Be sure that this examination has 3 pages.

The University of British Columbia

Final Examinations - April 2015

Mathematics 401

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Closed book examination | Time: $2\frac{1}{2}$ hours

Special Instructions: A two-sided single page of notes is allowed.

Marks

[20] **1.** Consider the ODE boundary value problem for $u(x)$ on $0 \le x \le 1$:

$$u'' - xu' - u = f(x) \,, \qquad u(0) = 0 \,, \quad u'(1) = 0 \,.$$

(i) Determine the homogeneous adjoint problem, and show analytically that this adjoint problem has no nontrivial solution.

(ii) Show how to represent $u(x)$ in terms of a Green's function $G(\xi, x)$ as $u(x) = \int_0^1 G(\xi, x) f(\xi)\, d\xi$. Write the conditions that $G(\xi, x)$ must satisfy. (**Do not calculate G explicitly**).

(iii) Show that upon multiplying the ODE for u by some function $p(x)$ the resulting problem can be made self-adjoint. Calculate the appropriate function $p(x)$.

[20] **2.** The free-space Green's function for $\Delta u - u = \delta(\mathbf{x} - \mathbf{x}_0)$ in 3-D is $u = Ae^{-r}/r$ for some constant A, where $r = |\mathbf{x} - \mathbf{x_0}|$.

(i) Derive the value of the constant A.

(ii) Let $\mathbf{x} \equiv (x, y, z)$. Use the method of images to determine an integral representation for the solution to the following PDE on a half-space:

$$\Delta u - u = 0 \,, \qquad -\infty < x < \infty \,, \quad -\infty < y < \infty \,, \quad 0 \le z < \infty \,,$$
$$u_z(x, y, 0) = f(x, y) \,; \qquad u \to 0 \quad \text{as} \quad (x^2 + y^2 + z^2) \to \infty \,.$$

(iii) Find a leading order approximation for u that is valid for $x^2 + y^2 + z^2 \to +\infty$.

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[20] **3.** Let $u = u(x)$ and consider the functional

$$I(u) = \int_0^L F(x, u, u', u'')\, dx$$

over all four times continuously differentiable functions $u(x)$ with $u(0) = u(L) = 0$.

(i) Show that the Euler-Lagrange equation associated with $I(u)$ is

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right) = 0\,.$$

What are the natural boundary conditions for u at $x = 0, L$?

(ii) Suppose that

$$F(x, u, u', u'') = \frac{1}{2}\left[u''\right]^2 + \frac{1}{2}\left[u'\right]^2 - \frac{\sigma}{(1+u)}\,,$$

where σ is a positive constant, and that $u(0) = u(L) = 0$ is prescribed. Write the associated Euler-Lagrange equation and natural boundary conditions for u explicitly. (This problem models the deflection of a beam in a micro-electrical-mechanical system).

(iii) Next, consider the eigenvalue problem

$$\left(p(x)u''\right)'' = \lambda r(x) u\,, \quad 0 \le x \le L\,; \qquad u(0) = u(L) = u'(0) = u'(L) = 0\,,$$

with $p(x) > 0$ and $r(x) > 0$ on $0 \le x \le L$. Find a variational principle, together with a simple trial function, that can be used to give an upper bound on the first eigenvalue λ_1. (**Do not calculate this bound explicitly**).

[20] **4.** Consider the following diffusion equation for $u(x,t)$:

$$u_t = u_{xx} + f(x,t)\,, \qquad 0 \le x < \infty\,, \quad t > 0\,,$$
$$u_x(0,t) = h(t)\,; \qquad u(x,0) = 0\,,$$

where we assume that f is such that $u \to 0$ and $u_x \to 0$ as $x \to \infty$ for any fixed $t > 0$.

(i) Show how to represent $u(x,t)$ in terms of an appropriate Green's function by deriving the PDE problem for the Green's function.

(ii) Give an analytical expression for the Green's function that is needed in (i).

(iii) If $h(t) = 0$ for $t \ge 0$ and $f(x,t) = \delta\left(x - (x_0 + vt)\right)$ for some constants $x_0 > 0$ and $v > 0$, where $\delta(z)$ is the Dirac delta function, find $u(x,t)$ using your integral representation from (i). (This problem models temperature distribution due to a localized heat source that moves with constant speed $v > 0$ along the positive x-axis.)

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[20] **5.** Consider the following eigenvalue problem for $\phi(x, y)$ in the 2-D elliptical-shaped domain $\Omega = \{(x, y) | \; x^2 + y^2/9 \leq 1\}$:

$$\nabla \cdot [p \nabla \phi] + \lambda \phi = 0 \,, \qquad \phi = 0 \quad \text{on} \quad (x, y) \in \partial\Omega \,.$$

Here $p(x, y) = 1 + x^2 + y^2$.

(i) Derive a simple upper and a lower bound for the first eigenvalue λ_0 of this problem by bounding p and the domain Ω. In bounding the domain use appropriate circular domains. State carefully the bounding principle that you are using.

(ii) Can you get tighter bounds for λ_0 by bounding Ω with rectangular domains? (Hint: it is an easy Calculus exercise to find the largest rectangle that can be inscribed within Ω.)

(iii) Now consider the time-dependent problem for $u(x, y, t)$ in Ω, where we instead impose a non-flux condition on $\partial\Omega$. The problem is formulated as

$$u_t = \nabla \cdot [p \nabla u] \,, \qquad \partial_n u = 0 \quad \text{on} \quad (x, y) \in \partial\Omega \,; \qquad u(x, y, 0) = u_0(x, y) \,,$$

where $p(x, y) = 1 + x^2 + y^2$. Show that for $t \to +\infty$, the solution to this problem has the approximate form

$$u(x, y, t) \approx A_0 + A_1 e^{-\lambda_1 t} \phi_1(x, y) + \cdots \,,$$

for some constants A_0, A_1, $\lambda_1 > 0$, and $\phi_1(x, y)$. Calculate A_0 explicitly. Also, formulate a variational principle for λ_1 and provide a simple trial function that can be used to provide an upper bound for λ_1 (**do not calculate the bound explicitly**).

[100] **Total Marks**

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