Math 401: Midterm March 11th, 2021; 40 Points; (M. Ward)

Instructions: **Open book and open online course notes from the internet. No collaboration or discussion of the problems with others. No posting of questions related to this quiz on Piazza or Chegg. Time limit: 1 hr for exam + 15 minute grace period for canvas upload.**

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1. (14 points) Consider the following differential equation for $u(x)$:

$$Lu \equiv u'' + u = f(x), \qquad 0 < x < \pi/2; \qquad u(0) = 0, \;\; u(\pi/2) = 0. \tag{1}$$

   (a) (3 points) Write down the problem that the Green's function for the operator L should satisfy.

   (b) (6 points) Find this Green's function explicitly and write the solution $u(x)$ in terms of it in as explicit a form as you can.

   (c) (5 points) Find the solvability condition on $f(x)$ that is needed for a solution $u(x)$ to exist when the boundary conditions for u are changed to $u(\pi/2) = 0$ and $u(0) = -u'(\pi/2)$.

Solution:

(a) Since L is formally self-adjoint with separated boundary conditions, the problem (1) is essentially self-adjoint. Therefore, the Green's function $v = v(t;x)$ satisfies

$$Lv \equiv v'' + v = \delta(t-x), \qquad 0 < t < \frac{\pi}{2}; \qquad v(0) = 0, \;\; v(\pi/2) = 0. \tag{2}$$

From Lagrange's identity the solution to (1) is

$$u(x) = \int_0^{\pi/2} v(t;x) f(t)\, dt. \tag{3}$$

(b) We now solve (2). The solutions to $Lv_H = 0$ are $v_H = \text{span}\{\sin(t-\alpha), \cos(t-\alpha)\}$ for any α. Therefore, we have upon imposing continuity across $t = x$ that

$$v = \begin{cases} A\sin(t)\sin(x-\pi/2), & \text{if } 0 \le t \le x \\ A\sin(t-\pi/2)\sin(x), & \text{if } x \le t \le \pi/2 \end{cases},$$

where A is to be found. By imposing the jump condition $v'(x^+) - v'(x^-) = 1$, we calculate using $\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$ that

$$A\left[\cos(x-\pi/2)\sin(x) - \cos(x)\sin(x-\pi/2)\right] = A\sin(x-(x-\pi/2)) = 1,$$

so that $A = 1/\left[\sin(\pi/2)\right] = 1$. We conclude that

$$v = \begin{cases} \sin(t)\sin(x-\pi/2), & \text{if } 0 \le t \le x \\ \sin(t-\pi/2)\sin(x), & \text{if } x \le t \le \pi/2 \end{cases}. \tag{4}$$

Finally, from (3) we get

$$u(x) = \sin\left(x - \frac{\pi}{2}\right)\int_0^x f(t)\sin(t)\, dt + \sin(x)\int_x^{\pi/2} f(t)\sin\left(t - \frac{\pi}{2}\right) dt. \tag{5}$$

(c) With the new boundary conditions, we need to find the adjoint problem. To determine the adjoint boundary conditions, we integrate by parts to get

$$\int_0^{\pi/2} v(u'' + u)\,dx = (vu' - uv')\,|_0^{\pi/2} + \int_0^{\pi/2} u(v'' + v)\,dx\,.$$

Putting in the boundary conditions $u(\pi/2) = 0$ and $u(0) + u'(\pi/2) = 0$, we get

$$\int_0^{\pi/2} v(u'' + u)\,dx = u(0)\left[v'(0) - v(\pi/2)\right] - v(0)u'(0) + \int_0^{\pi/2} u(v'' + v)\,dx\,, \qquad (6)$$

which provides the adjoint boundary conditions $v(0) = 0$ and $v'(0) = v(\pi/2)$. Therefore, the homogenous adjoint problem is

$$Lv \equiv v'' + v = 0\,, \qquad 0 < x < \pi/2\,; \qquad v(0) = 0\,,\ \ v'(0) = v(\pi/2)\ . \qquad (7)$$

The solution is readily found, up to a scalar multiple, as $v = \sin(x)$. As such, by using (6), the solvability condition on $f(x)$ is that

$$\int_0^{\pi/2} f(x)\sin(x)\,dx = 0\,. \qquad (8)$$

2. (10 points) Suppose in a 3-D half-space that $u(x, y, z)$ satisfies

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} - u = 0\,, \qquad -\infty < x < \infty\,, \quad -\infty < y < \infty\,, \quad z > 0\,,\\ u_z(x, y, 0) = f(x, y)\,; \qquad u \to 0 \quad \text{sufficiently fast as} \quad |\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2} \to +\infty\,. \end{aligned} \qquad (9)$$

(a) (4 points) Find the Green's function relevant to this problem by using the method of images.

(b) (4 points) Find an explicit integral representation for u in terms of this Green's function.

(c) (2 points) Next, suppose that $f(x, y)$ is identically zero on the range $x^2 + y^2 \geq R^2$ for some $R > 0$. For $|\mathbf{x}| = (x^2+y^2+z^2)^{1/2} \to \infty$, find an approximation for u in the form $u \sim Ae^{-|\mathbf{x}|}/|\mathbf{x}|$ as $|\mathbf{x}| \to \infty$ for some A to be found.

Solution:

(a) The Green's function problem will be to find $G(\mathbf{x}'; \mathbf{x})$ that satisfies

$$\Delta' G - G = \delta(\mathbf{x}' - \mathbf{x})\,, \quad \mathbf{x}' \in \mathbb{R}^{3+}\,; \qquad G_{z'} = 0\,, \qquad \text{on } z' = 0\,. \qquad (10)$$

Here $\mathbb{R}^{3+} \equiv \{\mathbf{x}' \mid z' \geq 0\}$, with $\mathbf{x}' = (x', y', z')$, and $\mathbf{x} = (x, y, z)$ with $z > 0$. The free-space Green's function, where the delta singularity is fixed at $\mathbf{x}' = \mathbf{x}$, is simply $G_f = -(4\pi|\mathbf{x}' - \mathbf{x}|)^{-1}e^{-|\mathbf{x}'-\mathbf{x}|}$. Since we must impose that $G_{z'} = 0$ on the plane $z' = 0$, to find the image function we must extend G to the lower half-space to be even in z'. In this way, we get

$$G = -\frac{1}{4\pi|\mathbf{x}' - \mathbf{x}|}e^{-|\mathbf{x}'-\mathbf{x}|} - \frac{1}{4\pi|\mathbf{x}' - \mathbf{x}_I|}e^{-|\mathbf{x}'-\mathbf{x}_I|}\,, \qquad (11)$$

where $\mathbf{x} \equiv (x, y, z)$ and $\mathbf{x}_I \equiv (x, y, -z)$.

(b) We use Green's second identity over the upper hemisphere Γ_R, defined by $|\mathbf{x}'| = R$ and $z' \geq 0$, to obtain upon taking the $R \to \infty$ limit that

$$\begin{aligned} \int_{\mathbb{R}^{3+}} \left[G\left(\Delta' u - u\right) - u\left(\Delta G' - G\right)\right]\,d\mathbf{x}' &= \lim_{R\to\infty} \int_{\Gamma_R} \left(G\partial_n u - u\partial_n G\right)\,dS\\ &= \quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(G\partial_n u - u\partial_n G\right)\,dx'dy'\,. \end{aligned}$$

Upon using the facts that $u \to 0$ and $G \to 0$ sufficiently fast on the boundary of the hemi-sphere, and noting that $\partial_n u = -u_{z'}$ on the infinite plane $z' = 0$, we obtain that

$$u(\mathbf{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') G|_{z'=0} \, dx' dy' \,, \tag{12}$$

where G, to be evaluated on the plane $z' = 0$, is given from (11) by

$$G|_{z'=0} = -\frac{1}{2\pi} \left[(x'-x)^2 + (y'-y)^2 + z^2 \right]^{-1/2} e^{-\left[(x'-x)^2+(y'-y)^2+z^2\right]^{1/2}} \,. \tag{13}$$

(c) Since $f(x', y') \equiv 0$ outside of a disk of radius R, we can let $|\mathbf{x}| \to \infty$ in (12) and (13) to obtain

$$u \sim \frac{A}{|\mathbf{x}|} e^{-|\mathbf{x}|} \,, \quad \text{as} \quad |\mathbf{x}| \to \infty \,, \quad \text{where} \quad A \equiv -\frac{1}{2\pi} \int_{\Omega_f} f(x', y') \, dx' dy' \,. \tag{14}$$

Here $\Omega_f \equiv \{(x', y') \,|\, (x')^2 + (y')^2 \leq R\}$ and $|\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2}$.

3. (12 points) Suppose in an infinite 3-D strip-like domain of width $H > 0$, and with no flux boundary conditions on $z = 0, H$, that $u(x, y, z)$ satisfies

$$\begin{aligned} &u_{xx} + u_{yy} + u_{zz} - u = \delta(x)\delta(y)\delta\left(z - H/2\right) \,, \quad -\infty < x < \infty \,, \quad -\infty < y < \infty \,, \quad 0 < z < H \,, \\ &u_z(x, y, 0) = 0 \,, \quad u_z(x, y, H) = 0 \,; \qquad u \text{ bounded as } |\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2} \to +\infty \,. \end{aligned} \tag{15}$$

(a) (3 points) Determine the set of 1-D eigenfunctions $\Phi_n(z)$ for $n \geq 0$ in the z direction that are appropriate for this problem.

(b) (4 points) By expanding $u(x, y, z) = \sum_{n=0}^{\infty} c_n(x, y) \Phi_n(z)$, derive a 2-D PDE for the coefficients $c_n(x, y)$.

(c) (3 points) Determine explicitly the solution to the 2-D PDE for $c_n(x, y)$ for each $n \geq 0$.

(d) (2 points) Calculate the depth average $\bar{u}(x, y)$, defined by $\bar{u} \equiv H^{-1} \int_0^H u(x, y, z) \, dz$.

Solution:

(a) The eigenfunction problem in the bounded z direction for $\Phi(z)$ is

$$\Phi'' + \lambda \Phi = 0 \,, \quad 0 < z < H \,; \qquad \Phi'(0) = \Phi'(H) = 0 \,.$$

We calculate $\Phi_0 = 1$, $\Phi_n = \cos\left(n\pi z / H\right)$ for $n \geq 1$ and $\lambda = n^2 \pi^2 / H^2$ for $n \geq 0$.

(b) We expand

$$u = c_0(x, y) + \sum_{n=1}^{\infty} c_n(x, y) \cos\left(\frac{n\pi z}{H}\right) \,,$$

where by orthogonality we identify that

$$c_0 = \frac{1}{H} \int_0^H u \, dz \,; \qquad c_n = \frac{2}{H} \int_0^H u \cos\left(\frac{n\pi z}{H}\right) dz \,, \qquad n = 1, 2, \ldots \,.$$

Upon multiplying the PDE by H^{-1} and integrating over $0 < z < H$, we obtain that c_0 satisfies

$$c_{0xx} + c_{0yy} - c_0 = \frac{1}{H} \delta(x) \delta(y) \,. \tag{16}$$

Similarly, by multiplying the PDE by $2H^{-1}\cos(n\pi z/H)$ and integrating over $0 < z < H$, we obtain upon integration by parts that

$$c_{nxx} + c_{nyy} - \left(\frac{n^2\pi^2}{H^2} + 1\right) c_n = \frac{2}{H}\cos\left(\frac{n\pi}{2}\right)\delta(x)\delta(y)\,. \tag{17}$$

(c) Recalling that $v_{xx} + v_{yy} - \kappa^2 v = \delta(x)\delta(y)$ has the free-space solution $v = -(2\pi)^{-1}K_0(\kappa|\mathbf{x}|)$, we identify with $|\mathbf{x}| \equiv (x^2+y^2)^{1/2}$ that

$$c_0 = -\frac{1}{2\pi H}K_0(|\mathbf{x}|)\,; \quad c_n = -\frac{1}{\pi H}\cos\left(\frac{n\pi}{2}\right)K_0(\beta_n|\mathbf{x}|)\,, \quad \text{where} \quad \beta_n = \sqrt{\frac{n^2\pi^2}{H^2} + 1}\,, \quad n \geq 1\,. \tag{18}$$

(d) We simply identify that $\bar{u} = c_0(x, y)$, where $c_0(x, y)$ is given in (18).

4. (4 points) Quick response questions:

(a) (2 points) Let Ω be a bounded 3-D domain with volume $V > 0$ and smooth boundary $\partial\Omega$ that contains the origin $\mathbf{x} = \mathbf{0}$. Find the value of the constant M for which the following problem has a solution:

$$\begin{aligned} &\Delta u = M\,, \quad \mathbf{x} \in \Omega\backslash\{\mathbf{0}\}\,; \qquad \partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \\ &u \sim \frac{2}{|\mathbf{x}|}\,, \quad \text{as} \quad \mathbf{x} \to \mathbf{0}\,. \end{aligned} \tag{19}$$

(b) (2 points) Find the solvability condition needed for $f(x)$ so that the following problem with periodic boundary conditions has a solution $u(x)$:

$$u'' + u = f(x)\,, \qquad 0 < x < 2\pi\,; \qquad u(0) = u(2\pi)\,, \; u'(0) = u'(2\pi)\,. \tag{20}$$

Solution:

(a) We recall the correspondence $\delta(\mathbf{x}) \longrightarrow -1/(4\pi|\mathbf{x}|)$ between a 3-D Delta singularity for Laplace's equation and the singular behavior of the free-space Green's function. Therefore, by scaling, we have the correspondence $-8\pi\delta(\mathbf{x}) \longrightarrow 2/|\mathbf{x}|$, so that we can write

$$\Delta u = M - 8\pi\delta(\mathbf{x})\,, \quad \mathbf{x} \in \Omega\,; \qquad \partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega\,.$$

Since $\mathbf{0} \in \Omega$ by assumption, we use the Divergence theorem and the no flux boundary conditions to get $MV - 8\pi = 0$, which yields $M = 8\pi/V$.

(b) With periodic boundary conditions, we can use Lagrange's identity to derive that we must have

$$\int_0^{2\pi} v f(x)\, dx = 0\,,$$

where $v(x)$ is any solution to $v'' + v = 0$ on $0 < x < 2\pi$, with periodic boundary conditions $v(0) = v(2\pi)$ and $v'(0) = v'(2\pi)$. There are two such independent solutions: $v = \sin(x)$ and $v = \cos(x)$. As such, the solvability condition is that we must have

$$\int_0^{2\pi} f(x)\sin(x)\, dx = 0\,, \qquad \text{and} \qquad \int_0^{2\pi} f(x)\cos(x)\, dx = 0\,. \tag{21}$$