|$ AND WRITE (1) AS

$$G_{rr} + \frac{2}{r} G_r - \frac{1}{c^2} G_{tt} = \frac{-\delta(r)\,\delta(t-0^+)}{4\pi r^2 c^2}$$

TAKING LAPLACE TRANSFORMS WE OBTAIN

$$\hat{G}_{rr} + \frac{2}{r}\hat{G}_r - \frac{s^2}{c^2}\hat{G} = \frac{-\delta(r)}{4\pi r^2 c^2}$$

FOR THE HOMOGENEOUS PROBLEM WE SET $\hat{G} = f(r)/r$ TO OBTAIN $f'' - \frac{s^2}{c^2} f = 0$. THE DECAYING SOLUTION HAS $f = Ae^{-sr/c}$. THUS $\hat{G} = Ae^{-sr/c}/r$. NOW SINCE WE MUST HAVE $\hat{G} \sim \frac{1}{4\pi r c^2}$ AS $r \to 0$ WE SET $A = \frac{1}{4\pi c^2}$

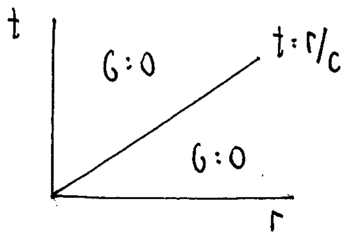
WE CONCLUDE THAT $\hat{G}(r,s) = \frac{1}{4\pi r c^2} e^{-sc/r}$.

NOW RECALLING $\mathcal{L}(\delta(t-B)) = \int_0^\infty e^{-st}\delta(t-B)\,dt = e^{-sB}$ WE SET $B = c/r$ TO OBTAIN

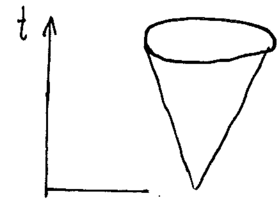
$$(4) \qquad G(r,t) = \frac{1}{4\pi r c^2}\,\delta(t - r/c) \quad \text{OR EQUIVALENTLY} \quad G = \frac{1}{4\pi|\underline{x}-\underline{x}_0|c^2}\,\delta\left(t - \frac{|\underline{x}-\underline{x}_0|}{c}\right).$$

WE OBSERVE THAT $G$ IS CONCENTRATED ONLY ON THE LINE $t = r/c = |\underline{x}-\underline{x}_0|/c$.

AND $G = 0$ OFF THIS LINE




THUS THE SIGNAL PROPAGATION IN 3-D IS "CRISP" AND IS CONCENTRATED ON THE BOUNDARY OF THE 3-D + 1-D CONE




**REMARK** THE G-FUNCTION PROBLEM (1) CAN ALTERNATIVELY BE EXPRESSED AS AN INITIAL VALUE PROBLEM OF THE FORM FOR $G(\underline{x}, \underline{x}_0, t)$

$$(5) \quad \begin{cases} \Delta G - \frac{1}{c^2} G_{tt} = 0 & \text{IN } \mathbb{R}^N,\ t > 0 \\ G = 0, \quad G_t = \delta(\underline{x} - \underline{x}_0) & \text{AT } t = 0. \end{cases}$$

TAKING LAPLACE TRANSFORMS AND RECALLING $\mathcal{L}(G_{tt}) = s^2\hat{G} - sG|_{t=0} - G_t|_{t=0}$ WE OBTAIN

$$\Delta\hat{G} - \frac{s^2}{c^2}\hat{G} + \frac{1}{c^2}\,\delta(\underline{x}-\underline{x}_0) = 0.$$

THIS YIELDS $\Delta \hat{G} - \frac{s^2}{c^2}\hat{G} = -\frac{1}{c^2}\delta(\underline{x}-\underline{x}_0)$

WHICH IS THE SAME PROBLEM AS OBTAINED BY TAKING LAPLACE TRANSFORMS OF (1).

AS A RESULT, THE SOLUTION OF (5) IN DIMENSIONS $N=1,2,3$ IS

$$\left.\begin{aligned}
G(x, x_0; t) &= \frac{1}{2c} H\left[t - \frac{|x-x_0|}{c}\right] && (1\text{-D})\\
G(x, x_0; t) &= \frac{1}{2\pi c^2} \frac{H(t - r/c)}{\sqrt{t^2 - r^2/c^2}} && (2\text{-D}) \quad \text{WITH } r = |\underline{x} - \underline{x}_0|\\
G(x; x_0, t) &= \frac{1}{4\pi r c^2}\,\delta(t - r/c) && (3\text{-D}) \quad \text{WITH } r = |\underline{x} - \underline{x}_0|.
\end{aligned}\right\} \quad (6).$$

NEXT, WE AIM TO FIND A GENERAL SOLUTION, ANALOGOUS TO D'ALEMBERT'S 1-D SOLUTION, FOR THE INITIAL VALUE PROBLEM FOR THE WAVE EQUATION IN 2-D AND 3-D. WE WOULD LIKE TO FIND A SOLUTION FORMULA FOR

$$(7)\quad \begin{cases} u_{tt} - c^2 \Delta u = 0, & \underline{x} \in \mathbb{R}^N,\ t>0\\ u(\underline{x},0) = g(\underline{x}), & u_t(\underline{x},0) = h(\underline{x}) \end{cases}$$

WHERE WE ASSUME EITHER $g, h$ BOTH HAVE COMPACT SUPPORT OR DECAY FAST ENOUGH AS $|\underline{x}| \to \infty$. IT IS CONVENIENT TO DECOMPOSE THE SOLUTION INTO TWO PIECES: LET $u(\underline{x},t) = u_1(\underline{x},t) + u_2(\underline{x},t)$ WHERE

$$(8)\quad \begin{cases} u_{1tt} - c^2 \Delta u_1 = 0\\ u_1(\underline{x},0) = 0, \quad u_{1t}(\underline{x},0) = h(x) \end{cases}$$

$$(9)\quad \begin{cases} u_{2tt} - c^2 \Delta u_2 = 0\\ u_2(\underline{x},0) = g(\underline{x}), \quad u_{2t}(x,0) = 0 \end{cases}$$

DEFINE $u_2(\underline{x},t) = \partial_t v(\underline{x},t)$. WE AIM TO FIND PROBLEM FOR $V$.

FROM PDE WE DIFFERENTIATE TO GET

$$(v_t)_{tt} - c^2 \Delta(v_t) = 0 \longrightarrow \partial_t\left[v_{tt} - c^2 \Delta v\right] = 0.$$

WE CHOOSE $v_{tt} - c^2 \Delta v = 0$. MOREOVER $u_2(\underline{x},0) = g(\underline{x}) \to \partial_t v(\underline{x},t)\big|_{t=0} = g(x)$.

FINALLY $u_{2t}(x,0)=0 \rightarrow \partial_{tt} v(\underline{x},t)\big|_{t=0}=0$. BUT FROM PDE WE HAVE $\partial_{tt} v(x,t)=c^2 \Delta v(\underline{x},t)$ AND SO $u_{2t}(x,0)=0 \rightarrow \Delta v(\underline{x},0)=0$.

THUS WE SHOULD TAKE $v(\underline{x},0)\equiv 0$.

IN CONCLUSION, IF WE LET $v(\underline{x},t)$ SATISFY

$$(10)\qquad \begin{aligned} &v_{tt}-c^2\Delta v=0 \\ &v(\underline{x},0)=0, \quad v_t(\underline{x},0)=g(x) \end{aligned}$$

THEN $u_2(x,t)=\dfrac{\partial}{\partial t} v(x,t)$, SOLVES (9).

IN THIS WAY, OUR DECOMPOSITION GIVES

$$(11)\qquad u(\underline{x},t)=u_1(\underline{x},t)+\frac{\partial}{\partial t} v(x,t)$$

WHERE $u_1$, $v$ SATISFY (8) AND (10), RESPECT.

NOW WE OBSERVE THAT (8) AND (10) ARE THE SAME PROBLEM BUT WITH DIFFERENT "DATA". IN OTHER WORDS LET $w=w_a(\underline{x},t)$ BE THE SOLUTION TO THE CANONICAL PROBLEM

$$(12)\quad \left\}\begin{aligned} &w_{tt}-c^2\Delta w=0 \\ &w(\underline{x},0)=0, \quad w_t(\underline{x},0)=a(\underline{x}). \end{aligned}\right.$$

THEN THE SOLUTION TO (7) IS GIVEN BY

$$(13)\qquad u(\underline{x},t)=w_h(\underline{x},t)+\frac{\partial}{\partial t} w_g(\underline{x},t).$$

THEREFORE, WE <u>ONLY</u> NEED TO SOLVE (12) FOR AN ARBITRARY $a(\underline{x})$ AND THEN SIMPLY REPLACE EITHER $a=h$ OR $a=g$ IN (13).

WE WILL SOLVE (12) BY USING OUR GREEN'S FUNCTION $G(\underline{x},x_0,t)$ WHICH SATISFIES

$$(14)\quad \left\{\begin{aligned} &\Delta G-\frac{1}{c^2}G_{tt}=0, \quad \underline{x}\in\mathbb{R}^N,\ t>0 \\ &G=0, \quad G_t=\delta(\underline{x}-\underline{x}_0) \quad \text{AT } t=0. \end{aligned}\right.$$

<u>LEMMA</u> THE SOLUTION TO (12) IS BY SUPERPOSITION

$$w(\underline{x},t) = \int_{\mathbb{R}^N} G(\underline{x},\underline{y},t)\, a(\underline{y})\, d\underline{y}. \qquad (14)$$

<u>PROOF</u> FOR $t>0$ WE CALCULATE

$$w_{tt} - c^2\Delta w = \int_{\mathbb{R}^N} (G_{tt} - c^2\Delta G)\, a(\underline{y})\, d\underline{y} = 0 \quad \text{SINCE} \quad G_{tt} - c^2 \Delta G = 0 \text{ FOR } t>0.$$

NOW $\quad w(0,t) = \int_{\mathbb{R}^N} G(0,\underline{y},t)\, a(\underline{y})\, d\underline{y} = 0 \quad$ SINCE $G=0$ AT $t=0$.

FINALLY $\quad w_t(x,t)\Big|_{t=0} = \int_{\mathbb{R}^N} G_t(\underline{x},\underline{y},t)\Big|_{t=0} a(\underline{y})\, dy = \int_{\mathbb{R}^N} \delta(\underline{x}-\underline{y})\, a(\underline{y})\, d\underline{y} = a(\underline{x}).$ ◻

THEREFORE USING THIS EXPRESSION (14) IN (13) WE OBTAIN THAT

$$u(\underline{x},t) = \int_{\mathbb{R}^N} G(\underline{x},\underline{y},t)\, h(\underline{y})\, d\underline{y} + \frac{\partial}{\partial t}\left( \int_{\mathbb{R}^N} G(\underline{x},\underline{y},t)\, g(\underline{y})\, d\underline{y} \right). \qquad (15)$$

THIS IS THE ANALOGUE OF D'ALEMBERT'S FORMULA IN 1-D. WE WILL NOW EXPLORE THE CONSEQUENCES OF (15) IN VARIOUS DIMENSIONS. IN (15) THE G-FUNCTIONS ARE GIVEN EXPLICITLY IN EQUATION (6).

NOW WE CONSIDER (15) IN VARIOUS DIMENSIONS

<u>ONE DIMENSION</u> RECALL $\quad G(x,y,t) = \frac{1}{2c} H\left[ t - \frac{|x-y|}{c} \right]$

WE CALCULATE

$$\int_{-\infty}^{\infty} G(x,y,t)\, h(y)\, dy = \frac{1}{2c} \int_{-\infty}^{\infty} H\left[ t - \frac{|x-y|}{c} \right] h(y)\, dy.$$

NOW $H=1$ IFF $\quad t - \frac{|x-y|}{c} > 0 \quad$ SO $\quad |x-y| < ct \longrightarrow -ct < x-y < ct$

OR EQUIVALENTLY $-x-ct < -y < ct - x \longrightarrow x-ct < y < x+ct.$

HENCE $$\int_{-\infty}^{\infty} G(x,y,t)\, h(y)\, dy = \frac{1}{2c} \int_{x-ct}^{x+ct} h(y)\, dy.$$

THEN $$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} G(x,y,t)\, g(y)\, dy = \frac{d}{dt}\left( \int_{x-ct}^{x+ct} \frac{1}{2c}\, g(y)\, dy \right) = \frac{1}{2c}\left[ g(x+ct)c - g(x-ct)(-c) \right]$$
$$= \frac{1}{2}\left[ g(x+ct) + g(x-ct) \right].$$

IN THIS WAY IN 1-D WE HAVE FROM (15) THAT

$$u(x,t) = \frac{1}{2}\left[ g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy . \qquad (16)$$

THIS IS THE 1-D D'ALEMBERT'S FORMULA.

<u>THREE DIMENSIONS</u> WE HAVE $G(\underline{x},\underline{y},t) = \frac{1}{4\pi c^2 |\underline{x}-\underline{y}|}\, \delta\left(t - |\underline{x}-\underline{y}|/c\right)$.

NOW IN (15) WE CALCULATE

$$I_1 = \int_{\mathbb{R}^3} G(\underline{x},\underline{y},t)\, h(\underline{y})\, d\underline{y} = \frac{1}{4\pi c^2}\int_{\mathbb{R}^3} \frac{\delta(t - r/c)}{r}\, h(\underline{y})\, d\underline{y} .$$

WHERE $r = |\underline{y}-\underline{x}|$. WE DO THIS INTEGRAL AS FOLLOWS:

$$I_1 = \frac{1}{4\pi c^2}\int_0^\infty \frac{\delta(t-r/c)}{r}\left(\int_{\partial B_r(\underline{x})} h(\underline{y})\, dS(\underline{y})\right) dr .$$

HERE $\partial B_r(\underline{x}) = \{\, \underline{y} \mid |\underline{y}-\underline{x}| = r \,\}$ IS THE BOUNDARY OF A BALL OF RADIUS $r$ CENTERED AT $\underline{y} = \underline{x}$. THE SURFACE AREA ELEMENT $dS(y)$ IS $dS(y) = r^2 \sin\phi \, d\phi \, d\varphi$ WHERE $r$ IS THE BALL RADIUS AND $\phi, \varphi$ ARE USUAL SPHERICAL ANGLES.

NOW SINCE $\delta(ax) = \frac{1}{a}\delta(x)$ FOR $a>0$ WE USE $\delta\left(t - \frac{r}{c}\right) = \delta\left(\frac{tc - r}{c}\right) = c\,\delta(tc - r)$ TO OBTAIN

$$I_1 = \frac{1}{4\pi c^2}\int_0^\infty \frac{c}{r}\, \delta(tc - r)\left(\int_{\partial B_r(\underline{x})} h(\underline{y})\, dS(\underline{y})\right) dr .$$

BY SIFTING PROPERTY WE OBTAIN

$$I_1 = \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(\underline{x})} h(\underline{y})\, dS(\underline{y}) .$$

THUS IF WE MULTIPLY TOP AND BOTTOM BY $t$ WE HAVE

$$I_1 = t\,\bar{h} \qquad \text{WHERE} \qquad \bar{h}(x,t) \equiv \frac{1}{4\pi c^2 t^2}\int_{\partial B_{ct}(\underline{x})} h(\underline{y})\, dS(\underline{y})$$

IS THE AVERAGE VALUE OF $h(\underline{y})$ OVER THE <u>SURFACE</u> OF A SPHERE OF RADIUS $ct$ CENTERED AT $\underline{y} = \underline{x}$.

NOW WITH AN IDENTICAL EVALUATION OF THE SECOND INTEGRAL IN (15) WE GET:

<u>THEOREM</u> (KIRCHOFF'S FORMULA). IN 3-D THE SOLUTION TO

$$u_{tt} - c^2 \Delta u = 0 \quad , \quad \underline{x} \in \mathbb{R}^3, \; t > 0$$
$$u(\underline{x},0) = g(\underline{x}), \quad u_t(\underline{x},0) = h(\underline{x})$$

IS SIMPLY

$$u(\underline{x},t) = t\bar{h} + \frac{\partial}{\partial t}\left(t\bar{g}\right) \qquad (17)$$

WHERE
$$\bar{h} \equiv \frac{1}{4\pi c^2 t^2}\int_{\partial B_{ct}(\underline{x})} h(\underline{y})\, dS(\underline{y}), \qquad \bar{g} \equiv \frac{1}{4\pi c^2 t^2}\int_{\partial B_{ct}(\underline{x})} g(\underline{y})\, dS(\underline{y}).$$

WITH $\bar{h} = h(\underline{x},t)$ AND $\bar{g} = g(\underline{x},t)$ □

THIS IS THE 3-D ANALOGOUE OF D'ALEMBERT'S FORMULA. WE WILL DISCUSS THE IMPLICATIONS OF THIS BELOW.

WE WILL FIRST REWRITE (17). WE LET $\underline{y} = \underline{x} + ct\,\underline{w}$ AND SO $\partial B_{ct}(\underline{x})$ BECOMES A BALL BOUNDARY OF RADIUS 1 IN $\underline{w}$-PLANE. $dS(\underline{y}) = c^2t^2\, dS(\underline{w})$ WHERE $dS(\underline{w}) \equiv \sin\phi\, d\phi\, d\varphi$ AND
$$\underline{w} = \begin{pmatrix} \cos\varphi \sin\phi \\ \sin\varphi \sin\phi \\ \cos\phi \end{pmatrix}.$$

IN THIS WAY
$$\bar{h} = \frac{1}{4\pi}\int_{\partial B_1(0)} h(\underline{x} + ct\,\underline{w})\, dS(\underline{w}).$$

AND SO (17) BECOMES

$$u(\underline{x},t) = \frac{t}{4\pi}\int_{\partial B_1(0)} h(\underline{x} + ct\,\underline{w})\, dS(\underline{w}) + \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{\partial B_1(0)} g(\underline{x} + ct\,\underline{w})\, dS(\underline{w})\right). \qquad (18)$$

WHICH IS AN EQUIVALENT WAY OF WRITING KIRCHOFF'S FORMULA.

WE NOW BRIEFLY EXPLORE THE GEOMETRICAL FEATURES ASSOCIATED WITH (17).

- WE WILL DISCUSS <u>HUYGENS PRINCIPLE</u> — CRISP WAVE PROPAGATION IN 3-D
- FINDING THE <u>WAVEFRONT</u> OF INITIAL DATA, WHICH HAS COMPACT SUPPORT.

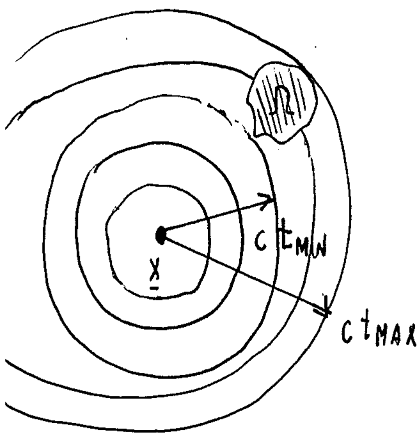
HUYGENS PRINCIPLE   SUPPOSE $g=0$ AND THAT $h$ HAS COMPACT SUPPORT IN SOME SET $\Omega$.

i.e. $h \neq 0$ IF $\underline{y} \in \Omega$ AND $h=0$ IF $\underline{y} \notin \Omega$.

WE ARE AN OBSERVER AT POINT $\underline{x}$ AND WE WANT TO SEE AT WHAT TIMES WILL WE "SEE" THE SIGNAL, i.e. FOR WHAT $t$ IS $u(\underline{x},t) \neq 0$.

OUR ANALYTICAL SOLUTION IS
$$u(\underline{x},t) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\underline{x})} h(\underline{y})\, dS(\underline{y}). \quad (*)$$

WE PLOT SPHERES OF RADIUS $ct$ CENTERED AT $\underline{x}$. ONLY THOSE SPHERES THAT INTERSECT $\Omega$ WILL GENERATE A NON-ZERO SIGNAL.




$\underline{y}$-SPACE

WE CONCLUDE THAT $u(\underline{x},t) \neq 0$ ONLY FOR $t_{MIN} < t < t_{MAX}$ AND $u(\underline{x},t)=0$ FOR $t < t_{MIN}$ AND $t > t_{MAX}$

THUS WE HAVE A SHARP SIGNAL PROPAGATION.

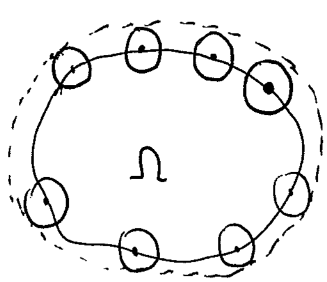
FOR $t_{MIN} < t < t_{MAX}$, $u(\underline{x},t)$ IS DETERMINED BY THE PORTION OF THE SPHERICAL SURFACE THAT LIES WITHIN $\Omega$.

i.e. $$u(\underline{x},t) = \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(\underline{x}) \cap \Omega} h(\underline{y})\, dS(\underline{y})$$

NEXT, WE ASK A DIFFERENT QUESTION. GIVEN $g=0$ AND

$h \neq 0$ IF $\underline{y} \in \Omega$ AND $h=0$ IF $\underline{y} \notin \Omega$.

WHAT IS THE BOUNDARY OF THE SET IN THE $\underline{y}$-SPACE AFTER TIME $t>0$, $t$ SMALL?




TAKE POINTS ON THE BOUNDARY OF $\Omega$, AND DRAW A SPHERE CENTERED AT THAT POINT OF RADIUS $ct$.

THE WAVEFRONT IS DEFINED AS THE ENVELOPE OF ALL OF THE TANGENT PLANES TO THE EDGE OF THE SPHERES. THIS REPRESENTS THE BOUNDARY OF THE REGION SEPARATING $u \neq 0$ FROM $u=0$.

EXAMPLE (FOCUSSING OF SINGULARITIES IN 3-D).

CONSIDER THE 3-D WAVE EQUATION

$$u_{tt} = c^2 \Delta u \qquad \underline{x} \in \mathbb{R}^3$$

WITH $\quad u(\underline{x}, 0) = \phi(|\underline{x}|), \quad u_t(\underline{x}, 0) = 0, \quad \underline{x} \in \mathbb{R}^3.$

(i) DERIVE A FORMULA FOR $u(0,t)$ USING KIRCHHOFF'S GENERAL SOLUTION

(ii) SUPPOSE THAT $\phi(r)$ IS A CONTINUOUS FUNCTION OF THE FORM

$$\phi(r) = \begin{cases} \sqrt{1-r} & \text{FOR } 0 \leq r \leq 1 \\ 0 & \text{FOR } r > 1 \end{cases}$$

CALCULATE $u(0,t)$ AND SHOW THAT $u(0,t) \to -\infty$ AS $t \to (1/c)^-$.

(REMARK: THE SINGULARITY OF $\phi'(r)$ AT $r=1$ HAS BEEN FOCUSSED AND LEADS TO A FINITE-TIME SINGULARITY).

SOLUTION FROM EQUATION (18) ON PAGE 8 WE HAVE THAT THE SOLUTION TO

$$u_{tt} - c^2 \Delta u = 0, \quad \underline{x} \in \mathbb{R}^3, \ t > 0$$

$$u(\underline{x}, 0) = \phi(|\underline{x}|), \quad u_t(\underline{x}, 0) = 0$$

IS

$$u(\underline{x}, t) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{\partial B_1(0)} \phi(|\underline{x} + ct\,\underline{w}|)\, dS(w)\right).$$

WHERE $\underline{w} = \begin{pmatrix} \cos\phi \sin\varphi \\ \sin\phi \sin\varphi \\ \cos\varphi \end{pmatrix}$ IN SPHERICAL COORDINATES AND $\partial B_1(0)$ IS THE BOUNDARY OF THE UNIT BALL. NOW AT $\underline{x} = 0$, $\phi(|\underline{x} + ct\,w|) = \phi(ct)$ SINCE $t > 0$, $|\underline{w}| = 1$

THUS $$u(0, t) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\phi(ct)\int_{\partial B_1(0)} dS(w)\right) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\phi(ct)\, 4\pi\right) = \frac{\partial}{\partial t}\left(t\,\phi(ct)\right).$$

THUS $u(\underline{0}, t) = \phi(ct) + ct\,\phi'(ct)$ IS THE SOLUTION.

(ii) NOW LET $\phi(r) = \begin{cases} \sqrt{1-r} & \text{FOR } 0 \leq r < 1 \\ 0 & \text{FOR } r > 1 \end{cases} \longrightarrow \phi'(r) = \begin{cases} -\frac{1}{2}\frac{1}{\sqrt{1-r}} & \text{FOR } 0 \leq r \leq 1 \\ 0 & \text{FOR } r > 1. \end{cases}$

THUS $\phi(ct) = \begin{cases} \sqrt{1-ct} & \text{FOR } 0 \leq ct < 1 \\ 0 & \text{FOR } ct > 1 \end{cases}$

HENCE FOR $t < 1/c$ WE HAVE $u(0,t) = \phi(ct) + ct\,\phi'(ct) = \sqrt{1-ct} + ct\left(-\frac{1}{2\sqrt{1-ct}}\right)$

WE GET $$u(0,t) = \sqrt{1-ct} - \frac{ct}{2\sqrt{1-ct}} \quad \text{FOR } t < 1/c.$$

OBSERVE THAT $u(0,t) \to -\infty$ AS $t \to (1/c)^-$.

## TWO DIMENSIONS

FOR THE 2-D PROBLEM WE HAVE

$$u(\underset{\sim}{x},t) = \int_{\mathbb{R}^2} G(\underset{\sim}{x},\underset{\sim}{y},t)\, h(\underset{\sim}{y})\, d\underset{\sim}{y} + \frac{\partial}{\partial t}\left( \int_{\mathbb{R}^2} G(\underset{\sim}{x},\underset{\sim}{y},t)\, g(\underset{\sim}{y})\, d\underset{\sim}{y} \right).$$

WHERE $G(\underset{\sim}{x},\underset{\sim}{y},t) = \dfrac{1}{2\pi c^2}\,\dfrac{H(t-r/c)}{\sqrt{t^2-r^2/c^2}} = \dfrac{H(t-r/c)}{2\pi c\sqrt{c^2t^2-r^2}}$ WITH $r = |\underset{\sim}{x}-\underset{\sim}{y}|$.

SINCE $H=1$ FOR $0<r<ct$ AND $H=0$ FOR $r>ct$ WE HAVE

$$u(\underset{\sim}{x},t) = \frac{1}{2\pi c}\int_0^{ct} \frac{1}{\sqrt{c^2t^2-r^2}} \left( \int_{\partial B_r(\underset{\sim}{x})} h(\underset{\sim}{y})\, dS(\underset{\sim}{y}) \right) dr + \frac{\partial}{\partial t}\left( \frac{1}{2\pi c}\int_0^{ct} \frac{1}{\sqrt{c^2t^2-r^2}} \left( \int_{\partial B_r(\underset{\sim}{x})} g(\underset{\sim}{y})\, dS(\underset{\sim}{y}) \right) dr \right)$$

WHERE $\partial B_r(\underset{\sim}{x})$ IS THE BOUNDARY OF THE DISK CENTERED AT $\underset{\sim}{x}$ OF RADIUS $r$.

i.e. $\partial B_r(\underset{\sim}{x}) \equiv \{ y \mid |\underset{\sim}{y}-\underset{\sim}{x}| = r \}$, AND $dS(y) = r\, d\varphi$ WITH $\underset{\sim}{y}-\underset{\sim}{x} = r\underset{\sim}{e}_\varphi$

WHERE $e_\varphi = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$. THEN WE HAVE PUTTING THE INTEGRALS TOGETHER:

$$u(\underset{\sim}{x},t) = \frac{1}{2\pi c}\int_{B_{ct}(\underset{\sim}{x})} \frac{1}{\sqrt{c^2t^2-r^2}}\, h(\underset{\sim}{y})\, d\underset{\sim}{y} + \frac{\partial}{\partial t}\left( \frac{1}{2\pi c}\int_{B_{ct}(\underset{\sim}{x})} \frac{g(\underset{\sim}{y})}{\sqrt{c^2t^2-r^2}}\, d\underset{\sim}{y} \right), \quad (19)$$

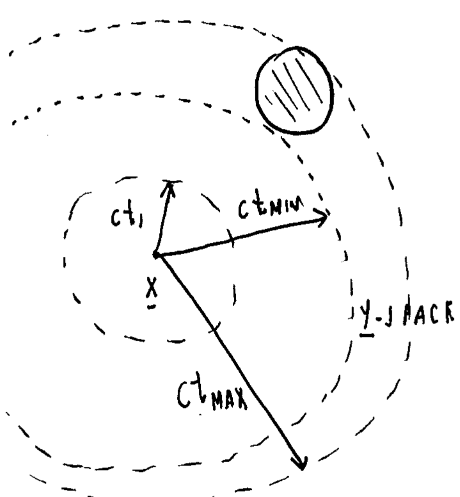
WITH $r = |\underset{\sim}{y}-\underset{\sim}{x}|$.

HERE $B_{ct}(\underset{\sim}{x}) \equiv \{ y \mid |\underset{\sim}{y}-\underset{\sim}{x}| \le ct \}$. WE EMPHASIZE HERE THAT THE INTEGRATION IN (19) IS OVER THE ENTIRE BALL $B_{ct}(\underset{\sim}{x})$ (i.e. A 2-D INTEGRAL) AND <u>NOT</u> AN INTEGRAL OVER THE BOUNDARY $\partial B_{ct}(\underset{\sim}{x})$. THIS EXPRESSION IS CALLED <u>POISSON'S FORMULA</u>.

TO ILLUSTRATE THE DIFFERENCE BETWEEN OUR 2-D CASE AND THE 3-D CASE CONSIDERED PREVIOUSLY SUPPOSE $g(\underset{\sim}{y}) \equiv 0$ AND

$$h(\underset{\sim}{y}) \neq 0 \text{ IN } \Omega \quad \text{AND} \quad h(\underset{\sim}{y}) = 0 \text{ OUTSIDE } \Omega.$$

CONSIDER A POINT $\underset{\sim}{x}$ AS SHOWN. WE DRAW A SERIES OF DISKS CENTERED AT $\underset{\sim}{x}$ OF RADIUS $ct$




WE HAVE
$$u(\underset{\sim}{x},t) = \frac{1}{2\pi c}\int_{B_{ct}(\underset{\sim}{x})} \frac{h(\underset{\sim}{y})\, d\underset{\sim}{y}}{\sqrt{c^2t^2-r^2}} \quad (20)$$

NOTICE $u(\underset{\sim}{x},t_1) = 0$

WE HAVE THAT $u(x,t) = 0$ UP UNTIL THE LEADING EDGE $t = t_{MIN}$. HOWEVER SINCE THE INTEGRAL IN (20) IS OVER <u>THE ENTIRE BALL</u> WE HAVE THAT $u(\underset{\sim}{x},t) \neq 0$ EVEN FOR $t > t_{MAX}$ (TRAILING EDGE). THUS THE SIGNAL DOES NOT TURN OFF FOR $t > t_{MAX}$.

WE CONCLUDE THAT THERE IS NO CRISP SIGNAL IN 2-D, I.E. THAT HUYGENS PRINCIPLE DOES NOT HOLD IN 2-D.

TO ILLUSTRATE THIS EFFECT PUT A CORK IN WATER AND THROW A STONE ON THE SURFACE OF THE WATER. THE CORK BOBS IN THE WATER WHEN HIT BY THE LEADING EDGE OF THE WAVEFRONT. IT CONTINUES TO OSCILLATE LONG AFTER THE TRAILING EDGE OF THE WAVEFRONT PASSES BY.

FINALLY IN 2-D WE TRY TO WRITE POISSON'S FORMULA (19) INTO A MORE CONVENIENT FORM. WE HAVE

$$u(\underline{x},t) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\left(\int_{B_{ct}(\underline{x})}\frac{g(\underline{y})}{\sqrt{c^2t^2-|\underline{x}-\underline{y}|^2}}\,d\underline{y}\right) + \frac{1}{2\pi c}\int_{B_{ct}(\underline{x})}\frac{h(\underline{y})}{\sqrt{c^2t^2-|\underline{x}-\underline{y}|^2}}\,d\underline{y} \qquad (21)$$

IN THE FIRST INTEGRAL WE PUT $\underline{y} = \underline{x} + ct\,\underline{z}$. THEN $d\underline{y} = c^2t^2\,d\underline{z}$.

WE OBTAIN
$$I_1 = \frac{1}{2\pi c}\frac{\partial}{\partial t}\left(\frac{c^2t^2}{ct}\int_{B_1(0)}\frac{g(\underline{x}+ct\,\underline{z})\,d\underline{z}}{\sqrt{1-|\underline{z}|^2}}\right)$$
$$= \frac{1}{2\pi}\frac{\partial}{\partial t}\left(t\int_{B_1(0)}\frac{g(\underline{x}+ct\,\underline{z})\,d\underline{z}}{\sqrt{1-|\underline{z}|^2}}\right)$$
$$I_1 = \frac{1}{2\pi}\left[\int_{B_1(0)}\frac{g(\underline{x}+ct\,\underline{z})}{\sqrt{1-|\underline{z}|^2}}\,d\underline{z} + t\int_{B_1(0)}\frac{\nabla g(\underline{x}+ct\,\underline{z})\cdot c\,\underline{z}}{\sqrt{1-|\underline{z}|^2}}\,d\underline{z}\right]$$

NOW CONVERT BACK USING $\underline{y} = \underline{x} + ct\,\underline{z}$.

WE OBTAIN
$$I_1 = \frac{1}{2\pi}c^2t^2\int_{B_1(0)}\frac{g(\underline{x}+ct\,\underline{z})\,d\underline{z}}{c^2t^2\sqrt{1-|\underline{z}|^2}} + \frac{(ct)^3}{2\pi(ct)^2}\int_{B_1(0)}\frac{\nabla g(\underline{x}+ct\,\underline{z})\cdot\underline{z}\,d\underline{z}}{\sqrt{1-|\underline{z}|^2}}$$

BUT $c^2t^2\,d\underline{z} = d\underline{y}$ AND $ct\,\underline{z} = \underline{y}-\underline{x}$. THIS GIVES

$$I_1 = \frac{1}{2\pi ct}\int_{B_{ct}(\underline{x})}\frac{g(\underline{y})\,d\underline{y}}{\sqrt{c^2t^2-|\underline{x}-\underline{y}|^2}} + \frac{1}{2\pi ct}\int_{B_{ct}(\underline{x})}\frac{\nabla g(\underline{y})\cdot(\underline{y}-\underline{x})\,d\underline{y}}{\sqrt{c^2t^2-|\underline{x}-\underline{y}|^2}} \qquad (22)$$

NOW SUBSTITUTING (22) INTO (21) WE FINALLY OBTAIN

$$u(\underline{x},t) = \frac{1}{2\pi ct}\int_{B_{ct}(\underline{x})}\frac{[g(\underline{y})+\nabla g(\underline{y})\cdot(\underline{y}-\underline{x})+t\,h(\underline{y})]}{\sqrt{c^2t^2-|\underline{x}-\underline{y}|^2}}\,d\underline{y}. \qquad (23)$$

THIS EXPRESSION IS ESPECIALLY USEFUL SINCE NO FURTHER DIFFERENTIATIONS ARE NEEDED. GIVEN $g(\underline{y})$ AND $h(\underline{y})$ WE CAN CALCULATE $u(\underline{x},t)$.

FINALLY WE WILL SHOW HOW TO DEAL WITH INHOMOGENEOUS TERMS IN THE PDE: WE WANT TO FIND THE SOLUTION TO

$$\left.\begin{aligned} u_{tt} &= c^2 \Delta u + F(\underline{x}, t) \qquad \underline{x} \in \mathbb{R}^N,\ t > 0 \\ u &= u_t = 0 \quad \text{AT } t = 0. \end{aligned}\right\} \quad (24)$$

WITH

WE WILL FOCUS ON THE 2-D OR 3-D CASES $N=2$ OR $N=3$.

A CONVENIENT WAY TO SOLVE (24) IS WITH DUHAMEL'S PRINCIPLE. WE FIRST ILLUSTRATE THIS METHODOLOGY WITH A SIMPLE ODE EXAMPLE.

CONSIDER THE ODE FOR $u(t)$ GIVEN FOR $a$ CONSTANT BY

$$u' + a u = f(t) \quad \text{IN } t > 0 \qquad (25)$$
$$u(0) = 0.$$

THE INTEGRATING FACTOR IS $e^{at}$ AND SO $(u e^{at})' = f(t) e^{at}$ AND WE OBTAIN $u e^{at} = \int_0^t f(\tau) e^{a\tau}\, d\tau$ OR $u = \int_0^t e^{-a(t-\tau)} f(\tau)\, d\tau$. THUS THE SOLUTION OPERATOR OF THE HOMOGENEOUS PROBLEM $u' + au = 0$ GIVEN BY $S(t) \equiv e^{-at}$ CAN BE USED AS

$$u(t) = \int_0^t S(t-\tau) f(\tau)\, d\tau. \qquad (26)$$

WE NOW SHOW HOW TO RECOVER (26) USING THE SOLUTION TO A PARAMETRIZED INITIAL VALUE PROBLEM. DEFINE $w(t;\tau)$ BY SOLUTION TO

$$\left.\begin{aligned} &w' + a w = 0 \quad \text{IN } t > \tau \\ &w = f(\tau) \quad \text{FOR } t = \tau, \quad \text{FOR ANY } \tau \text{ IN } [0,t]. \end{aligned}\right\} \quad (27)$$

WE THEN CLAIM THAT $u(t) = \int_0^t w(t;\tau)\, d\tau$. (28)

<u>PROOF</u> $u(0) = 0$ IS CLEAR AND $u'(t) = w(t;t) + \int_0^t w_t(t;\tau)\, d\tau$

THUS $$u' + au = w(t;t) + \underbrace{\int_0^t (w_t + aw)\, d\tau}_{=\,0} = f(t) + 0.$$

HENCE TO SOLVE (25) WE NEED ONLY SOLVE A FAMILY (PARAMETRIZED BY $\tau$) OF INITIAL VALUE PROBLEMS. THE SOLUTION TO (27) IS $w = f(\tau) e^{-a(t-\tau)}$ AND THEN $u(t) = \int_0^t w\, d\tau = \int_0^t f(\tau) e^{-a(t-\tau)}\, d\tau$. THIS IS DUHAMEL'S PRINCIPLE AND IT CAN BE USED IN OTHER CONTEXTS.

NOW WE APPLY A SIMILAR IDEA TO (24). LET $w(\underline{x},t;\tau)$ SATISFY THE FAMILY OF INITIAL VALUE PROBLEMS

$$\left.\begin{array}{l} w_{tt} = c^2 \Delta w \ , \quad \underline{x} \in \mathbb{R}^N, \ t \geq \tau \\ w(\underline{x},t;\tau) = 0, \quad w_t(\underline{x},t;\tau) = F(\underline{x},\tau) \ \text{ AT } \ t=\tau. \end{array}\right\} \quad (28)$$

FOR <u>ANY</u> $\tau$ IN $0 \leq \tau \leq t$. THEN WE CLAIM THAT

$$u(\underline{x},t) = \int_0^t w(\underline{x},t;\tau)\, d\tau \quad \text{SOLVES } (24)$$

<u>PROOF</u> WE WILL DO A FORMAL PROOF BY SIMPLY DIFFERENTIATING UNDER THE INTEGRAL SIGN. NOW CLEARLY $u(\underline{x},0)=0$ AND

$$u_t(\underline{x},t) = w(\underline{x},t;t) + \int_0^t w_t(\underline{x},t;\tau)\, d\tau.$$

BUT $w(\underline{x},t;t)=0$ AND SO $u_t(\underline{x},t) = \int_0^t w_t(\underline{x},t;\tau)\,d\tau$ SO THAT $u_t(\underline{x},0)=0$.

DIFFERENTIATE AGAIN:

$$u_{tt}(\underline{x},t) = w_t(x,t;t) + \int_0^t w_{tt}(x,t;\tau)\,d\tau = F(\underline{x},t) + \int_0^t w_{tt}(x,t;\tau)\,d\tau$$

BY (28). MOREOVER, $$c^2 \Delta u = \int_0^t c^2 \Delta w \, d\tau$$

AND SO $$u_{tt} - c^2 \Delta u = F(\underline{x},t) + \int_0^t \underbrace{(w_{tt} - c^2\Delta w)}_{=0}\, d\tau = F(\underline{x},t) \qquad \text{D.}$$

THUS $u(\underline{x},t)$ SATISFIES (24).

NOW WE WILL FIND SOLUTION TO (24) IN DIMENSION $N=3$. THE SOLUTION TO (28) IS GIVEN BY (17) WITH $t$ REPLACED BY $t-\tau$:

$$w(\underline{x},t;\tau) = \frac{1}{4\pi c^2 (t-\tau)} \int_{\partial B_{c(t-\tau)}(\underline{x})} F(\underline{y},\tau)\, dS(y)$$

WHERE $\partial B_{c(t-\tau)}(\underline{x}) = \{ \underline{y} \mid |\underline{y}-\underline{x}| \leq c(t-\tau) \}$. THEN

$$u(\underline{x},t) = \int_0^t w(\underline{x},t;\tau)\,d\tau = \int_0^t \frac{1}{4\pi c^2 (t-\tau)} \left( \int_{\partial B_{c(t-\tau)}(\underline{x})} F(\underline{y},\tau)\, dS(\underline{y}) \right) d\tau.$$

NOW CHANGE VARIABLES: LET

$$r = c(t-\tau) \quad \text{SO} \quad \tau = t - r/c \quad \text{SO} \quad d\tau = -dr/c.$$

THIS YIELDS $$u(\underline{x},t) = \int_{ct}^0 \frac{1}{4\pi c r} \left(-\frac{1}{c}\right) \left( \int_{\partial B_r(\underline{x})} F(\underline{y}, \underset{\uparrow \atop \tau = t - r/c}{\tau})\, dS(\underline{y}) \right) dr.$$

THIS YIELDS THAT

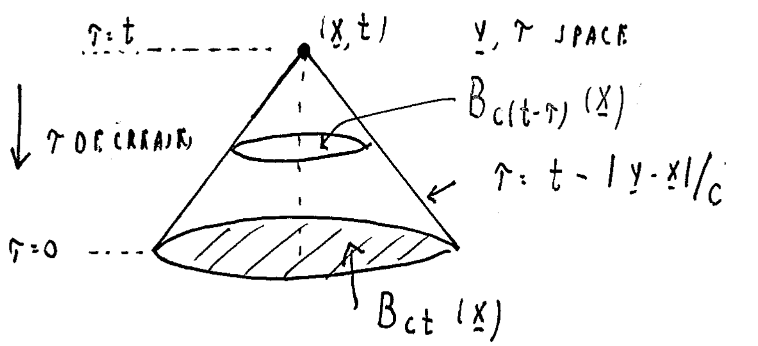
$$u(\underline{x},t) = \frac{1}{4\pi c^2}\int_0^{ct}\frac{1}{r}\left(\int_{\partial B_r(\underline{x})} F(\underline{y}, t-r/c)\, dS(\underline{y})\right) dr.$$

WHERE $\partial B_r(\underline{x}) \equiv \{ \underline{y} \mid |\underline{y}-\underline{x}| = r\}$. SINCE $r$ IS THEN INTEGRATED OVER $0<r<ct$ WE HAVE OUR FINAL RESULT IN 3-D. THE SOLUTION IN (3-D) TO (24) IS

$$u(\underline{x},t) = \frac{1}{4\pi c^2}\int_{B_{ct}(\underline{x})} \frac{F(\underline{y}, t-r/c)}{r}\, d\underline{y} \quad \text{WHERE } r = |\underline{y}-\underline{x}|. \qquad (29).$$

THIS IS THEN A VOLUME INTEGRAL OVER THE BALL $B_{ct}(\underline{x}) \equiv \{ \underline{y} \mid |\underline{y}-\underline{x}| \leq ct\}$.

__REMARK__ THIS IS CALLED THE __LORENZ SOLUTION__ AND THE INTEGRAND IS CALLED THE RETARDED POTENTIAL SINCE IT DEPENDS NOT ON $t$ BUT ON THE EARLIER TIME $t-r/c$. TO ILLUSTRATE (29) WE PLOT THE RETROGRADE LIGHT CONE




THE SOLUTION AT POINT $\underline{x}$ AND TIME $t$ DEPENDS ON VALUES OF $F$ ON THE CROSS-SECTION OF THIS LIGHT CONE THAT WAS CONTRIBUTED A TIME $|\underline{y}-\underline{x}|/c$ EARLIER i.e. THE SHADED REGION SHOWN. THIS IS A 3-D VOLUME INTEGRAL BUT SHOWN HERE AS A DISK.

FINALLY, WE DO THE 2-D ANALOGUE OF THIS RESULT.

FOR THE 2-D PROBLEM WE HAVE FROM SETTING $g=0$, $h = F(\underline{y},\tau)$ IN (21) AND REPLACING $t$ BY $t-\tau$ THAT

$$w(x,t;\tau) = \frac{1}{2\pi c}\int_{B_{c(t-\tau)}(\underline{x})} \frac{F(\underline{y},\tau)}{\sqrt{c^2(t-\tau)^2 - |\underline{x}-\underline{y}|^2}}\, d\underline{y}.$$

SO THAT

$$u(\underline{x},t) = \frac{1}{2\pi c}\int_0^t \left(\int_{B_{c(t-\tau)}(\underline{x})} \frac{F(\underline{y},\tau)}{\sqrt{c^2(t-\tau)^2 - |\underline{y}-\underline{x}|^2}}\, d\underline{y}\right) d\tau.$$

NOW LET $\sigma = c(t-\tau)$ BE A NEW VARIABLE. THEN $\tau = t - \sigma/c$ SO $d\tau = -\frac{1}{c}\, d\sigma$

AND SO

$$u(\underline{x},t) = \frac{1}{2\pi c}\int_{ct}^{0}\left(-\frac{1}{c}\right)\left(\int_{B_\sigma(\underline{x})} \frac{F(\underline{y}, t-\sigma/c)}{\sqrt{\sigma^2 - |\underline{y}-\underline{x}|^2}}\, d\underline{y}\right) d\sigma.$$

THIS YIELDS THE 2-D SOLUTION

$$u(\underline{x}, t) = \frac{1}{2\pi c^2} \int_0^{ct} \left( \int_{B_\sigma(\underline{x})} \frac{F(\underline{y}, t - \sigma/c)}{\sqrt{\sigma^2 - |\underline{y} - \underline{x}|^2}} \, d\underline{y} \right) d\sigma . \qquad (30)$$

IN CONTRAST TO THE 3-D CASE, THIS RESULT SAYS THAT $u(\underline{x}, t)$ IS DETERMINED BY THE <u>ENTIRE</u> LIGHT-CONE. (i.e. AN INTEGRAL OVER SPACE <u>AND</u> TIME).