

Math 401: Midterm March 8th-10th, 2022; 45 Points; (M. Ward)
Instructions: Open book and open online course notes from the internet. No collaboration or discussion of the problems with others. No posting of questions related to this quiz on Piazza or Chegg.

1. (15 points) Consider the following differential equation for $u(x)$ where $\lambda \geq 0$ and $L > 0$ are constants.

$$Lu \equiv u'' + \lambda u = f(x), \quad 0 < x < L; \quad u'(0) = 1, \quad u'(L) = 0. \quad (1)$$

(Careful: Notice the inhomogeneous boundary condition at $x = 0$.)

- (a) (6 points) Find the Green's function relevant for this problem and give a representation for the solution $u(x)$ in terms of $f(x)$. From your formula, for what values of λ does a Green's function not exist?
- (b) (5 points) For what values of λ is a condition on $f(x)$ required for there to be a solution? For these values of λ , find this solvability condition.
- (c) (4 points) Calculate the modified or generalized Green's function when $\lambda = 0$ and write the solution u for this case under the assumption that $\int_0^L f(x) dx = -1$.
2. (12 points) Suppose in a 3-D half-space that $u(x, y, z)$ satisfies

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= 0, & -\infty < x < \infty, & \quad -\infty < y < \infty, & \quad z > 0, \\ u_z(x, y, 0) &= f(x, y); & u &\rightarrow 0 \text{ sufficiently fast as } |\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2} \rightarrow +\infty. \end{aligned} \quad (11)$$

- (a) (4 points) Find the Green's function relevant to this problem by using the method of images.
- (b) (4 points) Find an explicit integral representation for u in terms of this Green's function.
- (c) (4 points) Next, suppose that $f(x, y)$ is identically zero on the range $x^2 + y^2 \geq R^2$ for some $R > 0$. For $|\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2} \rightarrow \infty$, find an approximation for u in the form

$$u \sim \frac{C}{|\mathbf{x}|} + \frac{\mathbf{p} \cdot \mathbf{x}}{|\mathbf{x}|^3} + \dots, \quad \text{as } |\mathbf{x}| \rightarrow \infty, \quad (12)$$

where the scalar C and the vector \mathbf{p} are to be found. Here \cdot is the usual dot product.

3. (12 points) Let $R > 0$ and $H > 0$ and suppose that in a finite cylinder $u(x, y, z)$ satisfies

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= 0, & \text{in } 0 \leq (x^2 + y^2)^{1/2} \leq R, & \quad 0 \leq z \leq H, \\ u_z(x, y, 0) &= -\delta(x - x_0)\delta(y - y_0), & u_z(x, y, H) &= 0, \\ u &= 0 & \text{on } (x^2 + y^2)^{1/2} = R, & \quad u \text{ bounded as } (x^2 + y^2)^{1/2} \rightarrow 0. \end{aligned} \quad (20)$$

Assume the location of the Dirac on the wall $z = 0$ satisfies $(x_0^2 + y_0^2)^{1/2} < R$.

- (a) (2 points) Give a sketch of the geometry for this problem. Give a two-sentence description of what this problem might model physically.
- (b) (5 points) Find an eigenfunction representation for u in terms of the eigenfunctions in the z direction. Determine, but not solve, the PDE problems for the coefficients in the eigenfunction expansion.
- (c) (5 points) Calculate $\bar{u}(x, y)$ explicitly, where we have defined $\bar{u} \equiv (1/H) \int_0^H u(x, y, z) dz$. Here \bar{u} is the spatial average of u over the length of the cylinder.

4. (6 points) Quick response questions:

- (a) (2 points) Let Ω be a sphere in 3-D centered at the origin $\mathbf{x} = \mathbf{0}$ of radius $r = 3$. Find the value of the constant M for which the following problem has a solution:

$$\begin{aligned} \Delta u &= M, & \mathbf{x} &\in \Omega \setminus \{\mathbf{0}\}; & \partial_n u &= 0, & \mathbf{x} &\in \partial\Omega, \\ u &\sim \frac{1}{|\mathbf{x}|}, & \text{as } \mathbf{x} &\rightarrow \mathbf{0}. \end{aligned} \tag{26}$$

- (b) (2 points) Suppose that $L > 0$. At what values of λ is a solvability condition needed for $f(x)$ so that the following problem with periodic boundary conditions has a solution $u(x)$?

$$u'' + \lambda u = f(x), \quad 0 < x < L; \quad u(0) = u(L), \quad u'(0) = u'(L). \tag{27}$$

At these values of λ determine the solvability condition(s) explicitly.

- (c) (2 points) Let Ω be a 2-D disk of radius $a > 0$ centered at the origin $\mathbf{x} = \mathbf{0}$. Explain why a simple application of the method of images cannot be used for finding the Green's function for the problem $\Delta G - G = \delta(\mathbf{x} - \mathbf{x}_0)$ in Ω with $G = 0$ on the boundary $\partial\Omega$ and $\mathbf{x}_0 \in \Omega$.