Math 401: Midterm March 8th-10th, 2022; 45 Points; (M. Ward)

Instructions: Open book and open online course notes from the internet. No collaboration or discussion of the problems with others. No posting of questions related to this quiz on Piazza or Chegg.

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1. (15 points) Consider the following differential equation for $u(x)$ where $\lambda \geq 0$ and $L > 0$ are constants.

$$Lu \equiv u'' + \lambda u = f(x)\,, \qquad 0 < x < L\,; \qquad u'(0) = 1\,,\;\; u'(L) = 0\,. \tag{1}$$

(Careful: Notice the inhomogeneous boundary condition at $x = 0$.)

(a) (6 points) Find the Green's function relevant for this problem and give a representation for the solution $u(x)$ in terms of $f(x)$. From your formula, for what values of λ does a Green's function not exist?

(b) (5 points) For what values of λ is a condition on $f(x)$ required for there to be a solution? For these values of λ, find this solvability condition.

(c) (4 points) Calculate the modified or generalized Green's function when $\lambda = 0$ and write the solution u for this case under the assumption that $\int_0^L f(x)\,dx = -1$.

Solution:

(a) Since L is formally self-adjoint with separated boundary conditions, the problem (1) is essentially self-adjoint. Therefore, the Green's function $v = v(t;x)$ satisfies

$$Lv \equiv v'' + v = \delta(t-x)\,, \qquad 0 < t < L,; \qquad v'(0) = 0\,,\;\; v'(L) = 0\,. \tag{2}$$

From Lagrange's identity the solution to (1) is

$$u(x) = v(0;x) + \int_0^L v(t;x) f(t)\,dt\,. \tag{3}$$

We now solve (2). The solutions to $Lv_H = 0$ are $v_H = \text{span}\{\sin(t-\alpha), \cos(t-\alpha)\}$ for any α. Therefore, we have upon imposing continuity across $t = x$ that

$$v = \begin{cases} A\cos\left(\sqrt{\lambda}t\right)\cos\left(\sqrt{\lambda}(x-L)\right)\,, & \text{if } 0 \leq t \leq x \\ A\cos\left(\sqrt{\lambda}(t-L)\right)\cos\left(\sqrt{\lambda}x\right)\,, & \text{if } x \leq t \leq L \end{cases}\,, \tag{4}$$

where A is to be found. By imposing the jump condition $v'(x^+) - v'(x^-) = 1$, we calculate using $\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$ that

$$A\sqrt{\lambda}\left[\sin\left(\sqrt{\lambda}x\right)\cos\left(\sqrt{\lambda}(x-L)\right) - \cos\left(\sqrt{\lambda}x\right)\sin\left(\sqrt{\lambda}(x-L)\right)\right] = 1\,,$$

$$A\sqrt{\lambda}\sin\left[\sqrt{\lambda}x - \sqrt{\lambda}(x-L)\right] = 1\,.$$

so that $A = 1/\left[\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right)\right]$. The Green's function does not exist when A is undefined. This occurs when

$$\lambda = 0\,, \quad \text{or} \quad \lambda = \frac{n^2\pi^2}{L^2}\,, \quad n = 1, 2, 3, \ldots \tag{5}$$

For λ not one of the values in (5), we use the formula for A in (4) and (3) to obtain

$$u(x) = \frac{\cos\left(\sqrt{\lambda}(x-L)\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right)}\int_0^L f(t)\cos(\sqrt{\lambda}t)\,dt + \frac{\cos\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}\sin\left(\sqrt{\lambda}L\right)}\int_0^L f(t)\cos\left(\sqrt{\lambda}(t-L)\right)\,dt$$

$$+\frac{\cos\left(\sqrt{\lambda}(x-L)\right)}{\sqrt{\lambda}\sin(\sqrt{\lambda}L)}\,.$$

(b) The homogeneous problem

$$L\Phi \equiv \Phi'' + \lambda\Phi = 0\,, \quad 0 < x < L\,; \quad \Phi'(0) = 0\,, \quad \Phi'(L) = 0\,,$$

has a nontrivial solution if and only if λ is one of the values in (5). We calculate explicitly

$$\Phi = 1\,, \quad \text{if} \quad \lambda = 0\,, \tag{6}$$

$$\Phi = \cos\left(\frac{n\pi x}{L}\right)\,, \quad \text{if} \quad \lambda = \frac{n^2\pi^2}{L^2}\,, \quad n = 1,2,3\,\ldots\,. \tag{7}$$

At one of these values of λ, we use Green's second identity with $Lu = f$, $L\Phi = 0$, and noting that $u'(0) = 1$, to get

$$\int_0^L \Phi L u\,dx - \int_0^L uL\Phi\,dx = (\Phi u' - \Phi' u)\,|_{x=0}^{L} = -\Phi(0)\,.$$

This yields the solvability condition

$$\int_0^L \Phi(x) f(x)\,dx = -\Phi(0)\,, \tag{8}$$

whenever λ is one of the values in (5). The corresponding Φ is given in (6) and (7).

(c) For $\lambda = 0$, we have $\Phi = 1$, and so we assume that the solvability condition $\int_0^L f(x)\,dx = -1$, as obtained from (8), holds. The modified or generalized Green's function $\mathcal{G}(t;x)$ satisfies

$$L\mathcal{G} \equiv \mathcal{G}'' = \delta(t-x) - \frac{1}{L}\,, \qquad 0 < t < L,; \qquad \mathcal{G}'(0) = 0\,,\ \mathcal{G}'(L) = 0\,, \tag{9}$$

where primes indicate derivatives in t. A simple calculation gives

$$\mathcal{G} = -\frac{t^2}{2L} + \begin{cases} c + x\,, & \text{if } 0 \le t \le x \\ c + t\,, & \text{if } x \le t \le L \end{cases}\,, \tag{10}$$

where c is an arbitrary constant. Lagrange's identity, with $u'' = f(t)$, given by

$$\int_0^L u''\mathcal{G}\,dt = \int_0^L u\left[\delta(t-x) - \frac{1}{L}\right] - u'(0)\mathcal{G}(0;x)$$

then yields that

$$u(x) = x + x\int_0^x f(t)\,dt + \int_x^L tf(t)\,dt + \kappa\,,$$

where κ is an arbitrary constant.

2. (12 points) Suppose in a 3-D half-space that $u(x,y,z)$ satisfies

$$\begin{aligned} &u_{xx} + u_{yy} + u_{zz} = 0\,, \qquad -\infty < x < \infty\,, \quad -\infty < y < \infty\,, \quad z > 0\,, \\ &u_z(x,y,0) = f(x,y)\,; \qquad u \to 0 \quad \text{sufficiently fast as} \quad |\mathbf{x}| = (x^2+y^2+z^2)^{1/2} \to +\infty\,. \end{aligned} \tag{11}$$

(a) (4 points) Find the Green's function relevant to this problem by using the method of images.

(b) (4 points) Find an explicit integral representation for u in terms of this Green's function.

(c) (4 points) Next, suppose that $f(x,y)$ is identically zero on the range $x^2+y^2 \geq R^2$ for some $R>0$. For $|\mathbf{x}| = (x^2+y^2+z^2)^{1/2} \to \infty$, find an approximation for u in the form

$$u \sim \frac{C}{|\mathbf{x}|} + \frac{\mathbf{p}\cdot\mathbf{x}}{|\mathbf{x}|^3} + \cdots \,, \qquad \text{as} \quad |\mathbf{x}| \to \infty \,, \tag{12}$$

where the scalar C and the vector $\mathbf{p}$ are to be found. Here $\cdot$ is the usual dot product.

Solution:

(a) The Green's function problem will be to find $G(\mathbf{x}';\mathbf{x})$ that satisfies

$$\Delta' G = \delta(\mathbf{x}'-\mathbf{x}) \,, \quad \mathbf{x}' \in \mathbb{R}^{3+} \,; \qquad G_{z'} = 0 \,, \qquad \text{on } z'=0 \,. \tag{13}$$

Here $\mathbb{R}^{3+} \equiv \{\mathbf{x}' \mid z' \geq 0\}$, with $\mathbf{x}' = (x',y',z')$, and $\mathbf{x} = (x,y,z)$ with $z>0$. The free-space Green's function, where the delta singularity is fixed at $\mathbf{x}'=\mathbf{x}$, is simply $G_f = -(4\pi|\mathbf{x}'-\mathbf{x}|)^{-1}$. Since we must impose that $G_{z'}=0$ on the plane $z'=0$, to find the image function we must extend G to the lower half-space to be even in z'. In this way, we get

$$G = -\frac{1}{4\pi|\mathbf{x}'-\mathbf{x}|} - \frac{1}{4\pi|\mathbf{x}'-\mathbf{x}_I|} \,, \tag{14}$$

where $\mathbf{x} \equiv (x,y,z)$ and $\mathbf{x}_I \equiv (x,y,-z)$.

(b) We use Green's second identity over the upper hemisphere Γ_R, defined by $|\mathbf{x}'| = R$ and $z' \geq 0$, to obtain upon taking the $R \to \infty$ limit that

$$\begin{aligned}\int_{\mathbb{R}^{3+}} \left[G\Delta' u - u\Delta G'\right] \, d\mathbf{x}' &= \lim_{R\to\infty} \int_{\Gamma_R} \left(G\partial_n u - u\partial_n G\right) \, dS \\ &= \quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(G\partial_n u - u\partial_n G\right) \, dx'dy' \,.\end{aligned}$$

Upon using the facts that $u \to 0$ and $G \to 0$ sufficiently fast on the boundary of the hemi-sphere, and noting that $\partial_n u = -u_{z'}$ on the infinite plane $z'=0$, we obtain that

$$u(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y') G|_{z'=0} \, dx'dy' \,. \tag{15}$$

Finally, we use (14) for G, and evaluate it on the plane $z'=0$ to get (14) by

$$G|_{z'=0} = -\frac{1}{2\pi} \left[(x'-x)^2 + (y'-y)^2 + z^2\right]^{-1/2} \,. \tag{16}$$

(c) Since $f(x',y') \equiv 0$ outside of a disk of radius R, we can let $|\mathbf{x}| \to \infty$ with $\mathbf{x} = (x,y,z)$ and label $\mathbf{x}' = (x',y',0)$ to obtain (as derived in the class notes) that

$$\frac{1}{|\mathbf{x}'-\mathbf{x}|} \sim \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \ldots \,, \quad \text{as} \quad |\mathbf{x}| \to \infty \,. \tag{17}$$

Upon substituting this expression into (15) and (16) we get for $|\mathbf{x}| \to \infty$ that

$$u \sim -\frac{1}{2\pi} \int_{\Omega_f} \left(\frac{1}{|\mathbf{x}|} + \frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3} + \ldots\right) f(x',y') \, dx'dy' \,. \tag{18}$$

In this way, we identify that

$$C = -\frac{1}{2\pi} \int_{\Omega_f} f(x',y') \, dx'dy' \,, \qquad \mathbf{p} = -\frac{1}{2\pi} \int_{\Omega_f} \mathbf{x}' f(x',y') \, dx'dy' \,. \tag{19}$$

Here $\Omega_f \equiv \{(x',y') \mid (x')^2 + (y')^2 \leq R\}$ and $|\mathbf{x}| = (x^2+y^2+z^2)^{1/2}$.

3. (12 points) Let $R > 0$ and $H > 0$ and suppose that in a finite cylinder $u(x, y, z)$ satisfies

$$\begin{gathered} u_{xx} + u_{yy} + u_{zz} = 0\,, \quad \text{in} \quad 0 \le (x^2+y^2)^{1/2} \le R\,, \qquad 0 \le z \le H\,, \\ u_z(x,y,0) = -\delta(x-x_0)\delta(y-y_0)\,, \qquad u_z(x,y,H) = 0\,, \\ u = 0 \quad \text{on} \quad (x^2+y^2)^{1/2} = R\,, \qquad u \quad \text{bounded as} \quad (x^2+y^2)^{1/2} \to 0\,. \end{gathered} \tag{20}$$

Assume the location of the Dirac on the wall $z = 0$ satisfies $(x_0^2 + y_0^2)^{1/2} < R$.

(a) (2 points) Give a sketch of the geometry for this problem. Give a two-sentence description of what this problem might model physically.

(b) (5 points) Find an eigenfunction representation for u in terms of the eigenfunctions in the z direction. Determine, but not solve, the PDE problems for the coefficients in the eigenfunction expansion.

(c) (5 points) Calculate $\bar{u}(x, y)$ explicitly, where we have defined $\bar{u} \equiv (1/H) \int_0^H u(x, y, z)\, dz$. Here $\bar{u}$ is the spatial average of u over the length of the cylinder.

Solution:

(a) The domain is a 3-D cylinder of height H and radius R with a localized flux on heat centered at some point (x_0, y_0) on the bottom $z = 0$ of the cylinder. We are asked to find the steady-state temperature under the condition that no heat escapes out the upper surface of the cylinder, and a zero temperature is given on the sides of the cylinder.

(b) The eigenfunction problem in the bounded z direction for $\Phi(z)$, corresponding to the homogeneous problem, is

$$\Phi'' + \lambda\Phi = 0\,, \quad 0 < z < H\,; \qquad \Phi'(0) = \Phi'(H) = 0\,.$$

We calculate $\Phi_0 = 1$, $\Phi_n = \cos(n\pi z/H)$ for $n \ge 1$ and $\lambda = \lambda_n = n^2\pi^2/H^2$ for $n \ge 0$.

(c) We expand

$$u = c_0(x,y) + \sum_{n=1}^{\infty} c_n(x,y) \cos\left(\frac{n\pi z}{H}\right)\,,$$

where by orthogonality we identify that

$$c_0 = \frac{1}{H}\int_0^H u\, dz\,; \qquad c_n = \frac{2}{H}\int_0^H u \cos\left(\frac{n\pi z}{H}\right) dz\,, \quad \text{for} \quad n = 1, 2, \ldots\,.$$

Upon multiplying the PDE by H^{-1} and integrating over $0 < z < H$, we obtain that

$$\begin{aligned} &\frac{1}{H}\int_0^H (u_{xx} + u_{yy})\, dz + \frac{1}{H}\int_0^H u_{zz}\, dz = 0\,, \\ &c_{0xx} + c_{0yy} - \frac{1}{H} u_{0z}|_{z=0} = 0\,, \\ &c_{0xx} + c_{0yy} + \frac{1}{H}\delta(\mathbf{x} - \mathbf{x}_0) = 0\,. \end{aligned}$$

This yields that c_0 satisfies

$$c_{0xx} + c_{0yy} = -\frac{1}{H}\delta(\mathbf{x} - \mathbf{x}_0)\,, \quad |\mathbf{x}| < R\,; \qquad c_0 = 0\,, \quad |\mathbf{x}| = R\,, \tag{21}$$

where $|\mathbf{x}| = \sqrt{x^2+y^2}$, $\mathbf{x} = (x,y)$ and $\mathbf{x}_0 = (x_0, y_0)$. Similarly, by multiplying the PDE by $2H^{-1}\cos(n\pi z/H)$ and integrating over $0 < z < H$, we obtain upon integration by parts that

$$c_{nxx} + c_{nyy} - \frac{2}{H} u_z|_{z=0} + \frac{2}{H}\left(\frac{n\pi}{H}\right)\int_0^H u_z \sin\left(\frac{n\pi z}{H}\right)\, dz = 0\,.$$

Integrating by parts one more time and using $u_z|_{z=0} = -\delta(x-x_0, y-y_0)$, we get for $n \geq 1$ that

$$c_{nxx} + c_{nyy} - \frac{n^2\pi^2}{H^2} c_n = -\frac{2}{H}\delta(x-x_0, y-y_0)\,, \quad |\mathbf{x}| < R\,; \qquad c_n = 0\,, \quad |\mathbf{x}| = R\,. \tag{22}$$

(d) We first identify that $\bar{u} = c_0(x,y)$, where $c_0(x,y)$ satisfies (21). This PDE for c_0 asks us to find the Green's function for a disk of radius R (apart from the $-1/H$ term). In this way, we write $c_0 = -H^{-1}G(\mathbf{x};\mathbf{x}_0)$, where G satisfies

$$\Delta G = \delta(\mathbf{x}-\mathbf{x}_0)\,, \quad |\mathbf{x}| < R\,; \qquad G = 0\,, \quad |\mathbf{x}| = R\,. \tag{23}$$

By the method of images we need an image point at $\mathbf{x}_I = \mathbf{x}_0 R^2/r^2$, where $r = |\mathbf{x}_0|$. We label $r_2 = |\mathbf{x} - \mathbf{x}_I|$ and $r_1 = |\mathbf{x} - \mathbf{x}_0|$. Now since $r_2 = (R/r)\, r_1$ when $|\mathbf{x}| = R$, we have

$$G = \frac{1}{2\pi}\ln r_1 - \frac{1}{2\pi}\ln\left(\frac{r r_2}{R}\right)\,, \tag{24}$$

so that c_0 is given by

$$c_0(\mathbf{x}) = -\frac{1}{2\pi H}\ln\left(\frac{r_1 R}{r r_2}\right)\,, \quad \text{where} \quad r_2 = |\mathbf{x}-\mathbf{x}_I|\,, \quad r_1 = |\mathbf{x}-\mathbf{x}_0|\,, \tag{25}$$

with $\mathbf{x}_I = \mathbf{x}_0 R^2/r^2$ and $r = |\mathbf{x}_0|$.

4. (6 points) Quick response questions:

(a) (2 points) Let Ω be a sphere in 3-D centered at the origin $\mathbf{x}=\mathbf{0}$ of radius $r=3$. Find the value of the constant M for which the following problem has a solution:

$$\begin{aligned} &\Delta u = M\,, \quad \mathbf{x}\in\Omega\backslash\{\mathbf{0}\}\,; \qquad \partial_n u = 0\,, \quad \mathbf{x}\in\partial\Omega\,, \\ &u \sim \frac{1}{|\mathbf{x}|}\,, \quad \text{as} \quad \mathbf{x}\to\mathbf{0}\,. \end{aligned} \tag{26}$$

(b) (2 points) Suppose that $L>0$. At what values of λ is a solvability condition needed for $f(x)$ so that the following problem with periodic boundary conditions has a solution $u(x)$?

$$u''+\lambda u = f(x)\,, \qquad 0<x<L\,; \qquad u(0)=u(L)\,, \quad u'(0)=u'(L)\,. \tag{27}$$

At these values of λ determine the solvability condition(s) explicitly.

(c) (2 points) Let Ω be a 2-D disk of radius $a>0$ centered at the origin $\mathbf{x}=\mathbf{0}$. Explain why a simple application of the method of images cannot be used for finding the Green's function for the problem $\Delta G - G = \delta(\mathbf{x}-\mathbf{x}_0)$ in Ω with $G=0$ on the boundary $\partial\Omega$ and $\mathbf{x}_0\in\Omega$.

Solution:

(a) We recall the correspondence $\delta(\mathbf{x}) \longrightarrow -1/(4\pi|\mathbf{x}|)$ between a 3-D Delta singularity for Laplace's equation and the singular behavior of the free-space Green's function. Therefore, by scaling, we have the correspondence $-4\pi\delta(\mathbf{x}) \longrightarrow 1/|\mathbf{x}|$, so that we can write

$$\Delta u = M - 4\pi\delta(\mathbf{x})\,, \quad \mathbf{x}\in\Omega\,; \qquad \partial_n u = 0\,, \quad \mathbf{x}\in\partial\Omega\,.$$

Since $\mathbf{0}\in\Omega$ by assumption, we use the Divergence theorem and the no flux boundary conditions to get $MV-4\pi=0$, where $V=4\pi(3)^3/3$ is the volume of a sphere of radius three. This yields that $M=1/9$.

(b) The solution to the homogeneous problem $\Phi''+\lambda\Phi=0$ is $\Phi=\text{span}\{\cos(\sqrt{\lambda}x),\sin(\sqrt{\lambda}x)\}$ when $\lambda>0$. Enforcing $u(0)=u(L)$ while using the cosine function gives $\cos(\sqrt{\lambda}L)=1$, so that $\sqrt{\lambda}L=2n\pi$ with $n\geq 1$ and n integer. Then, $u'(0)=u'(L)$ is automatically satisfied. Similarly, if we start with $u=\sin(\sqrt{\lambda}x)$ then $u(0)=u(L)$ yields $\sin(\sqrt{\lambda}L)=0$, so that $\sqrt{\lambda}L=2n\pi$. The condition $u'(0)=u'(L)$ is automatically satisfied. In summary, we have two independent nontrivial solutions to the homogeneous problem when $\lambda=4\pi^2n^2/L^2$ given by $\Phi=\sin(2n\pi x/L)$ and $\Phi=\cos(2n\pi x/L)$ where n is a non-negative integer. For $\lambda=0$, the nontrivial solution is simply $\Phi=1$. Using Lagrange's identity, the solvability condition(s) needed at these special values of λ are

$$\int_0^L f(x)\sin\left(\frac{2n\pi x}{L}\right)dx=0\,, \quad \text{and} \quad \int_0^L f(x)\cos\left(\frac{2n\pi x}{L}\right)dx=0\,, \quad \text{if} \quad \lambda=\frac{4\pi^2n^2}{L^2}>0\,,$$

$$\int_0^L f(x)\,dx=0\,, \quad \text{if} \quad \lambda=0\,.$$

(c) Consider the 2-D Green's function problem

$$\Delta G - G = \delta(\mathbf{x}-\mathbf{x}_0) \quad \mathbf{x}\in\Omega\,; \qquad G=0 \quad \mathbf{x}\in\partial\Omega\,,$$

where Ω is a disk centered at the origin of radius a and $\mathbf{x}_0\in\Omega$. We recall that $G_f = -(2\pi)^{-1}K_0(|\mathbf{x}-\mathbf{x}_0|)$ is the free-space Green's function and we try to add to it a harmonic conjugate H with image point centered at $\mathbf{x}_I = a^2\mathbf{x}_0/|\mathbf{x}_0|^2$, so that $|\mathbf{x}_I||\mathbf{x}_0|=a^2$. Therefore, we try

$$G(\mathbf{x};\mathbf{x}_0) = -\frac{1}{2\pi}K_0(|\mathbf{x}-\mathbf{x}_0|)+\frac{c}{2\pi}K_0(|\mathbf{x}-\mathbf{x}_I|)\,,$$

for some constant c. Although the geometric identity $|\mathbf{x}-\mathbf{x}_I| = a|\mathbf{x}-\mathbf{x}_0|/|\mathbf{x}_0|$ for $|\mathbf{x}|=a$ still holds, we can't use the form for G above to find a c for which $G=0$ whenever $|\mathbf{x}|=a$.