VARIATIONAL PROBLEMS FOR ODE'S

NOW WE CONSIDER VARIATIONAL PROBLEMS IN 1-D OF FINDING THE CURVE $y = f(x)$ ON $a \le x \le b$ WITH $y(a)$ AND $y(b)$ GIVEN THAT MINIMIZE THE FUNCTIONAL

$$I(y) = \int_a^b F(x, y, y')\, dx. \qquad (1)$$

WE WILL DERIVE A NECESSARY CONDITION THAT $y = y^*(x)$ IS A MINIMIZER (OR AT LEAST A STATIONARY POINT) OF $I(y)$.

WE PUT $y = y^* + \varepsilon w$ INTO (1) AND LINEAR IN ε. WE OBTAIN FROM TAYLOR SERIES THAT

$$J(\varepsilon) = I(y^* + \varepsilon w) = \int_a^b F(x, y^* + \varepsilon w, y^{*\prime} + \varepsilon w')\, dx = \int_a^b F(x, y^*, y^{*\prime})\, dx + \varepsilon \int_a^b [F_y(x, y^*, y^{*\prime}) w + F_{y'}(x, y^*, y^{*\prime}) w']\, dx + O(\varepsilon^2).$$

THUS

$$J(\varepsilon) = I(y^* + \varepsilon w) = I(y^*) + \varepsilon \int_a^b [F_y^* w + F_{y'}^* w']\, dx + O(\varepsilon^2), \quad \text{WHERE } F_y^* \equiv F_y(x, y^*, y^{*\prime}).$$

NOW A NECESSARY CONDITION FOR y^* TO BE A STATIONARY POINT OF $I(y)$ IS THAT $J'(\varepsilon)\big|_{\varepsilon=0} = 0$ SO THAT

$$\int_a^b (F_y^* w + F_{y'}^* w')\, dx = 0 \qquad \forall\, w \in C^2[a,b]. \qquad (2)$$

NOW WE WRITE $\frac{d}{dx}(F_{y'}^* w) = \frac{d}{dx}(F_{y'}^*) w + F_{y'}^* w'$ BY PRODUCT RULE SO THAT (2) BECOMES

$$0 = \int_a^b \left[\left(F_y^* w - \frac{d}{dx}(F_{y'}^*) w\right) + \frac{d}{dx}(F_{y'}^* w)\right] dx = \int_a^b \left(F_y^* - \frac{d}{dx} F_{y'}^*\right) w\, dx + F_{y'}^* w \Big|_a^b \qquad (3).$$

NOW SINCE $y(a), y(b)$ ARE SPECIFIED, I.E. $y(a) = y_a$, $y(b) = y_b$ WE HAVE $y^*(a) = y_a$, $y^*(b) = y_b$ AND SO $w(a) = w(b) = 0$. THUS IN (3), THE BOUNDARY-TERM VANISHES AND y^* IS A MINIMIZER (OR STATIONARY POINT) IF AND ONLY IF

$$\int_a^b \left(F_y^* - \frac{d}{dx} F_{y'}^*\right) w\, dx = 0 \qquad \forall\, w \in H \equiv \{ w \mid w \text{ IN } C^2[0,1],\ w(a) = w(b) = 0 \}.$$

BY THE FUNDAMENTAL LEMMA OF VARIATIONAL CALCULUS WE MUST HAVE THAT y^* SATISFIES EULER-LAGRANGE EQUATION:

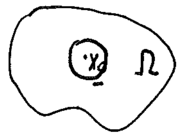
$$F_{y^*} - \frac{d}{dx} F_{y'}^* = 0 \quad \text{ON } a < x < b; \quad y^*(a) = y_a,\ y^*(b) = y_b \qquad (4)$$

WHICH IS A SECOND ORDER NONLINEAR ODE BOUNDARY VALUE PROBLEM.

WE HAVE SEEN THE FUNDAMENTAL LEMMA BEFORE IN OUR STUDY OF EIGENVALUE PROBLEMS.

LEMMA (FUNDAMENTAL LEMMA) IF $g(\underline{x})$ IS A CONTINUOUS FUNCTION ON Ω AND IF $\int_\Omega g(\underline{x})\, w(\underline{x})\, d\underline{x} = 0$ FOR ALL W ON Ω THEN $g(X) \equiv 0$ IN Ω.

PROOF SUPPOSE FOR SOME $\underline{X}_0 \in \Omega$ THAT $g(\underline{X}_0) \neq 0$. SINCE g IS CONTINUOUS, IN SOME BALL B AROUND $\underline{X}_0$ WE HAVE EITHER $g(X) > 0$ OR $g(x) < 0$. SO LET $w(\underline{x})$ BE A LITTLE BUMP CENTERED AT $\underline{X}_0$ WITH $W(X_0) = \text{SIGN}(g(X_0))$ AND $W > 0$ IN B, $W = 0$ OUTSIDE B.
THEN $\int_\Omega g(\underline{x})\, w(\underline{x})\, d\underline{x} > 0$.



IN SUMMARY, CONSIDER THE VARIATIONAL PROBLEM

$$\min_{y \in H} I(y) \quad \text{WITH} \quad I(y) = \int_a^b F(x, y, y')\, dx \quad \text{AND} \quad H \equiv \{\, y \mid y \in C^2[a,b] \ \text{WITH } y(a), y(b) \text{ FIXED} \,\}$$

THEN WITH $y = y^* + \varepsilon w$ WITH $w \in C^2[a,b]$ AND $w(a) = w(b) = 0$ WE HAVE THAT THE OPTIMIZER MUST SATISFY THE E-L EQUATION

$$F_y - \frac{d}{dx} F_{y'} = 0 \quad \text{WITH} \quad y(a), y(b) \text{ GIVEN}.$$

NOW SUPPOSE THAT $y(a)$ IS GIVEN BUT THAT NO CONDITION FOR y AT $x = b$ IS GIVEN. THEN FROM (3) WE HAVE THAT THE OPTIMIZER y SATISFIES THE E-L EQUATION

$$\left.\begin{aligned} &F_y - \frac{d}{dx} F_{y'} = 0 \quad \text{ON } a < x < b \\ &y(a) \text{ SPECIFIED AND } \quad F_{y'} = 0 \quad \text{AT } x = b \ \text{(NATURAL BC)}. \end{aligned}\right\} \quad (5)$$

REMARK IN (3), THE COMPETITOR W SATISFIES $W(a) = 0$. WE MUST HAVE THAT $\forall\, w \in H \equiv \{\, w \text{ IN } C^2[a,b],\ w(a) = 0 \,\}$

$$\int_a^b w\Big[F_{y^*} - \frac{d}{dx} F_{y'}^* \Big]\, dx + F_{y'}^* w \Big|_a^b = 0.$$

FOR COMPETITORS WITH $w(b) = 0$ WE MUST HAVE $F_y^* - \frac{d}{dx} F_{y'} = 0$. THEN FOR COMPETITORS WITH $w(b) \neq 0$ WE MUST HAVE $F_{y'}^* = 0$.

NOW WE WILL CONSIDER AN IMPORTANT SPECIAL CASE.

LEMMA 1 SUPPOSE THAT F IS INDEPENDENT OF X, I.E. $F = F(y, y')$, THEN THE E-L EQUATION GIVEN BY

$$F_y - \frac{d}{dx} F_{y'} = 0$$

HAS THE FIRST INTEGRAL $y' F_{y'} - F = C \quad \forall X$ WHERE C IS A CONSTANT.

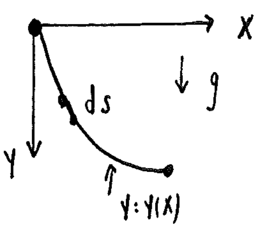
PROOF MULTIPLY BY y': $\quad y' F_y - y' \frac{d}{dx} F_{y'} = 0$. NOW ADD AND SUBTRACT $y'' F_{y'}$ SO THAT

$$y'' F_{y'} + y' F_y - y'\Big(\frac{d}{dx} F_{y'}\Big) - y'' F_{y'} = 0 \longrightarrow \frac{d}{dx}\big[-y' F_{y'} + F \big] = 0$$

$$\longrightarrow F - y' F_{y'} = C.$$

EXAMPLE (BRACHISTICHRONE)

WE WISH TO FIND THE PATH CONNECTING TWO POINTS IN (X, Y) PLANE THAT GIVES SMALLEST TRAVEL TIME FOR A PARTICLE FALLING UNDER GRAVITY, RELEASED FROM REST.



NOW THE POTENTIAL ENERGY IS $-mgy$

AND THE SPEED OF PARTICLE AT A POINT IS $\frac{1}{2}mv^2$.

THUS $\frac{1}{2}mv^2 = mgy$ SO $V = \sqrt{2gy}$.

NOW THE ARCLENGTH ELEMENT ds IS $ds = \sqrt{1+y'^2}\,dx$.

THE TOTAL TRAVEL TIME T IS $T = \int_{x_0}^{x_1} \frac{ds}{V} = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$.

LET'S TRY TO FIND THE OPTIMAL PATH THAT SATISFIES $y(0)=0$ AND $y(L)=L$.
WE MIGHT EXPECT THAT THE OPTIMAL PATH IS A QUARTER-CIRCLE, BUT THIS IS INCORRECT AS WE NOW SHOW. OUR PATH $y(x)$ IS TO MINIMIZE

$$T = \frac{1}{\sqrt{2g}}\int_0^L \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx \quad \text{WITH } y(0)=0, \text{ AND } y(L)=L.$$

HERE $F = \frac{\sqrt{1+y'^2}}{\sqrt{y}}$ IS INDEPENDENT OF X SO THAT BY LEMMA 1, THE EULER-LAGRANGE EQUATION HAS THE FIRST INTEGRAL $F - y'F_{y'} = C$.

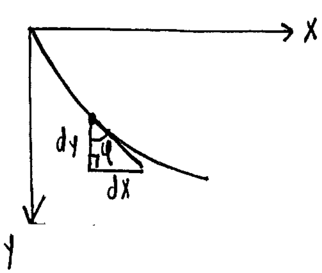
NOW $F(y,y') = y^{-1/2}\sqrt{1+y'^2}$. NOW $F_{y'} = y^{-1/2}\,y'\,(1+y'^2)^{-1/2}$ SO THAT

$$y^{-1/2}\sqrt{1+y'^2} - \frac{y^{-1/2}\,y'^2}{\sqrt{1+y'^2}} = C \;\rightarrow\; y^{-1/2}\frac{(1+y'^2-y'^2)}{\sqrt{1+y'^2}} = C.$$

THIS BECOMES $y^{-1/2} = C(1+y'^2)^{1/2}$. THIS GIVES $y^{-1} = C^2(1+y'^2)$.

WE SOLVE TO OBTAIN $y' = \sqrt{\frac{k-y}{y}}$ WITH $k = 1/C^2 > 0$.

NOW LET'S SOLVE THIS FIRST ORDER ODE GEOMETRICALLY. LET φ BE ANGLE BETWEEN THE NORMAL AND THE TANGENT LINE AS SHOWN



$dx/dy = \tan\varphi$ SO THAT $\frac{1}{\cot\varphi} = \sqrt{\frac{k-y}{y}}$.

THEN $\tan^2\varphi = \frac{y}{k-y}$ SO THAT $y[1+\tan^2\varphi] = k\tan^2\varphi$

$$\rightarrow \frac{y}{\cos^2\varphi} = k\frac{\sin^2\varphi}{\cos^2\varphi}.$$

φ = Angle between tangent to CURVE AND THE VERTICAL DIRECTION.

WE CONCLUDE THAT $y = K \sin^2 \varphi$.

NOW $dy/dx = \sqrt{\dfrac{K-y}{y}} = \dfrac{dy}{d\varphi} \Big/ dx/d\varphi$ SO THAT $\sqrt{\dfrac{K(1-\sin^2\varphi)}{K \sin^2\varphi}} = \dfrac{2K \sin\varphi \cos\varphi}{(dx/d\varphi)}$.

NOW SOLVING: $dx/d\varphi = \dfrac{2K \sin\varphi \cos\varphi}{\cot\varphi} = 2K \sin^2\varphi = K\left(1 - \cos(2\varphi)\right)$.

INTEGRATING ONCE WE OBTAIN

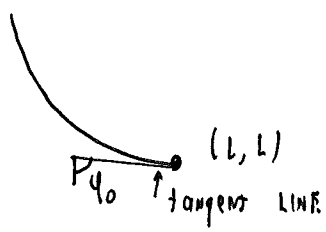
$$\left.\begin{aligned} x &= K\left(\varphi - \tfrac{1}{2}\sin(2\varphi)\right) \\ y &= K \sin^2\varphi = \tfrac{K}{2}\left(1 - \cos(2\varphi)\right) \end{aligned}\right\} \quad (*)$$

THIS PATH WHICH GOES THROUGH (0,0) IS THE PATH THAT GIVES SHORTEST TRAVEL TIME. THE PATH IS A CYCLOID, WHICH IS THE PATH OF A PARTICLE ON A BICYCLE TIRE.

FINALLY, WE MUST FIND K SO THAT $y(L) = L$.

THE PARAMETRIC FORM OF THE SOLUTION, GIVEN IN (*), SPECIFIES THE OPTIMAL PATH IN TERMS OF THE ANGLE FROM THE VERTICAL TO TANGENT LINE.

WE PUT $x = L$ AND $y = L$ IN (*) TO FIND K AND END ANGLE φ_0



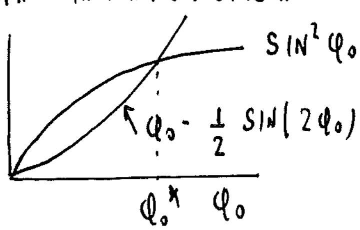
THUS $L = K \sin^2\varphi_0$

$$L = K\left(\varphi_0 - \tfrac{1}{2}\sin(2\varphi_0)\right).$$

WE ELIMINATE L TO OBTAIN THAT φ_0 SATISFIES

$$\sin^2\varphi_0 = \varphi_0 - \tfrac{1}{2}\sin(2\varphi_0) \quad (**)$$

A PLOT OF THE INTERSECTION POINTS IN (**) YIELDS



BY NEWTON'S METHOD WE GET

$\varphi_0^* \cong 1.206$

$\rightarrow$ WHICH gives $\varphi_0^* \cong 68.8°$.

EXAMPLE (HAMILTON'S PRINCIPLE)

LET $T(x, x')$ BE THE KINETIC ENERGY OF A PARTICLE AND $V(x, t)$ BE THE POTENTIAL ENERGY. THE LAGRANGIAN $L(x, x', t)$ IS DEFINED BY

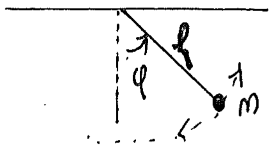
$$L(x, x', t) = T(x, x') - V(x, t)$$

HAMILTON'S LEAST ACTION PRINCIPLE IS THAT THE PARTICLE PATH $x(t)$ IS TO MINIMIZE THE ACTION $I = \int_{t_1}^{t_2} L(x, x', t)\, dt$. IN OTHER WORDS, $x(t)$ IS TO SATISFY THE E-L EQUATION

$$L_x - \frac{d}{dt} L_{x'} = 0$$

ASSOCIATED WITH L.

AS AN EXAMPLE CONSIDER THE NONLINEAR PENDULUM



THE KINETIC ENERGY IS $\frac{1}{2} m V^2$ WITH $V = \ell\dot{\varphi} \rightarrow T = \frac{1}{2} m (\ell^2 \dot{\varphi}^2)$

NOW $V = mg\ell(1-\cos\varphi)$ SO THAT $V = 0 \rightarrow \varphi = 0$ (BOTTOM)

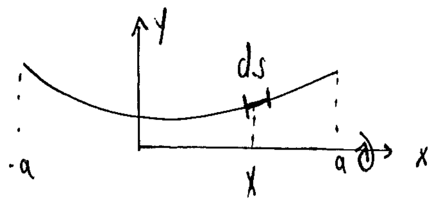
THUS $L = \frac{1}{2} m \ell^2 \dot{\varphi}^2 - mg\ell(1-\cos\varphi)$ AND $T = \int_{t_1}^{t_2} L(\varphi, \dot{\varphi})\, dt.$ $\quad L_\varphi = -mg\ell\sin\varphi$, $\quad L_{\dot{\varphi}} = m\ell^2\dot{\varphi}$

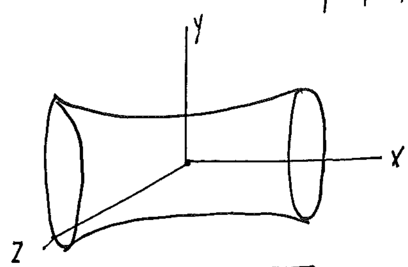
THE EL EQUATION IS $L_\varphi - \frac{d}{dt} L_{\dot{\varphi}} = 0 \rightarrow -mg\ell\sin\varphi - m\ell^2\ddot{\varphi} = 0.$

THUS $\ddot{\varphi} + \frac{g}{\ell}\sin\varphi = 0.$

EXAMPLE (CATENARY) (AREA OF SOAP FILM STRETCHED BETWEEN TWO END HOOPS) (MINIMAL SURFACE)

LET $y = y(x) > 0$ AND SUPPOSE $-a < x < a$. WE WANT TO FIND THE CURVE $y(x)$ THAT MINIMIZES THE SURFACE OBTAINED BY REVOLVING $y = y(x)$ AROUND X-AXIS.





NOW THE SURFACE AREA ELEMENT IS $dA = 2\pi y\sqrt{1+y'^2}\, dx.$

THE MINIMIZATION PROBLEM IS TO MINIMIZE $S = 2\pi\int_{-a}^{a} y\sqrt{1+y'^2}\, dx$ WITH $y(\pm a)$ PRESCRIBED. NOW $F(y, y') = y\sqrt{1+y'^2}$. FROM LEMMA 1, SINCE F IS INDEPENDENT OF X, WE HAVE THE FIRST INTEGRAL FOR THE E-L EQUATION GIVEN BY

$$F - y' F_{y'} = C.$$

THEN $F = y(1+y'^2)^{1/2}$, $F_{y'} = yy'(1+y'^2)^{-1/2}$. THUS,

$$y(1+y'^2)^{1/2} - yy'^2(1+y'^2)^{-1/2} = C$$

THEN $(1+y'^2)^{1/2}\left[\, y - yy'^2(1+y'^2)^{-1} \right] = C$

THIS GIVES $(1+y'^2)^{1/2}\, y\left[1 - \frac{y'^2}{1+y'^2} \right] = C$ SO $\frac{y}{\sqrt{1+y'^2}} = C.$

THIS YIELDS THAT $1 + y'^2 = \frac{y^2}{C^2}$ OR $y'^2 - y^2/C^2 = -1.$

NOW WE CAN PROCEED BY SOLVING FOR y' AND INTEGRATING OR ELSE MORE QUICKLY OBSERVE A FAMILY OF SOLUTIONS RECALLING $\cosh^2 x - \sinh^2 x = 1.$

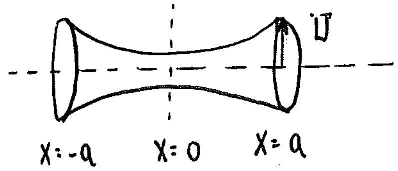
TRY $y = C\cosh\left[\frac{1}{C}(x-x_0)\right]$. THEN $y' = \sinh\left[\frac{1}{C}(x-x_0)\right]$

AND SO $y'^2 - y^2/C^2 = -1$, AS REQUIRED.

HENCE SOLUTIONS TO THE EL EQUATION ARE

$$y = C\cosh\left[\frac{1}{C}(x-x_0)\right] \text{ ON } -a<x<a.$$ THIS IS A CATENARY CURVE.

NOW WE WILL CONSIDER $y(a) = \bar{U}$ AND $y(-a) = \bar{U}$ SO THAT BY SYMMETRY $x_0 = 0$.



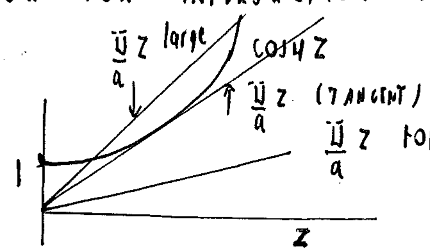
THUS, WE FIND C SO THAT

$$\bar{U} = C\cosh\left(\frac{a}{C}\right).$$

LET $Z = \frac{a}{C}$ SO $C = \frac{a}{Z}$. THEN $\bar{U} = \frac{a}{Z}\cosh(Z)$.

AS SUCH WE MUST LOOK FOR INTERSECTION POINTS TO $Z\left(\frac{\bar{U}}{a}\right) = \cosh Z$, IN $Z>0$.

WE PLOT AS FOLLOWS:



WE CONCLUDE THAT IF $\frac{\bar{U}}{a} >$ threshold, THEN $\exists$ TWO ROOTS Z_1, Z_2. QUESTION: WHICH ONE IS MINIMIZER?

MOREOVER, NO ROOTS IF $\frac{\bar{U}}{a} <$ threshold. HOW DO WE INTERPRET WHAT THIS MEANS?

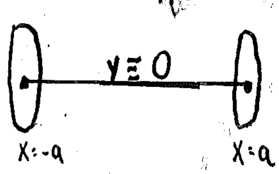
TO FIND THE THRESHOLD WE SET THE TANGENCY CONDITION

SOLVE $Z\left(\frac{\bar{U}}{a}\right) = \cosh Z$ together WITH $\frac{\bar{U}}{a} = \sinh Z$.

THIS GIVES $Z = \coth Z \longrightarrow Z \cong 1.19$

HENCE THE CRITICAL VALUE OF $\frac{\bar{U}}{a}$ IS $\sinh(1.19) \cong 1.49$

WE CONCLUDE THAT IF $\frac{\bar{U}}{a} > 1.49$ THERE ARE TWO STATIONARY POINTS FOR THE FUNCTIONAL AND NONE IF $\frac{\bar{U}}{a} < 1.49$. THE NON-EXISTENCE WHEN a IS LARGE SUGGESTS THAT PERHAPS WE SHOULD LOOK FOR A NONSMOOTH EXTREMAL CONSISTING OF TWO END HOOPS THAT ARE CIRCULAR CONNECTED BY A LINE WHERE $y \equiv 0$.

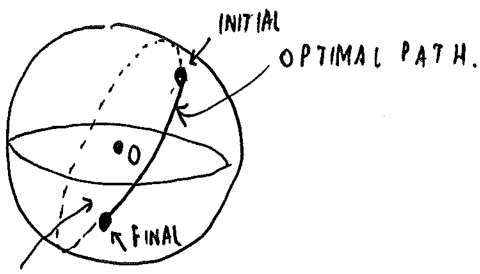


IN THE 1-D PROBLEM THIS MEANS $y = 0$ IN $-a<x<a$ AND IS NON-SMOOTH AT $x = \pm a$.

$\Rightarrow$ GOLDSHMIDT SOLUTION RUPTURE OF SOAP FILM

EXAMPLE (GEODESCICS)

GIVEN ANY TWO POINTS ON THE SURFACE OF THE UNIT SPHERE, CHARACTERIZE GEOMETRICALLY THE PATH THAT HAS SHORTEST DISTANCE. WE WILL SHOW THAT THE SHORTEST PATH IS A GREAT CIRCLE, SO THAT THE INITIAL POINT, FINAL POINT, AND CENTER OF THE SPHERE LIE ON A COMMON PLANE, THAT CONTAINS THE OPTIMAL PATH, i.e. THE GEODESCIC.



PLANE CONTAINING INITIAL POINT, FINAL POINT, AND CENTER OF SPHERE.

WE FIRST CALCULATE THE DISTANCE ELEMENT ON THE SPHERE.

RECALL IN SPHERICAL COORDINATES WITH A UNIT SPHERE

$$x = \cos\varphi \sin\phi \qquad y = \sin\varphi \sin\phi \qquad z = \cos\phi$$

$0 \le \phi \le \pi$
$0 \le \varphi \le 2\pi$

NOW
$$dx = -\sin\varphi \sin\phi \, d\varphi + \cos\varphi \cos\phi \, d\phi$$
$$dy = \cos\varphi \sin\phi \, d\varphi + \sin\varphi \cos\phi \, d\phi$$
$$dz = -\sin\phi \, d\phi$$

NOW $(dx)^2 + (dy)^2 = (\sin^2\varphi \sin^2\phi + \cos^2\varphi \sin^2\phi)(d\varphi)^2 + (\cos^2\varphi \cos^2\phi + \sin^2\varphi \cos^2\phi)(d\phi)^2$

SINCE $d\phi\, d\varphi$ TERM CANCELS. THIS GIVES $(dx)^2 + (dy)^2 = \sin^2\phi\,(d\varphi)^2 + \cos^2\phi\,(d\phi)^2$.

NOW $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \sin^2\phi\,(d\varphi)^2 + (\cos^2\phi + \sin^2\phi)(d\phi)^2$.

THUS $(ds)^2 = \sin^2\phi\,(d\varphi)^2 + (d\phi)^2$.

NOW WE THINK OF THE PATH WITH $\varphi = \varphi(\phi)$. HENCE $ds = \sqrt{1 + \sin^2\phi\,(\varphi')^2}\, d\phi$

THEN THE DISTANCE IS SIMPLY

$$D = \int_{\phi_0}^{\phi_1} \left(1 + \sin^2\phi\, \varphi'^2\right)^{1/2} d\phi .$$

NOW WITH $F(\phi, \varphi, \varphi') \equiv \left(1 + (\sin^2\phi)\varphi'^2\right)^{1/2}$.

THE EL EQUATION IS $F_\varphi - \dfrac{d}{d\phi} F_{\varphi'} = 0$. BUT $F_\varphi = 0$ SO $\dfrac{d}{d\phi} F_{\varphi'} = 0$

IMPLIES THAT

$F_{\varphi'} = C$. BUT $F_{\varphi'} = \dfrac{\varphi' \sin^2\phi}{\sqrt{1 + (\sin^2\phi)\varphi'^2}}$. OBSERVE THAT $C < 1$, MUST HOLD.

THIS GIVES $\varphi'^2 \sin^4\phi = C^2\left[1 + \sin^2\phi \; \varphi'^2\right]$

$$\longrightarrow \quad \varphi'^2 \sin^2\phi \left(\sin^2\phi - C^2\right) = C^2 .$$

WE CONCLUDE THAT $\varphi' = \dfrac{C}{\sin\phi\,(\sin^2\phi - C^2)^{1/2}} = \dfrac{C}{\sin^2\phi\,[1 - C^2\csc^2\phi]^{1/2}}$ SINCE ON $0<\phi<\pi$ $\sin\phi > 0$.

NOW PUT $u = \cot\phi$ AS A SUBSTITUTION. THEN $du = -\csc^2\phi\, d\phi$

THEN $\varphi = \displaystyle\int \frac{C}{\sin^2\phi\left[1 - C^2\csc^2\phi\right]^{1/2}}\, d\phi = -C\int \frac{du}{\left[1 - C^2\left(1+\cot^2\phi\right)\right]^{1/2}} = -C\int\frac{du}{\left[1 - C^2(1+u^2)\right]^{1/2}}$

HENCE $\varphi = -C\displaystyle\int \frac{du}{\sqrt{1-C^2-C^2u^2}} = -\frac{C}{C}\int\frac{du}{\sqrt{\frac{(1-C^2)}{C^2} - u^2}} = -\int\frac{du}{\sqrt{\frac{1-C^2}{C^2} - u^2}}$.

NOW LET $u = \sqrt{\dfrac{1-C^2}{C^2}}\,\sin t$ AS A FURTHER SUBSTITUTION.

THEN $\varphi = -\dfrac{\sqrt{\frac{1-C^2}{C^2}}}{\sqrt{\frac{1-C^2}{C^2}}}\displaystyle\int\frac{\cos t\, dt}{\sqrt{1-\sin^2 t}} = -\int dt.$

THUS $\varphi - \varphi_0 = -t$ WITH $u = a\sin t$ AND $a = \sqrt{\dfrac{1-C^2}{C^2}}$.

THEREFORE WE HAVE $u = a\sin[-(\varphi-\varphi_0)] = -a\sin(\varphi-\varphi_0)$. BUT $u = \cot\phi$

SO THAT $\cot\phi = -a\sin(\varphi-\varphi_0)$ WHERE $a = \sqrt{\dfrac{1-C^2}{C^2}}$.

TO VISUALIZE THIS SOLUTION WE PUT $\cot\phi = \dfrac{\cos\phi}{\sin\phi}$ AND MULTIPLY BY $\sin\phi$

THIS GIVES $\cos\phi = -a\sin\phi\,\sin(\varphi-\varphi_0) = -a\sin\phi\,[\sin\varphi\cos\varphi_0 - \cos\varphi\sin\varphi_0]$.

NOW RECALLING THAT $z = \cos\phi$, $x = \cos\varphi\sin\phi$, $y = \sin\varphi\sin\phi$ WE OBTAIN

$$z = -a\cos\varphi_0\, y + a\sin\varphi_0\, x.$$

THIS IS A PLANE THROUGH THE ORIGIN. THIS PLANE INTERSECTS THE SPHERE IN GREAT CIRCLE. WE CONCLUDE THAT THE OPTIMAL PATH ON THE SPHERE, CALLED THE GEODESIC, LIES ON A GREAT CIRCLE. GIVEN AN INITIAL POINT (ϕ_A, φ_A) AND A FINAL POINT (ϕ_B, φ_B) WE CAN SOLVE FOR a AND φ_0 IN

$$\cot\phi = -a\sin(\varphi-\varphi_0).$$

VARIATIONAL PROBLEMS FOR PDES

NOW WE STUDY VARIATIONAL PROBLEMS FOR PDES. WE WILL FOCUS ON SCALAR PROBLEMS IN 2-D AND ASSUME THAT MINIMIZERS AND STATIONARY POINTS ARE C^2 SMOOTH.

CONSIDER THE PROBLEM OF FINDING STATIONARY POINTS OR A MINIMIZER FOR THE FUNCTIONAL

$$I(u) = \int_\Omega F(x, y, u, u_x, u_y)\, dx\, dy, \quad \text{WITH } u = g \text{ ON } \partial\Omega \text{ SPECIFIED.} \Big\} \quad (V1)$$

WE NOW DERIVE A NECESSARY CONDITION TO MAKE u^* A MINIMIZER (OR STATIONARY POINT) OF $I(u)$ WITH $u \in C^2(\Omega)$.

WE LET $u = u^* + \varepsilon w$ WITH $u^* = g$ ON $\partial\Omega$ AND $w = 0$ ON $\partial\Omega$.

WE DEFINE $J(\varepsilon) = I(u^* + \varepsilon w) = \int_\Omega F(x, y, u^* + \varepsilon w, u_x^* + \varepsilon w_x, u_y^* + \varepsilon w_y)\, dx\, dy$

NOW TAYLOR EXPANDING AS $\varepsilon \to 0$:

$$F(x, y, u^* + \varepsilon w, u_x^* + \varepsilon w_x, u_y^* + \varepsilon w_y) = F(x, y, u^*, u_x^*, u_y^*) + \varepsilon\left[F_u^* w + F_{u_x}^* w_x + F_{u_y}^* w_y\right] + \ldots$$

WHERE (*) INDICATES EVALUATION AT $u = u^*$. THUS SUBSTITUTING GIVES

$$J(\varepsilon) = I(u^* + \varepsilon w) = I(u^*) + \varepsilon \int_\Omega \left(F_u^* w + F_{u_x}^* w_x + F_{u_y}^* w_y\right) dx\, dy + \ldots$$

NOW SINCE u^* IS A MINIMIZER (OR STATIONARY POINT), WE MUST HAVE $J'(0) = 0$, I.E. THAT

$$\int_\Omega \left(F_u^* w + F_{u_x}^* w_x + F_{u_y}^* w_y\right) dx\, dy = 0 \quad \forall\, w \in C^2(\Omega),\ w = 0 \text{ ON } \partial\Omega. \quad (V2)$$

NOW BY USING

$$\frac{\partial}{\partial x}\left(F_{u_x}^* w\right) = F_{u_x}^* w_x + \frac{\partial}{\partial x}\left(F_{u_x}^*\right) w$$

$$\frac{\partial}{\partial y}\left(F_{u_y}^* w\right) = F_{u_y}^* w_y + \frac{\partial}{\partial x}\left(F_{u_y}^*\right) w$$

THUS (V2) YIELDS

$$\int_\Omega \left(F_u^* w + \frac{\partial}{\partial x}\left(F_{u_x}^* w\right) + \frac{\partial}{\partial y}\left(F_{u_y}^* w\right) - \frac{\partial}{\partial x}\left(F_{u_x}^*\right) w - \frac{\partial}{\partial y}\left(F_{u_y}^*\right) w\right) dx\, dy = 0.$$

USING THE DIVERGENCE THEOREM WE GET:

$$\int_\Omega \left(F_u^* - \frac{\partial}{\partial x}\left(F_{u_x}^*\right) - \frac{\partial}{\partial y}\left(F_{u_y}^*\right)\right) w\, dx\, dy + \int_{\partial\Omega} \left(F_{u_x}^*, F_{u_y}^*\right) \cdot \hat{n}\, w\, ds = 0. \quad (V3)$$

NOW SINCE $w = 0$ ON $\partial\Omega$, THE BOUNDARY TERM VANISHES AND WE CONCLUDE THAT

$$\int_\Omega \left(F_u^* - \frac{\partial}{\partial x}\left(F_{u_x}^*\right) - \frac{\partial}{\partial y}\left(F_{u_y}^*\right)\right) w\, dx\, dy = 0 \quad \forall\, w \in C^2(\Omega).$$

WE CONCLUDE THAT u^* MUST SATISFY THE E-L EQUATION

$$F_u^* - \frac{\partial}{\partial x}\left(F_{u_x}^*\right) - \frac{\partial}{\partial y}\left(F_{u_y}^*\right) = 0 \quad \text{IN } \Omega. \quad (V4)$$

WITH $u^* = g$ ON $\partial\Omega$

NOW SUPPOSE THAT NO CONDITION FOR u^* ON $\partial\Omega$ IS SPECIFIED. THEN WE REQUIRE THAT FOR ANY $w \in C^1(\Omega)$, WE HAVE THAT (V3) HOLDS.

NOW $\forall w$ WITH $w = 0$ ON $\partial\Omega$, WE HAVE THAT

$$F_u^* - \frac{\partial}{\partial x}(F_{u_x}^*) - \frac{\partial}{\partial y}(F_{u_y}^*) = 0$$

WHILE FOR FUNCTIONS w WITH NO CONDITION ON $\partial\Omega$ WE MUST HAVE $(F_{u_x}^*, F_{u_y}^*) \cdot \hat{n} = 0$ ON $\partial\Omega$.

IN SUMMARY, A NECESSARY CONDITION THAT u^* IS A MINIMIZER OF $I(u)$ WITH NO CONDITIONS FOR u SPECIFIED ON $\partial\Omega$, WE MUST HAVE THAT u^* SATISFY

$$\left.\begin{aligned} &F_u^* - \frac{\partial}{\partial x} F_{u_x}^* - \frac{\partial}{\partial y} F_{u_y}^* = 0 \quad \text{IN } \Omega \\ &\text{SUBJECT TO THE NATURAL BOUNDARY CONDITIONS} \quad (F_{u_x}^*, F_{u_y}^*) \cdot \hat{n} = 0 \text{ ON } \partial\Omega. \end{aligned}\right\} \text{(V5)}$$

NOW CONSIDER A CLASS OF VARIATIONAL PROBLEMS WITH SURFACE ENERGY. MINIMIZE

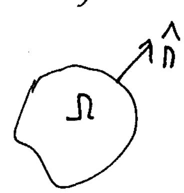
$$I(u) = \int_\Omega F(x, y, u, u_x, u_y)\, dx\, dy + \underbrace{\int_{\partial\Omega} G(x, y, u)\, dS}_{\rightarrow \text{ SURFACE ENERGY } \leftarrow} \qquad \text{(V6)}$$

WITH u UNSPECIFIED ON $\partial\Omega$. AS SUCH WE WILL WANT TO FIND NATURAL BC FOR u.

WE LET u^* BE A MINIMIZER AND WRITE $u = u^* + \varepsilon w$. DEFINE $J(\varepsilon) = I(u^* + \varepsilon w)$. WE CALCULATE FROM $J'(0) = 0$ THAT

$$\int_\Omega \left(F_u^* - \frac{\partial}{\partial x} F_{u_x}^* - \frac{\partial}{\partial y} F_{u_y}^* \right) w\, dx\, dy + \int_{\partial\Omega} \left[(F_{u_x}^*, F_{u_y}^*) \cdot \underline{\hat{n}} + G_u^* \right] w\, dS = 0$$

$\forall w \in C^2(\Omega)$. NOW IF NO CONDITION IS SPECIFIED FOR w ON $\partial\Omega$, WE HAVE THAT u^* MUST SATISFY



$$\left.\begin{aligned} &F_u^* - \frac{\partial}{\partial x} F_{u_x}^* - \frac{\partial}{\partial y} F_{u_y}^* = 0 \quad \text{IN } \Omega \\ &\text{SUBJECT TO THE NATURAL BOUNDARY CONDITIONS} \\ &(F_{u_x}^*, F_{u_y}^*) \cdot \underline{\hat{n}} + G_u^* = 0 \quad \text{ON } \partial\Omega. \end{aligned}\right\} \text{(V7)}$$

WE NOW CONSIDER A FEW EXAMPLES OF THE THEORY.

<u>EXAMPLE 1</u> AMONG ALL FUNCTIONS $u(x,y) \in C^2(\Omega)$ FIND u THAT MINIMIZES IN A BOUNDED DOMAIN Ω,

$$I(u) = \int_\Omega \left[\frac{1}{2}(u_x^2 + u_y^2) - pu \right] dx\, dy$$

(i) FOR $u = g$ ON $\partial\Omega$

(ii) NO CONDITION FOR u ON $\partial\Omega$ IS SPECIFIED.

<u>SOLUTION</u>

(i) WRITE $u = u^* + \varepsilon w$ WITH $u^* = g$ ON $\partial\Omega$ AND $w = 0$ ON $\partial\Omega$. WE OBTAIN FROM (V4) THAT u^* SATISFIES

$$F_{u^*} - \frac{\partial}{\partial x} F_{u_x^*} - \frac{\partial}{\partial y} F_{u_y^*} = 0 \quad \text{IN } \Omega \quad \text{WITH } u^* = g \text{ ON } \partial\Omega.$$

WE HAVE $F = \frac{1}{2}(u_x^2 + u_y^2) - pu$. NOW $F_{u_x} = u_x$, $F_{u_y} = u_y$ AND $F_u = -p$.

WE GET THAT $-p - \frac{\partial}{\partial x}(u_x) - \frac{\partial}{\partial y}(u_y) = 0 \rightarrow \Delta u = -p$ IN Ω

$u = g$ ON $\partial\Omega$.

(ii) NOW FOR (ii) WE HAVE THAT u SATISFIES THE E-L EQUATION SUBJECT TO THE NATURAL BOUNDARY CONDITIONS GIVEN IN (V5).

AS SUCH, WITH $(F_{u_x}, F_{u_y}) = (2u_x, 2u_y)$ WE OBTAIN THAT

$$\Delta u = -p \text{ IN } \Omega$$

$$\nabla u \cdot \hat{n} = 0 \text{ ON } \partial\Omega \rightarrow (\text{NATURAL BC}).$$

NOW WE SHOW THAT u^* IS ACTUALLY A MINIMIZER OF $I(u)$ AND NOT JUST A STATIONARY POINT. WE CALCULATE FOR $u = u^* + v$ WITH v A COMPETITOR THAT

$$I(u^* + v) = I(u^*) + \int_\Omega \left[\frac{1}{2}(2u_x^* v_x + 2u_y^* v_y) - pv \right] d\underline{x} + \frac{1}{2}\int_\Omega |\nabla v|^2 d\underline{x}$$

$$= I(u^*) + \int_\Omega [\nabla u^* \cdot \nabla v - pv]\, d\underline{x} + \frac{1}{2}\int_\Omega |\nabla v|^2 d\underline{x}$$

NOW USE $\nabla \cdot (v \nabla u^*) = v \Delta u^* + \nabla v \cdot \nabla u^*$. WE GET

$$I(u^* + v) = I(u^*) + \int_\Omega (\nabla \cdot (v \nabla u^*) - v \Delta u^* - pv)\, d\underline{x} + \frac{1}{2}\int_\Omega |\nabla v|^2 d\underline{x}$$

$$I(u^* + v) = I(u^*) + \int_\Omega [-\Delta u^* - p]\, v\, d\underline{x} + \int_{\partial\Omega} v \nabla u^* \cdot \hat{n}\, ds + \frac{1}{2}\int_\Omega |\nabla v|^2 d\underline{x}$$

NOW SINCE $\Delta u^* = -p$ IN Ω, WE OBTAIN

$$I(u^* + v) = I(u^*) + \int_{\partial\Omega} v \nabla u^* \cdot \hat{n}\, ds + \frac{1}{2}\int_\Omega |\nabla v|^2 d\underline{x}.$$

NOW IF $u^* = g$ ON $\partial\Omega \longrightarrow v = 0$ ON $\partial\Omega$. IF NO CONDITION ON $\partial\Omega$ IS SPECIFIED THEN THE NATURAL BOUNDARY CONDITIONS $\partial_n u^* = 0$ ON $\partial\Omega$ HOLD. IN EITHER CASE, $\int_{\partial\Omega} v \nabla u^* \cdot \hat{n}\, ds = 0$. THUS, WE CONCLUDE THAT

$$I(u^* + v) = I(u^*) + \frac{1}{2} \int_\Omega |\nabla v|^2 dx.$$

WE CONCLUDE THAT $I(u^* + v) > I(u^*)$, $\forall\, v \neq 0$.

THUS, WE CONCLUDE THAT u^* IS A <u>MINIMIZER</u> OF $I(u)$ AND NOT JUST A STATIONARY POINT

<u>EXAMPLE 2</u> SUPPOSE THAT $I(u) = \int_\Omega \left[Hu + \frac{1}{2} p (u_x^2 + u_y^2) \right] dx\,dy + \int_{\partial\Omega} \frac{1}{2} \sigma u^2 \, ds.$

HERE $p = p(x,y)$ AND H, σ ARE CONSTANTS. FIND THE E-L EQUATION FOR A MINIMIZER. NO CONDITION ON $\partial\Omega$ IS ASSUMED.

$F_u = H$

<u>SOLUTION</u> WE USE (V7), WITH $F = \frac{1}{2} p (u_x^2 + u_y^2)$ $\quad G = \frac{1}{2} \sigma u^2$. THUS $F_{u_x} = p u_x$, $F_{u_y} = p u_y$

NOW $\quad F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0 \longrightarrow \frac{\partial}{\partial x}\left[p \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y}\left[p \frac{\partial u}{\partial y} \right] = H \rightarrow \nabla \cdot (p \nabla u) = H$ IN Ω,

AND ON BOUNDARY WE HAVE $\quad (F_{u_x}, F_{u_y}) \cdot \hat{n} + G_u = 0$ ON $\partial\Omega$.

THIS GIVES, $\quad p \nabla u \cdot \hat{n} + \sigma u = 0$ ON $\partial\Omega$.

HENCE, $\quad \nabla \cdot (p \nabla u) = H$ IN Ω

AND $\quad p\, \partial u / \partial n + \sigma u = 0$ ON $\partial\Omega$,

IS THE PROBLEM THAT ANY MINIMIZER MUST SATISFY.

<u>RAYLEIGH-RITZ METHOD</u>

THE RAYLEIGH-RITZ METHOD IS A CLASSICAL BUT SIMPLE METHOD FOR APPROXIMATING SOLUTIONS TO VARIATIONAL PROBLEMS. SUPPOSE WE SEEK A MINIMUM TO

$$\min_{u \in \mathcal{H}} I(u)$$

WHERE I IS A FUNCTIONAL OF u AND $\mathcal{H}$ IS A CLASS OF FUNCTIONALS (WITH SOME BC'S). CONSIDER A CLASS OF FUNCTIONS $v_1, \ldots, v_n$, WITH $v_j \in \mathcal{H}$ AND LET

$$u^*(\underline{x}) = \sum_{j=1}^{n} c_j v_j(\underline{x}) \quad \text{SO THAT } u \in \mathcal{H}.$$

EXAMPLE 3 IN 1-D CONSIDER THE FUNCTIONAL

$$I(y) = \int_0^1 \left(y'^2 + (1+x)y^2 + 2y \right) dx \quad \text{WITH} \quad y(0) = y(1) = 0.$$

THE EL EQUATION WITH $F = (1+x)y^2 + 2y + y'^2$ IS $F_y - \frac{d}{dx} F_{y'} = 0 \rightarrow 2(1+x)y + 2 = 2y''$

SO THAT $y'' - (1+x)y = 1$, WITH $y(0) = y(1) = 0$.

NATURAL CHOICES FOR TRIAL FUNCTIONS ARE $y_n(x) = x^n(1-x)$ AND $y_n(x) = \sin(n\pi x)$.

NOW SUPPOSE WE TRY $y = c_1 y_1(x)$ WITH $y_1 = x(1-x)$.

WE CALCULATE $I(c_1 y_1) = c_1^2 \int_0^1 [(y_1')^2 + (1+x)y_1^2]\, dx + 2c_1 \int_0^1 y_1(x)\, dx$.

WE CALCULATE $\int_0^1 [(y_1')^2 + (1+x)y_1^2]\, dx = 23/60$ AND $2\int_0^1 y_1\, dx = 1/3$.

THIS YIELDS $J(c_1) \equiv I(c_1 y_1) = \frac{23}{60} c_1^2 + \frac{1}{3} c_1$.

NOW $J'(c_1) = 0 \rightarrow \frac{46}{60} c_1 = -\frac{1}{3} \Longrightarrow c_1 = c_1^* = -10/23$ THUS $c y_1 = -\frac{10}{23} x(1-x)$ IS

THE OPTIMUM FOR THIS TRIAL FUNCTION. THEN $J(c_1^*) = 23/60\,(10/23)^2 + \frac{1}{3}(-10/23)$.

NOW WE TRY TO IMPROVE THE APPROXIMATION.

IF WE PUT $y = c_1 x(1-x) + c_2 x^2(1-x)$ WE OBTAIN $J(c_1, c_2)$.

SETTING $J_{c_1} = 0$ AND $J_{c_2} = 0$ WE OBTAIN

$$\begin{aligned} 27/60\, c_1 + 27/140\, c_2 &= -1/6 \\ 27/140\, c_1 + 25/168\, c_2 &= -1/12 \end{aligned} \quad \rightarrow \quad \begin{aligned} c_1 &= c_1^* = -0.435 \\ c_2 &= c_2^* = .0111 \end{aligned}$$

SO $y = c_1^* x(1-x) + c_2^* x^2(1-x)$.

NOW IN GENERAL SUPPOSE THAT

$$y = \underline{c}^T \hat{\underline{y}} \quad \text{WITH} \quad \underline{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \hat{\underline{y}} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad \text{AND} \quad y_j = x^j(1-x).$$

THEN IF $I(y) = \int_0^1 \left(p(x) y'^2 + q(x) y^2 + 2r(x) y \right) dx$ WITH $y = 0$ AT $x = 0, 1$,

WE OBTAIN $I(y) = \int_0^1 \left(p(\underline{c}^T \hat{\underline{y}}') \hat{\underline{y}}'^T \underline{c} + q\, \underline{c}^T \hat{\underline{y}} \hat{\underline{y}}^T \underline{c} + 2r\, \underline{c}^T \hat{\underline{y}} \right) dx$

$$J(\underline{c}) \equiv I(y) = \underline{c}^T A \underline{c} + \underline{c}^T B \underline{c} + 2\underline{c}^T \underline{\Gamma} \quad \text{WHERE} \quad A \equiv \int_0^1 p\, \hat{\underline{y}}' \hat{\underline{y}}'^T dx \quad (n \times n \text{ MATRIX})$$

$$B = \int_0^1 q\, \hat{\underline{y}} \hat{\underline{y}}^T dx$$

$$\underline{\Gamma} = \int_0^1 r(x)\, \hat{\underline{y}}\, dx.$$

NOW TO OBTAIN THE OPTIMAL POINT WE CALCULATE THE JACOBIAN OF J TO OBTAIN THAT THE OPTIMAL $\underline{c} = \underline{c}^*$ SATISFIES

$$(A + B)\, \underline{c}^* = -\Gamma$$

THIS IS A LINEAR ALGEBRAIC SYSTEM FOR $\underline{c}^*$.

EXAMPLE 4 LET Ω BE UNIT SQUARE $0 \le x \le 1$, $0 \le y \le 1$. WE WANT TO MINIMIZE

$I(u) = \int_\Omega (|\nabla u|^2 + 2uf)\, d\underline{x}$ OVER $u \in H \equiv \{u \mid u \in C^2(\Omega),\ u = 0 \text{ ON } \partial\Omega\}$.

WE TAKE A TRIAL FUNCTION OF THE FORM

$$u(x,y) = \sum_{m=1}^N \sum_{n=1}^N a_{m,n} V_{m,n}(x,y) \quad \text{WHERE} \quad V_{m,n}(x,y) = \sin(m\pi x)\sin(n\pi y).$$

NOTICE THAT $V_{m,n} \in H$ AND WE WANT TO FIND COEFFICIENTS THAT MINIMIZE $I(u)$.

NOW $$u_x = \sum_{m=1}^N \sum_{n=1}^N m\pi a_{mn} \cos(m\pi x)\sin(n\pi y), \quad u_y = \sum_{m=1}^N \sum_{n=1}^N n\pi a_{mn} \sin(m\pi x)\cos(n\pi y).$$

THIS GIVES

$$u_x^2 + u_y^2 = \sum_{m=1}^N \sum_{n=1}^N \sum_{m'=1}^N \sum_{n'=1}^N \Big(\pi^2 m m' \cos(m\pi x)\cos(m'\pi x)\sin(n\pi y)\sin(n'\pi y)\, a_{mn} a_{m'n'} + a_{mn} a_{m'n'} \pi^2 n' n' \sin(m\pi x)\cos(n\pi y)\sin(m'\pi x)\cos(n'\pi y)\Big)$$

WE INTEGRATE OVER THE SQUARE Ω:

$$\int_\Omega (u_x^2 + u_y^2)\, dx\, dy = \sum_{m=1}^N \sum_{n=1}^N \sum_{m'=1}^N \sum_{n'=1}^N \pi^2 (mm' + nn')\, a_{mn} a_{m'n'} \int_0^1 \cos(m\pi x)\cos(m'\pi x)\, dx \cdot \int_0^1 \sin(n\pi y)\sin(n'\pi y)\, dy.$$

BUT $\int_0^1 \cos(m\pi x)\cos(m'\pi x)\, dx = \begin{cases} 1/2 & \text{IF } m = m' \\ 0 & \text{IF } m \ne m' \end{cases}$ AND $\int_0^1 \sin(n\pi y)\sin(n'\pi y)\, dy = \begin{cases} 1/2 & \text{IF } n = n' \\ 0 & \text{IF } n \ne n'. \end{cases}$

THUS, WE HAVE

$$\int_\Omega (u_x^2 + u_y^2)\, dx\, dy = \frac{\pi^2}{4} \sum_{m=1}^N \sum_{n=1}^N (n^2 + m^2)\, a_{mn}^2.$$

NOW WE HAVE IF $f = \sum_{m=1}^N \sum_{n=1}^N g_{m,n} \sin(m\pi x)\sin(n\pi y)$, WHERE $g_{mn} = 4\int_0^1\int_0^1 f \sin(m\pi x)\sin(n\pi y)\, dx\, dy$

WE CALCULATE $$2\int_\Omega u f\, d\underline{x} = 2\sum_{m=1}^N \sum_{n=1}^N a_{mn}\left(\int_0^1\int_0^1 f \sin(m\pi x)\sin(n\pi y)\, dx\, dy\right) = 2\sum_{m=1}^N \sum_{n=1}^N \frac{a_{mn} g_{mn}}{4}.$$

THIS YIELDS THAT $$I(u) = \int_\Omega (|\nabla u|^2 + 2uf)\, d\underline{x} = \frac{\pi^2}{4} \sum_{m=1}^N \sum_{n=1}^N (n^2 + m^2)\, a_{mn}^2 + \frac{1}{2} \sum_{m=1}^N \sum_{n=1}^N a_{mn} g_{mn} \equiv G.$$

HERE WE HAVE $2N$ UNKNOWNS $a_{11}, a_{12}, \ldots, a_{1N}, \ldots a_{NN}$, I.E. $\underline{a} = (a_{11}, \ldots, a_{NN})^T$

$$G(\underline{a}) \equiv \frac{\pi^2}{4} \sum_{m=1}^N \sum_{n=1}^N (n^2 + m^2)\, a_{mn}^2 + \frac{1}{2} \sum_{m=1}^N \sum_{n=1}^N a_{mn} g_{mn}. \quad \text{FINITE MINIMIZATION PROBLEM.}$$

NOW SET $\partial G / \partial a_{mn} = 0 \rightarrow \frac{\pi^2}{2}(m^2 + n^2)\, a_{mn} + g_{m,n}/2 = 0 \rightarrow a_{m,n} = \frac{-g_{mn}}{\pi^2 (m^2 + n^2)}$

WE CONCLUDE THAT THE OPTIMIZER IS SIMPLY $u = -\sum_{m=1}^N \sum_{n=1}^N \frac{g_{mn}}{\pi^2 (m^2 + n^2)} \sin(m\pi x)\sin(n\pi y)$

IS THE EIGENFUNCTION EXPANSION OF THE EULER LAGRANGE EQUATION $\Delta u = f$ IN Ω, $u = 0$ ON $\partial\Omega$.

<u>EXAMPLE</u> CONSIDER $\min_{w \in H} I(w)$ WITH $I(u) = \int_\Omega (|\nabla u|^2 - 2u)\, d\underline{x}$

AND Ω IS ELLIPSE $x^2/a^2 + y^2/b^2 \le 1$ WITH $H = \{ w \mid w = 0 \text{ on } \partial\Omega,\ w \text{ is } C^2(\Omega) \}$.

TAKE TRIAL FUNCTION $w = c[a^2b^2 - b^2x^2 - a^2y^2]$ AND FIND OPTIMUM c USING RAYLEIGH-RITZ.

<u>SOLUTION</u> $\int_\Omega |\nabla w|^2 d\underline{x} = \int_\Omega (4b^4x^2 + 4a^4y^2)\, d\underline{x}$. let $x = ar\cos\varphi$, $y = br\sin\varphi$ WITH $0 \le r \le 1$ AND $0 \le \varphi \le 2\pi$ BE NEW VARIABLES. THEN USING THE JACOBIAN $\int_\Omega (\ldots) d\underline{x} = \int (\ldots) J\, dr\, d\varphi$

AND $\det J = \det \begin{vmatrix} x_r & x_\varphi \\ y_r & y_\varphi \end{vmatrix} = \det \begin{pmatrix} a\cos\varphi & -ar\sin\varphi \\ b\sin\varphi & br\cos\varphi \end{pmatrix} = abr(\cos^2\varphi + \sin^2\varphi) = abr$,

WE OBTAIN $\int_\Omega |\nabla w|^2 d\underline{x} = c^2\int_\Omega (4b^4x^2 + 4a^4y^2)\, dx\, dy = 4abc^2\int_0^{2\pi}\int_0^1 (b^4a^2\cos^2\varphi + a^4b^2\sin^2\varphi)\, r^3\, dr\, d\varphi$.

SINCE $\int_0^1 r^3 dr = 1/4$ AND $\int_0^{2\pi} \cos^2\varphi\, d\varphi = \int_0^{2\pi} \sin^2\varphi\, d\varphi = \pi$ WE CALCULATE

$$\int_\Omega |\nabla w|^2 d\underline{x} = a^3b^3c^2\int_0^{2\pi} (b^2\cos^2\varphi + a^2\sin^2\varphi)\, d\varphi = \pi a^3b^3c^2(b^2 + a^2).$$

NOW WE CALCULATE $\int_\Omega 2w\, d\underline{x} = \int_{\tilde\Omega} 2w\, J\, dr\, d\varphi = \int_0^{2\pi}\int_0^1 2abr\, c[a^2b^2 - b^2a^2r^2\cos^2\varphi - r^2a^2b^2\sin^2\varphi]\, d\varphi$

$$= 4\pi c(a^3b^3)\int_0^1 r(1-r^2)\, dr = 4c\pi a^3b^3\left(\frac{1}{2} - \frac{1}{4}\right) = \pi c a^3b^3.$$

THUS, $J(c) \equiv \int_\Omega (|\nabla w|^2 - 2w)\, d\underline{x} = c^2\pi a^3b^3(b^2 + a^2) - \pi c a^3b^3$.

THE POINT c^* where $J(c)$ IS MINIMIZED IS $J'(c^*) = 0 \to 2c^*\pi a^3b^3(a^2+b^2) = \pi a^3b^3$.

SO $c^* = \dfrac{1}{2(b^2+a^2)}$ AND $w^* = \dfrac{1}{2(b^2+a^2)}[a^2b^2 - b^2x^2 - a^2y^2]$.

THUS THE OPTIMAL w^* IS $w^* = \dfrac{a^2b^2}{2(b^2+a^2)} - \dfrac{x^2b^2}{2(b^2+a^2)} - \dfrac{y^2a^2}{2(b^2+a^2)}$.

OBSERVE, CURIOUSLY THAT THE EL EQUATION FOR $I(u) = \int_\Omega (|\nabla u|^2 - 2u)\, d\underline{x}$, WHICH HAS $F \equiv u_x^2 + u_y^2 - 2u$ IS $F_u - \partial/\partial x\, F_{u_x} - \partial/\partial y\, F_{u_y} = 0$ WHICH IS

$$\Delta u = -1 \quad \text{WITH} \quad u = 0 \text{ on } \partial\Omega.$$

OBSERVE THAT FROM A DIRECT COMPUTATION THAT $\Delta w^* = -\dfrac{b^2}{a^2+b^2} - \dfrac{a^2}{b^2+a^2} = -1$

SO THAT w^* IS <u>EXACT</u> SOLUTION TO THE EL EQUATION.

NOW *KANTOROVICH'S METHOD* IS A MINOR VARIANT OF THE RAYLEIGH-RITZ PROCEDURE IF WE SEEK $\min_{u \in H} I(u)$ IN 2-D DOMAIN Ω, WE CONSIDER A TRIAL FUNCTION WHERE u IS SEPARABLE IN x, y, SPECIFIED EXPLICITY IN ONE DIRECTION BUT ALLOWED TO BE ARBITRARY IN SECOND DIRECTION. THE OPTIMUM OF THIS TRIAL FUNCTION IS FOUND BY A VARIATIONAL PROBLEM IN 1-D. WE CONSIDER TWO EXAMPLES.

EXAMPLE 5 LET $\Omega = \{(x,y) \mid 0 \le x \le a,\ -b < y < b\}$. WE AIM TO $\min_{u \in H} I(u)$ WHERE $H = \{u \mid u(0,y) = 0,\ u(x, \pm b) = 0$, NO CONDITION ON $x = a\}$ AND $I(u) = \int_0^a \int_{-b}^{b} (u_x^2 + u_y^2 - 2u)\,dx\,dy$.

WE WILL USE A TRIAL FUNCTION OF THE FORM $u_0 = (b^2 - y^2)\, v(x)$.

WE MUST HAVE $v(0) = 0$, AND THEN $u_0 \in H$.

NOW
$$I(u_0) = \int_0^a \int_{-b}^{b} \left[(b^2 - y^2)^2 v'^2 + 4y^2 v^2 - 2(b^2 - y^2) v \right] dx\,dy$$
$$= \int_0^a \left[2v'^2 \int_0^b (b^2 - y^2)^2 dy + 8v^2 \frac{y^3}{3}\Big|_0^b - 4v \int_0^b (b^2 - y^2)\,dy \right] dx$$
$$= \int_0^a \left(\frac{16}{15} b^5 v'^2 + \frac{8}{3} b^3 v^2 - \frac{8}{3} b^3 v \right) dx$$

THUS $I(u_0) = \int_0^a F(x, v, v')\,dx$ WITH $F(x, v, v') = \frac{16}{15} b^5 v'^2 + \frac{8}{3} b^3 v^2 - \frac{8}{3} b^3 v$

WE WILL MINIMIZE $I(u_0)$ WITH $v(0) = 0$, AND $v(a)$ UNSPECIFIED.

THE EL EQUATION IS $F_v - \frac{d}{dx} F_{v'} = 0$ WITH $v(0) = 0$.

WITH THE NATURAL BOUNDARY CONDITION $F_{v'} = 0$ AT $x = a$

NOW $F_v = \frac{16}{3} b^3 v - \frac{8}{3} b^3$, $F_{v'} = \frac{32}{15} b^5 v'$

THUS $F_v - \frac{d}{dx} F_{v'} = 0 \longrightarrow \frac{32}{15} b^5 v'' - \frac{16}{3} b^3 v + \frac{8b^3}{3} = 0$, $0 < x < a$

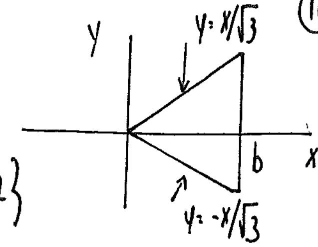
WITH $v(0) = 0$ AND $v'(a) = 0$.

THIS YIELDS THAT $v'' - \frac{5}{2b^2} v = -\frac{5}{4b^2}$, $0 < x < a$ WITH $v(0) = 0$ AND $v'(a) = 0$.

THE SOLUTION IS $v(x) = \frac{1}{2} + A \cosh\left[\sqrt{\frac{5}{2b^2}}\,(x - a) \right]$.

NOW $v(0) = 0 \rightarrow A = -\frac{1}{2} \operatorname{sech}\left[\sqrt{\frac{5}{2b^2}} \right]$. WE GET $v = \frac{1}{2}\left[1 - \frac{\cosh(\gamma(x-a))}{\cosh(\gamma a)} \right]$

WITH $\gamma = (5/2b^2)^{1/2}$, SO THAT $u_{0\,\text{OPTIMUM}} = \frac{1}{2}(b^2 - y^2)\left[1 - \frac{\cosh[\gamma(x-a)]}{\cosh[\gamma a]} \right]$.



EXAMPLE 6 ~~WE~~ SEEK TO APPROXIMATE

$$\min_{u \in H} I(u) \quad \text{WITH} \quad I(u) = \int_\Omega \left(|\nabla u|^2 - 2u \right) d\underline{x}$$

AND Ω IS THE TRIANGLE $\Omega = \{ (x,y) \mid |y| < x/\sqrt{3},\ 0 < x < b \}$, AND $H = \{ u \mid u = 0 \text{ ON } \partial\Omega \}$

WE ASSUME $u \in C^2(\Omega)$. NOTICE THAT THE EL EQUATION IS $\Delta u = 1$ IN Ω WITH $u = 0$ ON $\partial\Omega$.

NOW WE TAKE AS A TRIAL FUNCTION

$$u_0(x,y) = \left[y^2 - (x/\sqrt{3})^2 \right] v(x)$$

WHERE $v \in C^2(\Omega)$ AND $v(b) = 0$.

WE CALCULATE

$$u_{0x} = -\,2x/_3\, v + \left[y^2 - x^2/3 \right] v'$$

$$u_{0y} = 2yv$$

SO THAT $u_{0x}^2 + u_{0y}^2 = \dfrac{4x^2}{9} v^2 + \left[y^2 - x^2/3 \right]^2 v'^2 - \dfrac{4x}{3} v v' \left[y^2 - x^2/\sqrt{3} \right] + 4y^2 v^2$

NOTING THE $y \mapsto -y$ SYMMETRY WE GET

$$I(u_0) = 2 \int_0^b \left(\int_0^{x/\sqrt{3}} \left[\frac{4x^2}{9} v^2 + \left(y^2 - x^2/_3 \right)^2 v'^2 - \frac{4x}{3} v v' \left(y^2 - x^2/_3 \right) + 4y^2 v^2 - 2\left(y^2 - x^2/_3 \right) v \right] dy \right) dx.$$

NOW LET $y = sx/\sqrt{3}$ SO THAT $0 \le s \le 1$. THEN $dy = x/\sqrt{3}\, ds$. THUS

$$I(u_0) = \frac{2}{\sqrt{3}} \int_0^b \int_0^1 \left[\frac{4x^3}{9} v^2 + \frac{x^5}{9} (s^2 - 1)^2 v'^2 - \frac{4x^4}{9} v v' (s^2 - 1) + \frac{4}{3} x^3 v^2 s^2 - \frac{2x^3}{3} (s^2 - 1) v \right] ds\, dx$$

NOW USING $\int_0^1 (s^2 - 1)^2 ds = 8/15$, $\int_0^1 (s^2 - 1) ds = -2/3$, WE INTEGRATE OUT THE s-VARIABLE:

$$I(u_0) = \frac{2}{\sqrt{3}} \int_0^b \left[\frac{4x^3 v^2}{9} + \frac{8}{145} x^5 v'^2 + \frac{8}{27} x^4 v v' + \frac{4}{9} x^3 v^2 + \frac{4}{9} x^3 v \right] dx$$

$$I(u_0) = \frac{8}{9\sqrt{3}} \int_0^b F(x, v, v')\, dx \quad \text{WHERE} \quad F(x,v,v') = x^3 v^2 + \frac{2}{15} x^5 v'^2 + \frac{2}{3} x^4 v v' + x^3 v^2 + x^3 v$$

WE SIMPLIFY TO GET $F(x,v,v') = \dfrac{2}{15} x^5 v'^2 + \dfrac{2}{3} x^4 v v' + 2x^3 v^2 + x^3 v$.

NOW THE EL EQUATION IS $F_v - \dfrac{d}{dx} F_{v'} = 0$ WHERE $F_v = \dfrac{2x^4}{3} v' + 4x^3 v + x^3$

AND $F_{v'} = \dfrac{4}{15} x^5 v' + \dfrac{2}{3} x^4 v$. THUS $\left(\dfrac{4}{15} x^5 v' + \dfrac{2}{3} x^4 v \right)' = \dfrac{2x^4}{3} v' + 4x^3 v + x^3$.

THIS YIELDS $\dfrac{4}{15} x^5 v'' + \dfrac{4}{3} x^4 v' - \dfrac{4}{3} x^3 v = x^3 \longrightarrow x^2 v'' + 5xv' - 5v = \dfrac{15}{4}$, $0 < x < b$.

THIS IS EULER EQUATION. THE PARTICULAR SOLUTION IS $V_p = -3/4$ AND $V_H = x^\alpha \to \alpha(\alpha - 1) + 5\alpha - 5 = 0$

SO THAT $\alpha^2 + 4\alpha - 5 = (\alpha + 5)(\alpha - 1) = 0 \to \alpha = -5, 1$. HENCE $V = C_1 x^{-5} + C_2 x - 3/4$.

WE WANT V BOUNDED AS $x \to 0$ AND $v(b) = 0 \longrightarrow V = 3/4\, (x/b - 1)$.

HENCE THE OPTIMUM u_0 IS $u_0 = \dfrac{3}{4} \left(x/b - 1 \right) \left(y^2 - x^2/3 \right)$.

NEXT, WE WILL CONSIDER <u>VARIATIONAL PROBLEMS WITH HIGHER DERIVATIVES</u>.

WE SEEK TO MINIMIZE $I(u) = \int_\Omega F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\, dx\, dy$ WITH $u \in C^4$

AND Ω A BOUNDED 2-D DOMAIN. WE WILL CONSIDER BOTH PRESCRIBED AND NATURAL BOUNDARY CONDITIONS FOR u.

WE FIRST DERIVE A NECESSARY CONDITION FOR u^* TO BE A MINIMIZER. LET $u = u^* + \varepsilon w$ AND WE CALCULATE

$$J(\varepsilon) \equiv I(u^* + \varepsilon w) - I(u^*) = \varepsilon \int_\Omega \left(F_u^* w + F_{u_x}^* w_x + F_{u_y}^* w_y + F_{u_{xx}}^* w_{xx} + F_{u_{xy}}^* w_{xy} + F_{u_{yy}}^* w_{yy} \right) dx\, dy + O(\varepsilon^2).$$

NOW SET $J'(\varepsilon)\big|_{\varepsilon=0} = 0$ SO THAT u^* MUST SATISFY

$$\int_\Omega \left(F_u^* w + F_{u_x}^* w_x + F_{u_y}^* w_y + F_{u_{xx}}^* w_{xx} + F_{u_{xy}}^* w_{xy} + F_{u_{yy}}^* w_{yy} \right) dx\, dy = 0 \qquad (V8)$$

NOW USE
$$\partial_x [F_{u_{xx}}^* w_x] = \partial_x (F_{u_{xx}}^*) w_x + F_{u_{xx}}^* w_{xx}$$
$$\partial_y [F_{u_{yy}}^* w_y] = \partial_y (F_{u_{yy}}^*) w_y + F_{u_{yy}}^* w_{yy}.$$

ALSO
$$F_{u_{xy}}^* w_{xy} = \frac{1}{2}\left[\partial_x (F_{u_{xy}}^* w_y) + \partial_y (F_{u_{xy}}^* w_x) - \partial_y (F_{u_{xy}}^*) w_x - \partial_x (F_{u_{xy}}^*) w_y \right]$$

THEN
$$F_{u_{xx}}^* w_{xx} + F_{u_{yy}}^* w_{yy} + F_{u_{xy}}^* w_{xy} = \partial_x \left[F_{u_{xx}}^* w_x + \frac{1}{2} F_{u_{xy}}^* w_y \right] + \partial_y \left[F_{u_{yy}}^* w_y + \frac{1}{2} F_{u_{xy}}^* w_x \right]$$
$$- \partial_x (F_{u_{xx}}^*) w_x - \partial_y (F_{u_{yy}}^*) w_y - \frac{1}{2} \partial_y (F_{u_{xy}}^*) w_x - \frac{1}{2} \partial_x (F_{u_{xy}}^*) w_y.$$

NOW PUTTING THIS INTO (V8) WE OBTAIN:

$$\int_\Omega \left[F_u^* w + \left(F_{u_x}^* - \partial_x (F_{u_{xx}}^*) - \frac{1}{2} \partial_y (F_{u_{xy}}^*) \right) w_x + \left(F_{u_y}^* - \partial_y (F_{u_{yy}}^*) - \frac{1}{2} \partial_x (F_{u_{xy}}^*) \right) w_y \right] dx\, dy$$
$$+ \int_\Omega \left[\partial_x \left(F_{u_{xx}}^* w_x + \frac{1}{2} F_{u_{xy}}^* w_y \right) + \partial_y \left(F_{u_{yy}}^* w_y + \frac{1}{2} F_{u_{xy}}^* w_x \right) \right] dx\, dy = 0. \quad (V9)$$

NOW DEFINE $\underline{K} \equiv (K_1, K_2)^T$ WITH $K_1 = F_{u_{xx}}^* w_x + \frac{1}{2} F_{u_{xy}}^* w_y$, $K_2 = F_{u_{yy}}^* w_y + \frac{1}{2} F_{u_{xy}}^* w_x$

AND $\underline{G} = (G_1, G_2)^T$ WITH
$$G_1 = F_{u_x}^* - \partial_x (F_{u_{xx}}^*) - \frac{1}{2} \partial_y (F_{u_{xy}}^*)$$
$$G_2 = F_{u_y}^* - \partial_y (F_{u_{yy}}^*) - \frac{1}{2} \partial_x (F_{u_{xy}}^*). \qquad (V10)$$

THEN (V9) BECOMES,

$$\int_\Omega \left[F_u w + G_1 w_x + G_2 w_y + \nabla \cdot \underline{K} \right] dx\, dy = 0. \qquad (V11)$$

NOW WE USE OUR FAMILIAR IDENTITIES
$$G_1 w_x = \frac{\partial}{\partial x}(G_1 w) - \left(\frac{\partial}{\partial x} G_1\right) w, \qquad G_2 w_y = \frac{\partial}{\partial y}(G_2 w) - \frac{\partial}{\partial y}(G_2) w.$$

THUS PUTTING THIS INTO (VII) AND USING THE DIVERGENCE THEOREM WE OBTAIN

$$\int_\Omega (F_u^* - \partial_x G_1^* - \partial_y G_2^*)\, w\, d\underline{x} + \int_{\partial\Omega} (\underline{G}^* \cdot \hat{\underline{n}}\, w + \underline{H} \cdot \hat{\underline{n}})\, ds = 0. \quad \text{(VI2)}.$$

NOW WE USE (VI0) AND $\hat{\underline{n}} = (n_1, n_2)$ TO WRITE

$$\underline{H} \cdot \hat{\underline{n}} = F^*_{u_{xx}} w_x n_1 + \frac{1}{2} F^*_{u_{xy}} w_y n_1 + F^*_{u_{yy}} w_y n_2 + \frac{1}{2} F^*_{u_{xy}} w_x n_2$$

$$= (\nabla w)^T A \hat{\underline{n}} \quad \text{WHERE} \quad A = \begin{pmatrix} F^*_{u_{xx}} & \frac{1}{2} F^*_{u_{xy}} \\ \frac{1}{2} F^*_{u_{xy}} & F^*_{u_{yy}} \end{pmatrix}, \quad \nabla w = \begin{pmatrix} w_x \\ w_y \end{pmatrix}. \quad \text{(VI3)}$$

IN THIS WAY, u^* IS A MINIMIZER OR STATIONARY POINT OF $I(u)$ ONLY IF

$$\int_\Omega [F_u - \partial_x G_1 - \partial_y G_2]\, w\, d\underline{x} + \int_{\partial\Omega} (\underline{G} \cdot \hat{\underline{n}}\, w + (\nabla w)^T A \hat{\underline{n}})\, ds = 0 \quad \forall w. \quad \text{(VI4)}$$

HERE $\underline{G} = (G_1, G_2)^T$ WITH

$$G_1 = F^*_{u_x} - \partial_x F^*_{u_{xx}} - \frac{1}{2} \partial_y F^*_{u_{xy}}$$

$$G_2 = F^*_{u_y} - \partial_y F^*_{u_{yy}} - \frac{1}{2} \partial_x F^*_{u_{xy}}.$$

<u>CASE I</u> SUPPOSE THAT $w = 0$ ON $\partial\Omega$ AND $\partial_n w = 0$ ON $\partial\Omega$.

THIS MEANS THAT IN H WE HAVE SPECIFIED u^* AND $\partial_n u^*$ ON $\partial\Omega$.

SINCE $\partial_n w = \nabla w \cdot \hat{\underline{n}} = 0$ WE CONCLUDE THAT WITH $w = 0$ ON $\partial\Omega$, ∇w IS $\perp$ TO $\partial\Omega$, AS SUCH $\partial_n w = |\nabla w| = \underline{0}$. HENCE $\nabla w = 0$ ON $\partial\Omega$. WE CONCLUDE THAT THE BOUNDARY TERM IN (VI4) VANISHES AND WE GET

$$F_{u^*} - \partial_x F^*_{u_x} + \partial_{xx} F^*_{u_{xx}} - \partial_y F^*_{u_y} + \partial_{yy} F^*_{u_{yy}} + \partial_x \partial_y F^*_{u_{xy}} = 0 \quad \text{(V15)}$$

WITH u^*, $\partial_n u^*$ SPECIFIED ON $\partial\Omega$.

<u>CASE II</u> SUPPOSE u^* IS SPECIFIED ON $\partial\Omega$ BUT NO OTHER CONDITION ON $\partial\Omega$ IS GIVEN. THEN FROM (VI4), $w = 0$ ON $\partial\Omega$ IS REQUIRED. THIS MEANS THAT ON $\partial\Omega$. $\nabla w = \pm |\nabla w| \hat{n}$ SINCE ∇w IS $\perp$ TO LEVEL CURVE $w = 0$ AS SUCH FROM (V14) WE HAVE THAT $(\nabla w)^T A \hat{\underline{n}} = \pm |\nabla w| \hat{\underline{n}}^T A \underline{n}$. IN ORDER FOR THIS TO VANISH $\forall\ |\nabla w|$ WE MUST HAVE

$$\hat{\underline{n}}^T A \hat{\underline{n}} = 0 \text{ ON } \partial\Omega. \text{ THIS IS THE NATURAL BC.} \quad \text{(VI6)}$$

CASE III IF NO BC FOR u^* ARE GIVEN ON $\partial\Omega$ THEN u^* SATISFIES (V15)
AND THE NATURAL BOUNDARY CONDITIONS

$$\underline{G}\cdot\hat{\underline{n}} = 0 \text{ ON } \partial\Omega \quad \text{AND} \quad \hat{\underline{n}}^T A\,\hat{\underline{n}} = 0 \text{ ON } \partial\Omega.$$

WE NOW CONSIDER A SIMPLER IMPORTANT SPECIAL CASE OF THE FORMULA.

SPECIAL CASE (F INDEPENDENT OF u_{xy}). SUPPOSE THAT $F = F(x, y, u, u_x, u_y, u_{xx}, u_{yy})$ WITH u SPECIFIED ON $\partial\Omega$. WE THEN OBTAIN FROM (V15) AND (V16) THAT u^* SATISFIES

$$\left.\begin{aligned} &F_u^* - \frac{\partial}{\partial x}(F_{u_x}^*) + \frac{\partial^2}{\partial x^2}(F_{u_{xx}}^*) - \frac{\partial}{\partial y}(F_{u_y}^*) + \frac{\partial^2}{\partial y^2}(F_{u_{yy}}^*) = 0 \quad \text{IN } \Omega \\ &\text{WITH } u^* \text{ SPECIFIED ON } \partial\Omega \text{ AND } \cancel{(F_{u_{xx}}^*\ F_{u_{yy}}^*)\cdot\hat{\underline{n}}} = 0 \text{ ON } \partial\Omega. \\ &F_{u_{xx}}^* n_1^2 + F_{u_{yy}} n_2^2 = 0 \text{ ON } \partial\Omega \text{ WITH } \hat{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \end{aligned}\right\} \quad (V17)$$

IF IN FACT WE HAVE THAT $F = F(x, u, u_x, u_{xx})$, I.E. NO DEPENDENCE ON y, THEN (V17) SIMPLIFIES FURTHER TO

$$\left.\begin{aligned} &F_u^* - \frac{\partial}{\partial x}(F_{u_x}^*) + \frac{\partial^2}{\partial x^2}(F_{u_{xx}}^*) = 0 \quad \text{IN } a \le x \le b \\ &u^* \text{ GIVEN AT } x = a, b; \qquad F_{u_{xx}}^* = 0 \text{ AT } x = a, b. \end{aligned}\right\} \quad (V18).$$

WE NOW GIVE A FEW EXAMPLES:

EXAMPLE 7 SUPPOSE THAT $I(u) = \int_0^1 \left[\frac{1}{2} u_{xx}^2 + V(u)\right] dx$ WITH $u(0) = u(1) = 0$ BUT NO OTHER CONDITIONS SPECIFIED AT $x = 0, 1$. FIND E-L EQUATION AND BOUNDARY CONDITIONS.

WE HAVE $F = F(x, u, u_{xx}) = \frac{1}{2} u_{xx}^2 + V(u)$ AND F INDEPENDENT OF u_x.
FROM (V18) WE CALCULATE $F_u = V'(u)$, $F_{u_{xx}} = u_{xx}$. THUS, u^* SATISFIES

$$u_{xxxx} + V'(u) = 0 \quad \text{IN } 0 < x < 1; \quad u(0) = u(1) = 0$$

$$\text{AND} \quad u_{xx}(0) = u_{xx}(1) = 0 \quad (\text{NATURAL BC})$$

EXAMPLE 8 CONSIDER A THIN PLATE WITH A LOAD $p(x, y)$ THAT CAUSES IT TO SAG. THE DEFLECTION $u(x, y)$ OF THE PLATE IS GIVEN BY

$$\min_{u \in H} I(u) \quad \text{WITH} \quad I(u) = \frac{1}{2}\int_\Omega (u_{xx} + u_{yy})^2\, d\underline{x} - \int_\Omega p\, u\, d\underline{x}$$

WHERE $H = \{ u \mid u \in C^4(\Omega), \text{ WITH } u = u_n = 0 \text{ ON } \partial\Omega \}$.

NOW SINCE $F = F(x, y, u, u_{xx}, u_{yy})$ THE EL EQUATION FROM (VI7) IS OBTAINED WITH $F_u = -p, \quad F_{u_{xx}} = (u_{xx} + u_{yy}) = F_{u_{yy}}$.

THUS FROM (VI7), THE EL EQUATION IS

$$-p + \partial_{xx}(u_{xx} + u_{yy}) + \partial_{yy}(u_{xx} + u_{yy}) = 0 \quad \text{IN } \Omega$$

$$u = 0 \text{ ON } \partial\Omega, \quad u_n = 0 \text{ ON } \partial\Omega.$$

THIS YIELDS THAT $\quad -p + (\partial_{xx} + \partial_{yy})(\partial_{xx} + \partial_{yy})u = 0$

$$\text{OR} \quad \Delta^2 u = p \quad \text{IN } \Omega$$

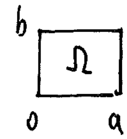
$$\text{WITH} \quad u = u_n = 0 \text{ ON } \partial\Omega.$$

NOW WE CAN ALSO USE A RAYLEIGH-RITZ PROCEDURE TO ESTIMATE THE MINIMUM ENERGY IN THE CASE OF A CONSTANT LOAD WHERE p = CONSTANT (INDEPENDENT OF x, y).

WE CONSIDER A TRIAL FUNCTION

$$u_0 = c_0 \left[1 - \cos\left(2\pi x/a\right)\right]\left[1 - \cos\left(2\pi y/b\right)\right]$$

WHERE WE TAKE $\Omega = \{(x,y) \mid 0 \le x \le a, \; 0 \le y \le b\}$.



NOTICE THAT $u_0 = 0$ ON $x = 0, a$; AND $y = 0, b$. ALSO $u_{0x} = 0$ ON $x = 0, a$; $u_{0y} = 0$ ON $y = 0, b$.
THUS $u_0 \in H$. WE SEEK TO DETERMINE c_0 TO MINIMIZE $I(u_0)$.

WE HAVE $\quad \min\limits_{u \in H} I(u) \le \min\limits_{c_0} I(u_0)$.

WE CALCULATE $\quad J(c_0) = I(u_0) = \frac{1}{2}\int_0^a \int_0^b \left[(\Delta u_0)^2 - 2p u_0\right] dx\, dy.$

PUT IN THE FORM FOR u_0 AND ASSUME p IS A CONSTANT. WE CALCULATE AFTER SOME ALGEBRA THAT

$$J(c_0) = \frac{2\pi^4 c_0^2}{a^3 b^3}\left[3(a^4 + b^4) + 2a^2 b^2\right] - p\, c_0\, a b.$$

WE SET $J'(c_0) = 0$ TO OBTAIN $\quad c_0 = c_0^* = \dfrac{p a^4 b^4}{4\pi^4\left[3(a^4 + b^4) + 2a^2 b^2\right]}$.

NOW SINCE $J''(c_0) > 0$, c_0^* IS INDEED A MINIMUM OF $J(c_0)$. THE OPTIMIZER FOR $I(u_0)$ IS OBTAINED WITH $\quad u_0 = u_0^* = c_0^*\left[1 - \cos\left(\frac{2\pi x}{a}\right)\right]\left[1 - \cos\left(\frac{2\pi y}{b}\right)\right]$.

VARIATIONAL PROBLEMS WITH CONSTRAINTS

CONSIDER THE CALCULUS PROBLEM OF MINIMIZING $f(x_1,..,x_n)$ SUBJECT TO m CONSTRAINTS OF THE FORM $g_i(x_1,...,x_n)=0$ FOR $i=1,..,m$ WITH $m<n$. THE IDEA IS TO INTRODUCE LAGRANGE MULTIPLIER PARAMETERS $\lambda_1,..,\lambda_m$ AND CONSTRUCT

$$\hat{f}(x_1,..,x_n) = f(x_1,..,x_n) + \lambda_1 g_1(x_1,..,x_n) + \lambda_m g_m(x_1,..,x_n).$$

NOW IF f HAS A MINIMUM AT $x_1^0,..,x_n^0$ SUBJECT TO $g_i = 0$ FOR $i=1,..,m$ IT IS NECESSARY THAT

$$\nabla \hat{f}(x_1,..,x_n) = 0 \quad \longrightarrow \quad \nabla f|_{\underline{x}^0} = -\underline{\lambda}^T \nabla g|_{\underline{x}^0} \quad \text{WITH} \quad g \equiv \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix}, \ \lambda \equiv \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \ \text{AND} \ \underline{x}^0 \equiv \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix}$$

TOGETHER WITH

$$g_1 = .. = g_m = 0 \text{ AT } \underline{x}^0.$$

OVERALL THIS YIELDS $n+m$ EQUATIONS FOR THE $n+m$ UNKNOWNS $\underline{x}^0$ AND λ. FOR A PROOF SEE ANY CALCULUS TEXT.

WE NOW ILLUSTRATE THIS APPROACH FOR A CALCULUS OF VARIATIONS PROBLEM OF MINIMIZING $I(u) = \int_\Omega F(x,y,u,u_x,u_y)\,d\underline{x}$ WITH $u=0$ ON $\partial\Omega$ SUBJECT TO THE CONSTRAINT THAT $\int_\Omega g(x,y,u,u_x,u_y)\,d\underline{x} = M$ WHERE M IS CONSTANT. WE CAN REDUCE THIS TO THE LAGRANGE MULTIPLIER PROBLEM AS FOLLOWS:

LET $u = u^* + \varepsilon_1 w + \varepsilon_2 v$ WHERE $w = v = 0$ ON $\partial\Omega$.

WE WANT TO MINIMIZE

$$J(\varepsilon_1,\varepsilon_2) = \int_\Omega F(x,y,u^*+\varepsilon_1 w+\varepsilon_2 v,\ u_x^*+\varepsilon_1 w_x+\varepsilon_2 v_x,\ u_y^*+\varepsilon_1 w_y+\varepsilon_2 v_y)\,d\underline{x}$$

SUBJECT TO $G(\varepsilon_1,\varepsilon_2) = M$ WHERE

$$G(\varepsilon_1,\varepsilon_2) \equiv \int_\Omega g(x,y,u^*+\varepsilon_1 w+\varepsilon_2 v,\ u_x^*+\varepsilon_1 w_x+\varepsilon_2 v_x,\ u_y^*+\varepsilon_1 w_y+\varepsilon^2 v_y)\,d\underline{x}$$

THIS BECOMES A LAGRANGE MULTIPLIER PROBLEM. WE INTRODUCE

$$\hat{J}(\varepsilon_1,\varepsilon_2) = J(\varepsilon_1,\varepsilon_2) + \lambda[G(\varepsilon_1,\varepsilon_2) - M].$$

THE VARIATIONAL PROBLEM IS TO SET $\hat{J}_{\varepsilon_1} = 0$ AND $\hat{J}_{\varepsilon_2} = 0$ AT $\varepsilon_1 = 0$, $\varepsilon_2 = 0$.

THIS IS EQUIVALENT TO REWRITING THE AUGMENTED FUNCTIONAL AS

$$\hat{J} = \int_\Omega F(x,y,u,u_x,u_y)\,d\underline{x} + \lambda\left(\int_\Omega g(x,y,u,u_x,u_y)\,d\underline{x} - M\right)$$

AND DETERMINING THE LAGRANGE MULTIPLIER λ AND MINIMIZER u^* FROM A NECESSARY CONDITION FOR A MINIMUM:

$$F_u - \frac{\partial}{\partial x}F_{u_x} - \frac{\partial}{\partial y}F_{u_y} + \lambda\left(g_u - \frac{\partial}{\partial x}g_{u_x} - \frac{\partial}{\partial y}g_{u_y}\right) = 0$$

SUBJECT TO $\int_\Omega g\,d\underline{x} = M$,

ANOTHER APPROACH TO VIEW VARIATIONAL PROBLEMS WITH CONSTRAINTS IS AS FOLLOWS:

SUPPOSE WE WANT TO MINIMIZE

$$I(u) = \int_\Omega F(x, y, u, u_x, u_y)\, d\underline{x} \quad \text{FOR } u \in H \text{ WITH } H \equiv \{ u \mid u \in C^2(\Omega),\ u = 0 \text{ ON } \partial\Omega \}$$

SUBJECT TO $M(u) = \int_\Omega G(x, y, u, u_x, u_y)\, d\underline{x} = \underset{\text{PRESCRIBED}}{\text{CONSTANT}}$. (ONE CONSTRAINT).

<u>CLAIM 1</u> IF u^* SOLVES THE VARIATIONAL PROBLEM, THEN u^* IS A CRITICAL POINT FOR THE FUNCTIONAL $I(u) + \lambda M(u)$, i.e. u^* SATISFIES

$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} + \lambda\left(G_u - \frac{\partial}{\partial x} G_{u_x} - \frac{\partial}{\partial y} G_{u_y}\right) = 0 \quad \text{SUBJECT TO } \int_\Omega G\, d\underline{x} = \text{PRESCRIBED CONSTANT}$$

<u>PROOF</u> SUPPOSE $u^*(\underline{x})$ IS A MINIMIZER. THUS $M(u^*) = C$ AND u^* MINIMIZES $I(u)$ AMONG ALL FUNCTIONS $u \in H$ WITH $M(u) = C$. IN PARTICULAR LET $u_\sigma \in H$ BE A ONE-PARAMETER FAMILY OF FUNCTIONS IN H WITH $M(u_\sigma) = C$ AND $u_0 = u^*$ (i.e. AT $\sigma = 0$ IT IS THE MINIMIZER). THEN $I(u_\sigma)$ (BEING A FUNCTION OF σ) IS MINIMIZED AT $\sigma = 0$ AND SO

$$0 = \frac{d}{d\sigma} I(u_\sigma)\Big|_{\sigma=0} = \frac{d}{d\sigma}\int_\Omega F(\underline{x}, u_\sigma, \partial_x u_\sigma, \partial_y u_\sigma)\, d\underline{x}\Big|_{\sigma=0} = \int_\Omega \left(F_u \frac{\partial u_\sigma}{\partial \sigma}\Big|_{\sigma=0} + F_{u_x} \frac{\partial}{\partial x}\frac{\partial u_\sigma}{\partial \sigma} + F_{u_y}\frac{\partial}{\partial y}\frac{\partial u_\sigma}{\partial\sigma}\Big|_{\sigma=0} d\underline{x}\right)$$

NOW LABEL $\zeta \equiv \frac{\partial u_\sigma}{\partial \sigma}\Big|_{\sigma=0}$. NOTICE THAT $\zeta = 0$ ON $\partial\Omega$, AND SO USING DIV. THEOREM

$$\int_\Omega \left[F_u^* - \frac{\partial}{\partial x} F_{u_x}^* - \frac{\partial}{\partial y} F_{u_y}^*\right] \zeta(\underline{x})\, d\underline{x} = 0$$

WHERE $*$ INDICATES EVALUATION AT $\sigma = 0$, i.e. AT $u^*(\underline{x})$.

NOW IF $\zeta(\underline{x})$ COULD BE ANY SMOOTH FUNCTION ON Ω (WITH ZERO BC ON $\partial\Omega$), WE WOULD CONCLUDE AS BEFORE THAT $F_u^* - \frac{\partial}{\partial x} F_{u_x}^* - \frac{\partial}{\partial y} F_{u_y}^*$ MUST BE ZERO, AND THAT WOULD BE THE E-L EQUATION. HOWEVER, DUE TO THE CONSTRAINT $M(u_\sigma) = C$, $\zeta(\underline{x})$ CANNOT BE JUST ANY FUNCTION. INDEED SINCE $M(u_\sigma) = C \ \forall \sigma$, THEN BY DIFFERENTIATING WRT σ WE HAVE

$$0 = \frac{d}{d\sigma} M(u_\sigma)\Big|_{\sigma=0} = \int_\Omega \left(G_u^* - \frac{\partial}{\partial x} G_{u_x}^* - \frac{\partial}{\partial y} G_{u_y}^*\right) \zeta(\underline{x})\, d\underline{x}$$

THEREFORE WE HAVE THAT, WITH $f(\underline{x}) = F_u^* - \frac{\partial}{\partial x} F_{u_x}^* - \frac{\partial}{\partial y} F_{u_y}^*$ AND $g(\underline{x}) = G_u^* - \frac{\partial}{\partial x} G_{u_x}^* - \frac{\partial}{\partial y} G_{u_y}^*$, WE HAVE $\int_\Omega f(\underline{x})\zeta(\underline{x})\, d\underline{x} = 0$ FOR ANY $\zeta(\underline{x})$ WITH $\int_\Omega g(\underline{x})\zeta(\underline{x})\, d\underline{x} = 0$.

WHAT CAN WE CONCLUDE ABOUT $f(\underline{x})$ FROM THIS? WE CLAIM THAT

<u>CLAIM 2</u> $f(\underline{x}) + \lambda g(\underline{x}) = 0$ FOR SOME λ REAL.

ONCE WE ESTABLISH CLAIM 2, THEN CLAIM 1 IS COMPLETE. LET $\Lambda(\underline{x})$ BE ANY SMOOTH FUNCTION IN Ω, AND DECOMPOSE IT AS

$$\Lambda(\underline{x}) = K g(\underline{x}) + \hat{\Lambda}(\underline{x}) \quad \text{WHERE} \quad \int_\Omega \hat{\Lambda}\, g \, d\underline{x} = 0.$$

THUS $\hat{\Lambda}$ IS $\perp$ TO g. BY MULTIPLYING BY $g(x)$ AND INTEGRATING, $\int_\Omega \Lambda(\underline{x}) g(\underline{x})\, d\underline{x} = K \int_\Omega g(\underline{x}) g(\underline{x}) d\underline{x}$ OR EQUIVALENTLY $K = (\Lambda, g)/(g,g)$. THUS,

$$\Lambda(\underline{x}) = \frac{(\Lambda, g)}{(g,g)} g(\underline{x}) + \hat{\Lambda}(\underline{x}).$$

NOW SINCE $\hat{\Lambda}$ IS $\perp$ TO g, WE MUST HAVE (SETTING $\xi = \Lambda$) THAT

$$\int_\Omega f(\underline{x}) \tilde{\Lambda}(\underline{x})\, d\underline{x} = 0. \longrightarrow \int_\Omega f(x) \left[\Lambda(\underline{x}) - \frac{(\Lambda, g)}{(g,g)} g(\underline{x}) \right] d\underline{x} = 0.$$

INTERCHANGING INTEGRATION TO GET $\int_\Omega [\cdots] \Lambda(\underline{x})\, d\underline{x} = 0$ WE CONCLUDE THAT

$$\int_\Omega \left[f(\underline{x}) - \frac{(f,g)}{(g,g)} g(\underline{x}) \right] \Lambda(\underline{x})\, d\underline{x} = 0 \quad . \quad \forall \ \Lambda(\underline{x}).$$

WE CONCLUDE THAT $\quad f(\underline{x}) - \dfrac{(f,g)}{(g,g)} g(\underline{x}) = 0$ IN Ω.

IN OTHER WORDS $\exists\ \lambda$ SUCH THAT THE OPTIMIZER u^* SATISFIES

$$\left(F_u^* - \frac{\partial}{\partial x} F_{u_x}^* - \frac{\partial}{\partial y} F_{u_y}^* \right) + \lambda \left(G_u^* - \frac{\partial}{\partial x} G_{u_x}^* - \frac{\partial}{\partial y} G_{u_y}^* \right) = 0$$

$$\text{WITH} \quad \int_\Omega G(\underline{x}, u^*, u_x^*, u_y^*)\, d\underline{x} = C.$$

<u>EXAMPLE</u> MINIMIZE $\int_\Omega (u_x^2 + u_y^2)\, dx\, dy$ SUBJECT TO $u = 0$ ON $\partial\Omega$ WITH $\int_\Omega u^2\, dx\, dy = 1$.

WE LET $\hat{J}(u) = \int_\Omega (u_x^2 + u_y^2)\, dx\, dy + \lambda \left[\int_\Omega u^2\, dx\, dy - 1 \right] = \int_\Omega (u_x^2 + u_y^2)\, dx\, dy + \lambda \int_\Omega \left(u^2 - \frac{1}{|\Omega|} \right) dx\, dy.$

THE EL EQUATION FOR $\hat{J}(u)$ IS

$$2\lambda u - \frac{\partial}{\partial x}(2u_x) - \frac{\partial}{\partial y}(2u_y) = 0 \longrightarrow \begin{array}{l} u_{yy} + u_{xx} - \lambda u = 0. \text{ IN } \Omega. \\ u = 0 \text{ ON } \partial\Omega \end{array}$$

THUS $\lambda = -\sigma$ WHERE σ IS AN EIGENVALUE OF $\Delta u + \sigma u = 0$ IN Ω; $u = 0$ ON $\partial\Omega$, WITH CORRESPONDING EIGENFUNCTION $u = \sigma$, NORMALIZED BY $\int_\Omega \phi^2\, d\underline{x} = 1$.

THEN $\quad I(\phi) = \int_\Omega |\nabla \phi|^2\, d\underline{x} = -\int_\Omega \phi \Delta \phi\, d\underline{x} = \sigma \int_\Omega \phi^2\, d\underline{x} = \sigma.$

THUS $I(\phi)$ IS MINIMIZED WHEN $\sigma = \sigma_1$, THE SMALLEST EIGENVALUE.

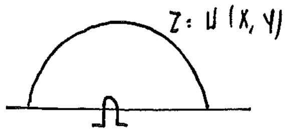
EXAMPLE (GEOMETRY) CONSIDER THE MINIMUM SURFACE AREA PROBLEM OF MINIMIZING

$$J(u) \equiv \int_\Omega (1 + u_x^2 + u_y^2)^{1/2}\, dx\, dy \quad \text{WITH} \quad u = 0 \text{ ON } \partial\Omega$$

SUBJECT TO A FIXED VOLUME CONSTRAINT THAT $\int_\Omega u\, dx\, dy = V_0$

FIND THE E-L EQUATION AND SOLVE IT WHEN $\Omega = \{(x,y) \mid x^2 + y^2 \leq r_0^2\}$.

SOLUTION $J(u)$ IS THE SURFACE AREA OF THE FUNCTION $z = u(x,y)$ IN 3-D ABOVE Ω AND THE SURFACE IS ATTACHED TO THE $z = 0$ PLANE AT $u = 0$ ON $\partial\Omega$.



(SIDE-VIEW). THEN V_0 IS THE VOLUME OF THE "DOME" BELOW $u(x,y)$ AND ABOVE $u = 0$.

USING LAGRANGE'S MULTIPLIERS WE WRITE THE AUGMENTED FUNCTIONAL AS

$$\hat{J}(u) = \int_\Omega (1 + u_x^2 + u_y^2)^{1/2}\, d\underline{x} + \lambda\left(\int_\Omega u\, d\underline{x} - V_0\right) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, d\underline{x} + \lambda \int_\Omega \left(u - \frac{V_0}{|\Omega|}\right) d\underline{x}$$

NOW $F \equiv \sqrt{1 + |\nabla u|^2} + \lambda\left(u - \frac{V_0}{|\Omega|}\right)$ AND SO $F_u = \lambda$

$$F_{u_x} = \frac{u_x}{\sqrt{1 + |\nabla u|^2}}, \quad F_{u_y} = \frac{u_y}{\sqrt{1 + |\nabla u|^2}}.$$

THIS YIELDS THAT

$$F_u - \frac{d}{dx} F_{u_x} - \frac{d}{dy} F_{u_y} = 0 \quad \text{SO THAT} \qquad T \equiv \frac{\nabla}{\sqrt{1 + |\nabla u|^2}}$$

$$(1) \quad \left\{ \begin{array}{l} \nabla \cdot (T u) = \lambda \\ u = 0 \text{ ON } \partial\Omega \quad \text{AND} \quad \int_\Omega u\, d\underline{x} = V_0 \end{array} \right.$$

THE LAGRANGE MULTIPLIER PARAMETER λ IS TO BE FOUND TO SATISFY THE VOLUME CONSTRAINT.

NOW FROM GEOMETRY $2H = -\nabla \cdot (T u) \equiv \nabla_3 \cdot \hat{n}$ WHERE $\hat{n} = \frac{(\nabla u, -1)}{\sqrt{1 + |\nabla u|^2}}$

AND ∇_3 IS THE 3-D GRADIENT $\nabla_3 = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ AND $\hat{n}$ IS UNIT NORMAL TO $z = u(x,y)$ IN 3-D. HERE H IS MEAN CURVATURE OF SURFACE.

AS A RESULT, IF WE WANT $\nabla \cdot (T u) =$ CONSTANT, THEN WE SHOULD TAKE $z = u(x,y)$ TO CORRESPOND TO PART OF A SPHERE IN 3-D FOR WHICH WE KNOW $H =$ CONSTANT. IN PARTICULAR, FOR A SPHERE OF RADIUS c_0, i.e. $x^2 + y^2 + z^2 = c_0^2$ THEN $H = 1/c_0$.

THIS SUGGESTS THAT IF Ω IS THE DISK $x^2+y^2=r_0^2$ WE SHOULD LOOK FOR u IN THE FORM $(u+A)^2+x^2+y^2=c_0^2$. WHEN $x^2+y^2=r_0^2 \to u=0 \longrightarrow A=\sqrt{c_0^2-r_0^2}$. THUS,

$$u(x,y)=(c_0^2-x^2-y^2)^{1/2}-(c_0^2-r_0^2)^{1/2} \qquad (2)$$

FOR THEN $\nabla\cdot(Tu)$ SHOULD BE CONSTANT AND $u=0$ ON $\partial\Omega$ WHERE $x^2+y^2=r_0^2$.

WE CALCULATE $u_x=-x\,(c_0^2-x^2-y^2)^{-1/2}$, $u_y=-y\,(c_0^2-x^2-y^2)^{-1/2}$

SO $1+u_x^2+u_y^2 = 1+\dfrac{x^2+y^2}{c_0^2-x^2-y^2} = \dfrac{c_0^2}{c_0^2-x^2-y^2}$, $\longrightarrow \sqrt{1+|\nabla u|^2}=\dfrac{c_0}{\sqrt{c_0^2-x^2-y^2}}$.

THUS, $\dfrac{u_x}{\sqrt{1+|\nabla u|^2}}=-\dfrac{x}{c_0}$, $\dfrac{u_y}{\sqrt{1+|\nabla u|^2}}=-\dfrac{y}{c_0}$ SO THAT $Tu=\dfrac{1}{c_0}(-x,-y)$.

HENCE $\nabla\cdot(Tu)=-2/c_0$. THIS IMPLIES FROM (1) THAT $\lambda=-2/c_0$.

NOW WE FIND c_0 BY IMPOSING VOLUME CONSTRAINT. WE INTEGRATE IN POLAR COORDINATES TO GET

$$\int_0^{2\pi}\int_0^{r_0}\left([c_0^2-r^2]^{1/2}-(c_0^2-r_0^2)^{1/2}\right)r\,dr=V_0 \longrightarrow 2\pi\int_0^{r_0}\left[r\,(c_0^2-r^2)^{1/2}-(c_0^2-r_0^2)^{1/2}r\right]dr=V_0.$$

WE INTEGRATE TO OBTAIN

$$-\frac{1}{3}(c_0^2-r^2)^{3/2}\Big|_0^{r_0}-(c_0^2-r_0^2)^{1/2}\frac{r_0^2}{2}=\frac{V_0}{2\pi}\longrightarrow +\frac{1}{3}c_0^3-\frac{1}{3}(c_0^2-r_0^2)^{3/2}-\frac{r_0^2}{2}(c_0^2-r_0^2)^{1/2}=\frac{V_0}{2\pi}$$

REWRITING THIS WE OBTAIN

$$c_0^3-(c_0^2-r_0^2)^{1/2}\left[(c_0^2-r_0^2)+\frac{3r_0^2}{2}\right]=\frac{3V_0}{2\pi}.$$

HENCE WE HAVE A NONLINEAR ALGEBRAIC EQUATION FOR c_0 IN TERMS OF V.

$$\frac{3V_0}{2\pi}=c_0^3-(c_0^2-r_0^2)^{1/2}\left[c_0^2+r_0^2/2\right].$$

NOW INTRODUCE A NEW PARAMETER γ BY $c_0^2=\gamma r_0^2 \longrightarrow c_0=\sqrt{\gamma}\,r_0$.

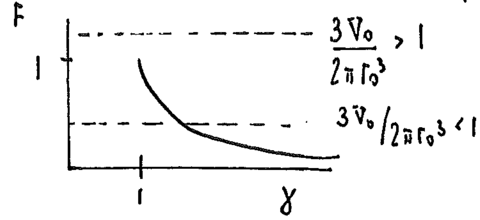
THEN WE OBTAIN THAT

$$\frac{3V_0}{2\pi r_0^3}=F(\gamma)\equiv\gamma^{3/2}-(\gamma-1)^{1/2}(\gamma+1/2) \qquad (3)$$

NOW IF (3) HAS A SOLUTION γ^* FOR A GIVEN V, IT FOLLOWS THAT THE SOLUTION TO (1) IS GIVEN BY (2) WHERE $c_0=\sqrt{\gamma^*}\,r_0$.

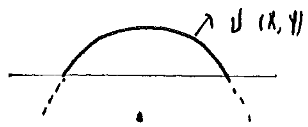
NOW FOR $\gamma > 1$ WE CAN SHOW THAT $F'(\gamma) < 0$ AND $F(1) = 1$.

LOOKING FOR INTERSECTION WITH $3\bar{V}_0 / 2\pi r_0^3$ WE HAVE

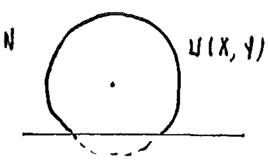


WE CONCLUDE FROM (3) THAT IF $\bar{V}_0 < \bar{V}_{CR} \equiv 2\pi r_0^3 / 3$ ($\frac{1}{2}$ A SPHERE OF RADIUS r_0)
THEN $\exists$ A UNIQUE VALUE OF γ SUCH THAT (3) HAS A SOLUTION γ^*. IN THIS CASE $\exists$ A UNIQUE $C_0 = C_0^*$ AND WE CONCLUDE THAT $u = [C_0^{*2} - x^2 - y^2]^{1/2} - (C_0^{*2} - r_0^2)^{1/2}$.

AS SUCH WE CAN HAVE A PICTURE AS SHOWN



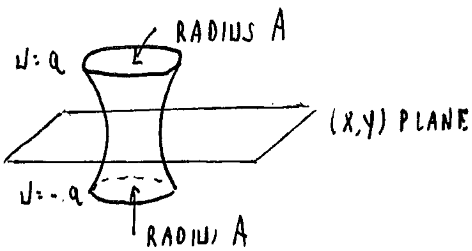
AND NOT



$\Leftarrow$ THIS YIELDS A MULTI-VALUED

FOR OTHER APPLICATIONS A SPHERE MAY NOT BE ONLY SHAPE THAT GIVES $\nabla \cdot (Tu) = $ CONSTANT.
IN GENERAL $\nabla \cdot (Tu) = -2H$ $\quad H = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$ IS MEAN CURVATURE OF THE SURFACE AT POINT $\underline{x}$, WITH r_1, r_2 BEING THE PRINCIPAL RADII OF CURVATURE AT (x, y).

<u>REMARK</u> CONSIDER A LIQUID BRIDGE SYMMETRIC IN Z-DIRECTION WITH A CIRCULAR CROSS-SECTION AS SHOWN



IT CAN BE SHOWN THAT THE CATENOID (SEE 1-D NOTES)

$$\cosh\left(u/C_0 + C_1\right) = \frac{1}{C_0}(x^2 + y^2)^{1/2}$$

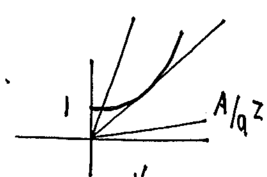
SATISFIES $\nabla \cdot (Tu) = $ CONSTANT.

TO SATISFY THE B.C. :

$$\left.\begin{aligned} \cosh\left(a/C_0 + C_1\right) &= A/C_0 \\ \cosh\left(-a/C_0 + C_1\right) &= A/C_0 . \end{aligned}\right\} (*)$$

WE MUST TAKE $C_1 = 0$ IN (*) SO THAT $\cosh\left(a/C_0\right) = A/C_0$.

THIS IS A NONLINEAR ALGEBRAIC EQUATION FOR C_0 IN TERMS OF a AND A.
LET $z = a/C_0$ SO THAT $\frac{A}{a} z = \cosh z$ DETERMINES z IN TERMS OF A/a.



$\exists$ EITHER 0, 1, OR 2 SOLUTIONS, TWO SOLUTIONS IF A/a LARGE ENOUGH. LET $r = (x^2 + y^2)^{1/2}$.
FINALLY $\cosh\left(u/C_0\right) = \frac{1}{C_0}(x^2 + y^2)^{1/2} \longrightarrow r = C_0 \cosh\left(u/C_0\right)$. THE VOLUME CONSTRAINT IS $\bar{V}_0 = \pi \int_{-a}^{a} (r(u))^2 \, du \longrightarrow \bar{V}_0 = 2\pi C_0^2 \int_0^a \cosh^2\left(u/C_0\right) du$.

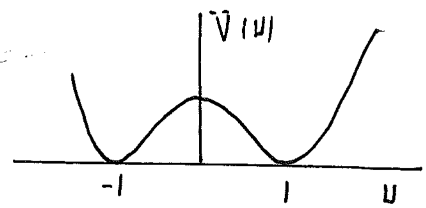
<u>QUESTION</u>: GIVEN $\bar{V}_0 > 0$ DOES $\exists$ A VALUE OF C_0 AND A SUCH THAT
<u>BOTH</u> $\cosh\left(a/C_0\right) = A/C_0$ AND THE CONSTRAINT $\bar{V}_0 = 2\pi C_0^2 \int_0^a \cosh^2\left(u/C_0\right) du$
CAN BE SATISFIED? $\rightarrow$ NONLINEAR ALGEBRAIC SYSTEM FOR A AND C_0 IN TERMS OF $a, \bar{V}_0$.

APPLICATIONS (PHYSICS)

IN THE CONTINUUM THEORY OF PHASE TRANSITIONS WE ENCOUNTER VARIATIONAL PROBLEMS OF THE FORM IN 2-D:

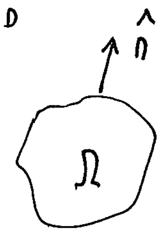
$$\text{MIN} \left[\int_\Omega \left[\frac{\varepsilon^2}{2} |\nabla u|^2 + V(u) \right] d\underline{x} \right] \quad \text{SUBJECT TO THE MASS CONSTRAINT} \int_\Omega u \, d\underline{x} = M$$

WHERE M IS CONSTANT. THE POTENTIAL $V(u)$ HAS A DOUBLE-WELL STRUCTURE OF THE FORM



- PROTOTYPICAL IS $V(u) = \frac{1}{4}(1-u^2)^2$.
- ε IS SMALL AND POSITIVE
- NO BC ARE GIVEN $\longrightarrow$ NATURAL BC HOLD.

TO DERIVE THE EULER LAGRANGE EQUATION WE BEGIN WITH THE AUGMENTED FUNCTIONAL



$$I(u) \equiv \int_\Omega F(x, y, u, u_x, u_y)\, d\underline{x} + \lambda \int_\Omega \left[G(x,y,u) - \frac{M}{|\Omega|} \right] d\underline{x}$$

WHERE $F = \frac{\varepsilon^2}{2} |\nabla u|^2 + V(u)$, $\quad G = u$, $\quad$ WITH $|\nabla u|^2 = u_x^2 + u_y^2$.

THE E.L. EQUATION IS
$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} + \lambda (G_u) = 0, \qquad (A1)$$

AND THE NATURAL BOUNDARY CONDITIONS ARE $(F_{u_x}, F_{u_y}) \cdot \hat{\underline{n}} = 0$ ON $\partial\Omega$.

WE CALCULATE
$$F_u = V'(u), \quad F_{u_x} = \varepsilon^2 u_x, \quad F_{u_y} = \varepsilon^2 u_y, \quad G_u = 1.$$

THUS WE OBTAIN THAT (A1) BECOMES
$$V'(u) - \varepsilon^2 \Delta u + \lambda = 0, \qquad \nabla u \cdot \hat{n} = \frac{\partial u}{\partial n} = 0 \text{ ON } \partial\Omega.$$

THUS THE MINIMIZER WOULD SATISFY
$$\left.\begin{aligned} &\varepsilon^2 \Delta u - V'(u) = \lambda \text{ IN } \Omega \\ &\partial_n u = 0 \text{ ON } \partial\Omega \\ &\int_\Omega u \, d\underline{x} = M \end{aligned}\right\} \quad (A2)$$

WHERE THE MASS CONSTRAINT IS THE EFFECTIVE EQUATION FOR λ.

THIS PDE (A2) IS CALLED THE CONSTRAINED ALLEN-CAHN EQUATION.

THE CORRESPONDING TIME DEPENDENT PROBLEM IS FOR $U(x,y,t)$

$$\left.\begin{aligned} &U_t = \varepsilon^2 \Delta U - V'(U) - \lambda \\ &\partial_n U = 0 \text{ on } \partial\Omega \text{ AND } \int_\Omega U \, d\underline{x} = M, \end{aligned}\right\} \quad (A3)$$

WITH SOME INITIAL CONDITION $U(\underline{x},0)$.

WE DEFINE THE TIME-DEPENDENT ENERGY $E[U(\underline{x},t)]$ BY

$$E(U) \equiv \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla U|^2 + V(U) \right) d\underline{x}.$$

IT IS CLEAR THAT $E \geq 0$ ALWAYS. WE NOW CLAIM THE FOLLOWING

<u>CLAIM</u> $\quad \dfrac{d}{dt} E(U) = -\displaystyle\int_\Omega (U_t)^2 \, d\underline{x} \leq 0$ IF U SATISFIES (A3).

THIS INDICATES THAT U EVOLVES IN (A3) SO AS TO MINIMIZE THE ENERGY U.

<u>PROOF</u> $\quad \dfrac{d}{dt} E = \displaystyle\int_\Omega \left(\varepsilon^2 \nabla U \cdot \nabla U_t + V'(U) U_t \right) d\underline{x}$

RECALL THAT $\nabla \cdot [V \nabla W] = \nabla V \cdot \nabla W + V \Delta W$ FOR ANY V, W. LET $W = U$, $V = U_t$

SO THAT $\quad \nabla U \cdot \nabla U_t = \nabla \cdot [U_t \nabla U] - U_t \Delta U.$

WE SUBSTITUTE TO GET

$$\begin{aligned} \frac{dE}{dt} &= \int_\Omega \left(-\varepsilon^2 U_t \Delta U + \varepsilon^2 \nabla \cdot (U_t \nabla U) + V'(U) U_t \right) d\underline{x} \\ &= \int_\Omega \underbrace{\left[V'(U) - \varepsilon^2 \Delta U \right]}_{\text{USE PDE}} U_t \, d\underline{x} + \varepsilon^2 \int_{\partial\Omega} U_t \underbrace{\nabla U \cdot \hat{n}}_{=0} \, ds \\ &= \int_\Omega (-U_t - \lambda) \, U_t \, d\underline{x} + 0 \\ &= -\lambda \frac{d}{dt} \underbrace{\left(\int_\Omega U \, dx \right)}_{= M} - \int_\Omega U_t^2 \, d\underline{x} = -\lambda \frac{dM}{dt} - \int_\Omega U_t^2 \, d\underline{x}. \end{aligned}$$

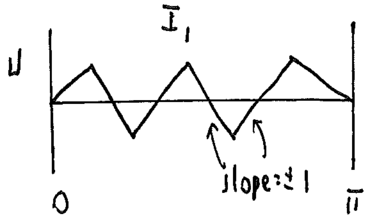
SINCE M IS CONSTANT, WE GET $\quad dE/dt = -\displaystyle\int_\Omega U_t^2 \, d\underline{x} \leq 0.$

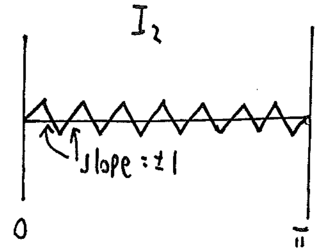
THE NEXT APPLICATION INVOLVES FORMATION OF MICROSTRUCTURE IN MATERIAL SCIENCE (VISCOELASTICITY). CONSIDER THE FUNCTIONAL IN 1-D FOR $U(x)$

$$I(U) = \int_0^\pi \left(\frac{1}{4} (U_x^2 - 1)^2 + \frac{\alpha U^2}{2} \right) d\underline{x} \quad \text{WITH } U(0) = 0, \; U(\pi) = 0 \text{ AND } \alpha > 0.$$

THE "MINIMIZER" OF $I(U)$ WANTS TO DECREASE U (DUE TO $\alpha U^2/2$ TERM) BUT SIMULTANEOUSLY HAVE $U_x^2 \simeq 1 \rightarrow U_x \simeq \pm 1$, WHICH IS IMPOSSIBLE.

THE ENERGY $I(u)$ DECREASES THE MORE SAW-TOOTHED THE FUNCTION BECOMES (WITH SMALLER u)

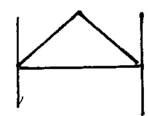




$I_2 < I_1$ (smaller energy).

IT IS CLEAR THAT THE "MINIMIZER" IS NOT ACHIEVED IN THE SPACE OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS. TO ILLUSTRATE THIS CONSIDER

$$V(x) = \begin{cases} x & \text{if } 0 \leq x \leq \pi/2 \\ \pi - x & \text{if } \pi/2 \leq x \leq \pi. \end{cases}$$



DEFINE A SAW-TOOTH PROFILE $u^K(x)$ BY

$$u^K(x) = \frac{1}{K} V(Kx) \quad \text{WITH } K \text{ AN INTEGER AND LET } V(x+\pi) = V(x).$$

AS $K\uparrow$ THE AMPLITUDE DECREASES AND $u^K(x)$ ON $0 < x < \pi$ BECOMES MORE JAGGED. WE NOW CALCULATE THE ENERGY ALONG $u^K(x)$.

NOW $u^K_x = V'(Kx) = \{ \pm 1$ SO THAT $(u^K_x)^2 - 1 = 0$.

WE OBTAIN
$$I(u^K(x)) = \int_0^{\pi} \frac{\alpha}{2} (u^K(x))^2 dx = \frac{\alpha}{2K^2} \int_0^{\pi} (V(Kx))^2 dx = \frac{\alpha}{2K^3} \int_0^{K\pi} (V(t))^2 dt.$$
let $t = Kx$

NOW BY PERIODICITY $V(x+\pi) = V(x)$ WE HAVE

$$I(u^K(x)) = \frac{\alpha}{2K^3} K \int_0^{\pi} (V(t))^2 dt = \frac{\alpha}{2K^2} \left[\int_0^{\pi/2} t^2 dt + \int_{\pi/2}^{\pi} (\pi - t)^2 dt \right]$$
$$= \frac{\alpha}{2K^2} \left[\int_0^{\pi} t^2 dt + \int_{\pi/2}^{\pi} (\pi^2 - 2\pi t) dt \right] = \frac{\alpha}{2K^2} \left[\frac{\pi^3}{3} + \pi^2 \left(\frac{\pi}{2}\right) - \pi \left(\pi^2 - \frac{\pi^2}{4} \right) \right]$$
$$= \frac{\alpha}{2K^2} \left[\frac{\pi^3}{3} + \frac{\pi^3}{2} - \pi^3 + \frac{\pi^3}{4} \right] = \frac{\alpha \pi^3}{24 K^2}.$$

WE CONCLUDE THAT $I(u^K(x)) = \dfrac{\pi^3 \alpha}{24K^2} \longrightarrow 0$ AS $K \to \infty$ SO THAT THE ENERGY DECREASES AS THE FUNCTION BECOMES INCREASINGLY SAWTOOTHED BUT WITH A SMALLER AMPLITUDE. THE "MINIMIZER" IS NOT A DIFFERENTIABLE FUNCTION.

NOW IF WE CONSIDER THE TIME-DEPENDENT PDE FOR $u(x,t)$ GIVEN BY

$$u_{tt} = (u_x^3 - u_x + \beta u_{xt})_x - \alpha u, \quad \beta > 0, \ \alpha > 0$$
WITH $u(0,t) = u(\pi,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0.$ (AA)

$\beta > 0$ IS A VISCO-ELASTIC DAMPING PARAMETER.

THEN FOR STEADY STATES WE HAVE THAT u SATISFIES

$$(u_x^3 - u_x)_x = \alpha u \quad \text{ON} \quad 0 < x < \pi \quad \text{WITH} \quad u(0) = u(\pi) = 0.$$

THIS IS THE EL EQUATION FOR $I(u) = \int_0^\pi \left(\frac{1}{4}(u_x^2 - 1)^2 + \frac{\alpha u^2}{2} \right) dx$.

NOW IF WE DEFINE AN ENERGY $E(u) = \int_0^\pi \left(\frac{1}{2} u_t^2 + \frac{1}{4}(u_x^2 - 1)^2 + \frac{\alpha}{2} u^2 \right) dx$

THEN WE CLAIM THAT

<u>CLAIM</u>: $dE/dt = -\beta \int_0^\pi (u_{xt})^2 \, dx \leq 0$. THIS SUGGESTS THAT STARTING WITH SMOOTH INITIAL DATA $u(x,0) = f(x)$ WE HAVE THAT $E \to 0$ AS $t \to \infty$, WHICH SUGGESTS THAT u EVOLVES TO AN INCREASINGLY JAGGED SAWTOOTH FUNCTION AS $t \uparrow$.

<u>PROOF</u> WE CALCULATE:

$$dE/dt = \int_0^\pi \left(u_t u_{tt} + (u_x^2 - 1) u_x u_{xt} + \alpha u u_t \right) dx.$$

NOW SINCE $u_{tt} = (u_x^3 - u_x + \beta u_{xt})_x - \alpha u$ WE SUBSTITUTE FOR u_{tt} TO GET

$$dE/dt = \int_0^\pi (u_x^3 - u_x + \beta u_{xt})_x u_t + (u_x^2 - 1) u_x u_{xt}.$$

$$= \int_0^\pi \left[(u_x^3 - u_x)_x u_t + (u_x^3 - u_x) u_{xt} + \beta (u_{xxt} u_t) \right] dx.$$

NOW $\frac{d}{dx}\left[(u_x^3 - u_x) u_t \right] = (u_x^3 - u_x)_x u_t + (u_x^3 - u_x) u_{xt}$

AND $\frac{d}{dx}(u_{xt} u_t) = u_{xxt} u_t + (u_{xt})^2$.

THIS YIELDS $$dE/dt = \int_0^\pi \frac{d}{dx}\left[(u_x^3 - u_x) u_t \right] dx + \int_0^\pi \left(\beta \frac{d}{dx}(u_{xt} u_t) - \beta (u_{xt})^2 \right) dx$$

WHICH BECOMES

$$\frac{dE}{dt} = (u_x^3 - u_x) u_t \Big|_{x=0}^{\pi} + \beta u_{xt} u_t \Big|_{x=0}^{\pi} - \beta \int_0^\pi (u_{xt})^2 \, dx.$$

SINCE $u(0,t) = 0$ AND $u(\pi, t) = 0 \rightarrow u_t(0,t) = u_t(\pi, t) = 0 \quad \forall t$.

THIS YIELDS THAT $$dE/dt = -\beta \int_0^\pi (u_{xt})^2 \, dt \leq 0.$$