Math 401: Midterm March 11th, 2021; 40 Points; (M. Ward)

Instructions: Open book and open online course notes from the internet. No collaboration or discussion of the problems with others. No posting of questions related to this quiz on Piazza or Chegg. Time limit: 1 hr for exam + 15 minute grace period for canvas upload.

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1. (14 points) Consider the following differential equation for $u(x)$:

$$Lu \equiv u'' + u = f(x)\,, \qquad 0 < x < \pi/2\,; \qquad u(0) = 0\,, \;\; u(\pi/2) = 0\,. \tag{1}$$

   (a) (3 points) Write down the problem that the Green's function for the operator L should satisfy.

   (b) (6 points) Find this Green's function explicitly and write the solution $u(x)$ in terms of it in as explicit a form as you can.

   (c) (5 points) Find the solvability condition on $f(x)$ that is needed for a solution $u(x)$ to exist when the boundary conditions for u are changed to $u(\pi/2) = 0$ and $u(0) = -u'(\pi/2)$.

2. (10 points) Suppose in a 3-D half-space that $u(x, y, z)$ satisfies

$$\begin{aligned} &u_{xx} + u_{yy} + u_{zz} - u = 0\,, \qquad -\infty < x < \infty\,, \quad -\infty < y < \infty\,, \quad z > 0\,, \\ &u_z(x, y, 0) = f(x, y)\,; \qquad u \to 0 \;\; \text{sufficiently fast as} \;\; |\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2} \to +\infty\,. \end{aligned} \tag{9}$$

   (a) (4 points) Find the Green's function relevant to this problem by using the method of images.

   (b) (4 points) Find an explicit integral representation for u in terms of this Green's function.

   (c) (2 points) Next, suppose that $f(x, y)$ is identically zero on the range $x^2 + y^2 \geq R^2$ for some $R > 0$. For $|\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2} \to \infty$, find an approximation for u in the form $u \sim Ae^{-|\mathbf{x}|}/|\mathbf{x}|$ as $|\mathbf{x}| \to \infty$ for some A to be found.

3. (12 points) Suppose in an infinite 3-D strip-like domain of width $H > 0$, and with no flux boundary conditions on $z = 0, H$, that $u(x, y, z)$ satisfies

$$\begin{aligned} &u_{xx} + u_{yy} + u_{zz} - u = \delta(x)\delta(y)\delta(z - H/2)\,, \quad -\infty < x < \infty\,, \quad -\infty < y < \infty\,, \quad 0 < z < H\,, \\ &u_z(x, y, 0) = 0\,, \quad u_z(x, y, H) = 0\,; \qquad u \;\; \text{bounded as} \;\; |\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2} \to +\infty\,. \end{aligned} \tag{15}$$

   (a) (3 points) Determine the set of 1-D eigenfunctions $\Phi_n(z)$ for $n \geq 0$ in the z direction that are appropriate for this problem.

   (b) (4 points) By expanding $u(x, y, z) = \sum_{n=0}^{\infty} c_n(x, y)\Phi_n(z)$, derive a 2-D PDE for the coefficients $c_n(x, y)$.

   (c) (3 points) Determine explicitly the solution to the 2-D PDE for $c_n(x, y)$ for each $n \geq 0$.

   (d) (2 points) Calculate the depth average $\bar{u}(x, y)$, defined by $\bar{u} \equiv H^{-1} \int_0^H u(x, y, z)\, dz$.

4. (4 points) Quick response questions:

   (a) (2 points) Let Ω be a bounded 3-D domain with volume $V > 0$ and smooth boundary $\partial\Omega$ that contains the origin $\mathbf{x} = \mathbf{0}$. Find the value of the constant M for which the following problem has a solution:

$$\begin{aligned} &\Delta u = M\,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{0}\}\,; \qquad \partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \\ &u \sim \frac{2}{|\mathbf{x}|}\,, \quad \text{as} \quad \mathbf{x} \to \mathbf{0}\,. \end{aligned} \tag{19}$$

   (b) (2 points) Find the solvability condition needed for $f(x)$ so that the following problem with periodic boundary conditions has a solution $u(x)$:

$$u'' + u = f(x)\,, \qquad 0 < x < 2\pi\,; \qquad u(0) = u(2\pi)\,, \;\; u'(0) = u'(2\pi)\,. \tag{20}$$