

## Math 401: Final Exam, April 26th, 2021; (M. Ward)

**Instructions:** Open book and open online course notes from canvas or from the internet. However, no collaboration or discussion of the problems with others. No posting of questions related to this exam on Piazza or Chegg or with others.

**Time limit:** 150 minutes for exam + 15 minute grace period for canvas upload.

**Total Points: 80 Points Maximum**

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1. (15 points) Consider the following ODE boundary value problem for  $u(x)$  on  $0 \leq x \leq 1$ :

$$Lu \equiv u'' - xu' - u = f(x), \quad 0 \leq x \leq 1; \quad u'(0) = 1, \quad u'(1) = 0. \quad (1)$$

- (a) (6 points) Determine the **homogeneous** adjoint problem, and show analytically that this adjoint problem has no nontrivial solution. (Hint: It is helpful to note that  $(xv)' = xv' + v$  once you find the adjoint problem).
- (b) (4 points) Show how to represent the solution to (1) in terms of the Green's function  $G(\xi, x)$  for the adjoint operator  $L^*$ . Write **only** all the conditions that  $G(\xi, x)$  must satisfy. **Do not calculate  $G$  explicitly.**
- (c) (5 points) Now consider the same problem in (1) except where the boundary conditions have been modified to  $u'(0) = 1$  and  $u'(1) = u(1)$ . Show that the homogeneous adjoint problem now has a nontrivial solution and find a solvability condition on  $f(x)$  that is needed for a solution  $u$  to exist.
2. (25 points) Let  $\Omega = \{\mathbf{x} \equiv (x, y) \mid x^2/a^2 + y^2/b^2 \leq 1\}$  be an ellipse in 2-D, and consider the eigenvalue problem for the Laplacian, written for  $u(\mathbf{x})$  with  $\mathbf{x} \equiv (x, y)$  as

$$\Delta u + \lambda u = 0, \quad \mathbf{x} \in \Omega; \quad u = 0, \quad \mathbf{x} \in \partial\Omega. \quad (2)$$

We will assume without loss of generality that  $a \geq b > 0$ .

- (a) (3 points) Write a **variational principle** for the smallest positive eigenvalue, denoted by  $\lambda_1$ , for (2), which clearly identifies the trial function space  $\mathcal{H}_0$ .
- (b) (6 points) By using the specific trial function  $w = a^2b^2 - b^2x^2 - a^2y^2$ , calculate an explicit upper bound for  $\lambda_1$ . (Hint: in performing the integrations it is much easier to use the elliptical coordinates  $(r, \theta)$  defined by  $x = ar \cos \theta$  and  $y = br \sin \theta$ , but make sure to not forget including the Jacobian).
- (c) (5 points) Determine the best upper and lower bounds for  $\lambda_1$  that you can get with rectangular domains that are either inside or outside  $\Omega$ . From your expressions, is your upper bound for  $\lambda_1$  obtained in this way better than that obtained in part (b)?
- (d) (3 points) Next consider the same problem (2), but now with Neumann boundary conditions on  $\partial\Omega$ , written as

$$\Delta u + \sigma u = 0, \quad \mathbf{x} \in \Omega; \quad \partial_n u = 0, \quad \mathbf{x} \in \partial\Omega. \quad (3)$$

Show that the smallest eigenvalue, denoted by  $\sigma_1$ , for this problem is  $\sigma_1 = 0$ .

- (e) (3 points) Write a **variational principle** for the second smallest eigenvalue  $\sigma_2$  of (3), which clearly identifies the required trial space  $\mathcal{H}$ .
- (f) (5 points) Next show that  $w = x$  is an admissible trial function for part (e). Then, use this trial function to obtain an upper bound for  $\sigma_2$ . (Hint: elliptical coordinates  $(r, \theta)$  are again convenient for calculating the integrals).

3. (12 points) Let  $u = u(x)$  and consider the 1-D functional

$$I(u) = \int_0^1 F(x, u, u', u'') dx \quad (4)$$

over all four times continuously differentiable functions  $u(x)$  with  $u(0) = u(1) = 0$ .

- (a) (5 points) Derive from first principles that the Euler-Lagrange equation associated with  $I(u)$  is

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \left( \frac{\partial F}{\partial u'} \right) + \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial u''} \right) = 0. \quad (5)$$

What are the natural boundary conditions for  $u$  at  $x = 0, 1$ ?

- (b) (3 points) Suppose that

$$F(x, u, u', u'') = \frac{1}{2} (u'')^2 + \frac{1}{2} (u')^2 - p(x)u, \quad (6)$$

where  $p(x) > 0$  on  $0 \leq x \leq 1$ . Using the result in part (a) write the associated Euler-Lagrange equation and **natural boundary conditions** for  $u$  explicitly.

- (c) (4 points) Let  $u^*(x)$  denote the solution to the Euler-Lagrange equation for the functional  $F$  in part (b). By calculating the difference  $I(u^* + v) - I(u^*)$  prove that  $u^*$  is a **global minimizer** for  $I(u)$  amongst all functions  $v(x)$  with  $v(0) = v(1) = 0$  that are sufficiently smooth.

4. (12 points) With an absorbing boundary at  $x = 0$ , the probability density function  $p(x, t)$  for a Brownian particle on the half-line  $0 < x < \infty$  with constant diffusivity  $D > 0$ , and which starts at some point  $x = L > 0$  at time  $t = 0$ , satisfies the diffusion equation

$$\begin{aligned} p_t &= Dp_{xx}, & 0 < x < \infty, & \quad t > 0; & \quad p(x, 0) &= \delta(x - L), \\ p(0, t) &= 0, & p(x, t) &\rightarrow 0 & \text{as } x &\rightarrow \infty. \end{aligned} \quad (7)$$

Here  $\delta(x - L)$  is the usual Dirac-Delta function.

- (a) (6 points) Determine the explicit solution for  $p(x, t)$  by the method of images.  
 (b) (3 points) From your formula in part (a) calculate  $T(t)$  defined by  $T(t) \equiv Dp_x(0, t)$ , and make a very **quick** sketch of a plot of  $T(t)$  versus  $t$  for  $t \geq 0$  (for any fixed  $L$ ).  
 (c) (3 points) Show from integrating the PDE in (7) that we must have  $\int_0^\infty T(t) dt = 1$ .

(Interpretation:  $T(t)$  is the probability that the Brownian particle starting from  $x = L$  at  $t = 0$  first reaches the absorbing boundary at  $x = 0$  between times  $t$  and  $t + dt$ . The fact that  $\int_0^\infty T(t) dt = 1$  means that in 1-D the Brownian walker must always eventually become absorbed by the boundary at  $x = 0$ , no matter how large  $L$  is).

5. (16 points) Quick response questions: (**Hint: Only a few lines of calculation or clear reasoning are needed for each of these problems**).

- (a) (4 points) Let  $\Omega$  be the unit disk in 2-D centered at the origin. Find the value of the constant  $S$  for which the following problem has a solution:

$$\begin{aligned} \Delta u &= 1, \quad \mathbf{x} \in \Omega \setminus \{\mathbf{0}\}; & \partial_n u &= 1, \quad \mathbf{x} \in \partial\Omega, \\ u &\sim S \log |\mathbf{x}| \quad \text{as } \mathbf{x} \rightarrow \mathbf{0}. \end{aligned} \tag{8}$$

Here  $\partial_n u$  denotes the outward normal derivative on  $\partial\Omega$ . (Note the non-vanishing flux on  $\partial\Omega$ ).

- (b) (4 points) Let  $\Omega = \{\mathbf{x} \equiv (x, y, z) \mid x^2 + y^2 + z^2 \leq 1, \text{ with } z \geq 0\}$  be a hemisphere (or half-sphere) of radius one in 3-D centered at the origin. Let  $G(\mathbf{x}; \xi)$  satisfy

$$\begin{aligned} \Delta G &= \delta(\mathbf{x} - \xi), \quad \mathbf{x} \in \Omega, \\ G &= 0 \text{ on } x^2 + y^2 + z^2 = 1, \text{ for } z > 0; & G_z &= 0 \text{ on } z = 0 \text{ for } x^2 + y^2 \leq 1. \end{aligned} \tag{9}$$

Here  $\xi$  is any point strictly inside the hemisphere. By using the method of images, find an explicit formula for  $G$ . (Note: the boundary condition for  $G$  differs on different parts of the boundary).

- (c) (4 points) Let  $\Omega$  be the unit sphere in 3-D centered at the origin, and consider the following eigenvalue problem for  $\lambda$ :

$$\Delta u + \lambda e^{-|\mathbf{x}|^2} u = 0, \quad \mathbf{x} \in \Omega; \quad u = 0, \quad \mathbf{x} \in \partial\Omega, \tag{10}$$

where  $\mathbf{x} \equiv (x, y, z)$  and  $|\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2}$ . Prove that for the  $n^{\text{th}}$  largest eigenvalue  $\lambda_n$  of this problem we have  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

- (d) (4 points) Let  $\lambda_1$  be the **smallest** positive eigenvalue of (10) in part (c). Suppose that only on the upper half of the boundary of the sphere we replace the condition  $u = 0$  with the no-flux condition  $\partial_n u = 0$ . Does the corresponding lowest eigenvalue for this modified problem decrease or increase in comparison with  $\lambda_1$ ? Clearly explain your reasoning.