MATH 401 MIDTERM SOLUTIONS

PROBLEM 1

(i) $Lu = u'' + \lambda u = f(x)$ WITH $u'(0) = u'(L) = 0$ IS SELF-ADJOINT.

THE HOMOGENEOUS PROBLEM $\phi'' + \lambda\phi = 0$, $\phi'(0) = \phi'(L) = 0$ WITH $\lambda \geq 0$

HAS NONTRIVIAL SOLUTIONS $\phi_0 = 1$ WHEN $\lambda = 0$

$\phi_n = \cos\left(\frac{n\pi x}{L}\right)$ WHEN $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, ..$

THUS WHEN $\lambda = 0$, THE SOLVABILITY CONDITION IS $\int_0^L f(x)\,dx = 0$

WHEN $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, ..$ THE SOLVABILITY CONDITION IS $\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0$.

(ii) SUPPOSE $\lambda > 0$ AND $\lambda \neq \lambda_n$ FOR ANY $n = 1, 2, ...$

THE GREEN'S FUNCTION $v(t; x)$ SATISFIES

$$v'' + \lambda v = \delta(t - x) \quad \text{in } 0 < t < L$$

$$v'(0) = v'(L) = 0.$$

THE HOMOGENEOUS PROBLEM HAS $v = \text{SPAN}\{\cos(\sqrt{\lambda}(t + \phi)), \sin(\sqrt{\lambda}(t + \phi))\}$.

TO MAKE $v' = 0$ AT $t = 0, L$, WE HAVE

$$v = \begin{cases} A\cos(\sqrt{\lambda}\,t), & 0 \leq t \leq x \\ B\cos(\sqrt{\lambda}(t - L)), & \text{on } x \leq t \leq L. \end{cases}$$

NOW CONTINUITY AT $x = t$ HOLDS IF

$$v = \begin{cases} C\cos(\sqrt{\lambda}(x - L))\cos(\sqrt{\lambda}\,t), & 0 \leq t \leq x \\ C\cos(\sqrt{\lambda}(t - L))\cos(\sqrt{\lambda}\,x), & x \leq t \leq L. \end{cases}$$

THE JUMP CONDITION $v'|_{x+} - v'|_{x-} = 1$ REQUIRES

$$\sqrt{\lambda}\,C\left[-\sin(\sqrt{\lambda}(x - L))\cos(\sqrt{\lambda}\,x) + \cos(\sqrt{\lambda}(x - L))\sin(\sqrt{\lambda}\,x)\right] = 1$$

BUT $\sin(A - B) = \sin A\cos B - \cos A\sin B$ SO THAT

$$-\sqrt{\lambda}\,C\sin(-\sqrt{\lambda}\,L) = 1 \longrightarrow C = \frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\,L)}.$$

HENCE

$$v = \frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\,L)}\begin{cases} \cos(\sqrt{\lambda}(x - L))\cos(\sqrt{\lambda}\,t), & 0 < t < x \\ \cos(\sqrt{\lambda}(t - L))\cos(\sqrt{\lambda}\,x), & x \leq t \leq L. \end{cases}$$

THE INTEGRAL REPRESENTATION IS OBTAINED FROM

$$\int_0^L vLu\,dt = \underbrace{(vu' - uv')\Big|_0^L}_{=0} + \int_0^L uLv\,dt$$

THUS,

$$u(x) = \int_0^L v(t; x)f(t)\,dt$$

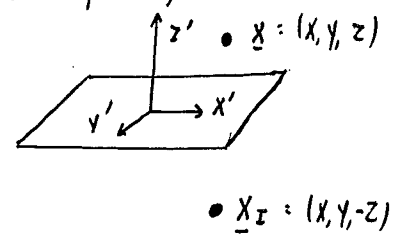
(iii) FOR $\lambda = 0$, THE MODIFIED GREEN'S FUNCTION IS

$$v'' = \delta(t - x) - \frac{1}{L} \quad \text{ON } 0 < t < L; \quad v'(0) = v'(L) = 0.$$

PROBLEM 2 SUPPOSE THAT

$$\Delta u = 0 \quad \text{IN } -\infty < x < \infty,\ -\infty < y < \infty,\ z > 0$$

WITH $u_z(x,y,0) = f(x,y)$ AND $u \sim C/r$ AS $r = (x^2+y^2+z^2)^{1/2} \to \infty$.



(i) NOW THE G-FUNCTION $G(\underline{x}'; \underline{x})$ SATISFIES

$$\begin{cases} \Delta' G = \delta(\underline{x}' - \underline{x}) & \text{IN } -\infty < x', y' < \infty,\ z' > 0 \\ G_{z'} = 0 & \text{ON } z' = 0. \end{cases}$$

NOW THE FREE SPACE G FUNCTION IS $G^f = -\dfrac{1}{4\pi |\underline{x}' - \underline{x}|}$.

BY METHOD OF IMAGES WE NEED AN IMAGE CHARGE AT $\underline{x}_I = (x, y, -z)$ WITH SAME SIGN.

$$|\underline{x}' - \underline{x}| = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$$

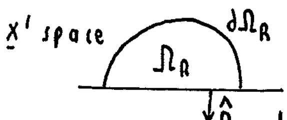
$$|\underline{x}' - \underline{x}_I| = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'+z)^2}$$

THUS $$G(\underline{x}'; \underline{x}) = -\frac{1}{4\pi |\underline{x}' - \underline{x}|} - \frac{1}{4\pi |\underline{x}' - \underline{x}_I|}.$$

THIS SATISFIES $G_{z'} = 0$ ON $z' = 0$.

(ii) NOW USE GREEN'S SECOND IDENTITY BY INTEGRATING OVER A HEMI-SPHERE OF RADIUS R.

WE GET



$$\Delta' u = 0 \quad \text{IN } z' \geq 0$$

$$u_{z'} = f(x', y') \quad \text{ON } z' = 0$$

THEN
$$\lim_{R\to\infty} \int_{\Omega_R} (G \Delta' u - u \Delta' G)\, d\underline{x}' = \lim_{R\to\infty} \int_{\partial\Omega_R} \left(G \frac{\partial u}{\partial r}\Big|_{r=R} - u \frac{\partial G}{\partial r}\Big|_{r=R} \right) ds + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right)\Big|_{z'=0} dx'\, dy'.$$

NOW ON $\partial\Omega_R$, $ds = O(R^2)$ AND $G \sim O(1/R)$

WHILE $u \sim O(1/R)$. HENCE $$\lim_{R\to\infty} \int_{\partial\Omega_R} \left(G \frac{\partial u}{\partial r}\Big|_{r=R} - u \frac{\partial G}{\partial r}\Big|_{r=R} \right) ds = 0.$$

NOW $\partial G/\partial n = -\partial G/\partial z'$ ON $z' = 0$, AND $\partial u/\partial n = -\partial u/\partial z'$ ON $z' = 0$.

HENCE, $$\lim_{R\to\infty} \int_{\Omega_R} -u\, \delta(\underline{x}' - \underline{x})\, d\underline{x}' = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G \frac{\partial u}{\partial z'}\Big|_{z'=0} dx'\, dy', \quad \text{WITH } \frac{\partial u}{\partial z'}\Big|_{z'=0} = f(x', y').$$

THEREFORE, $$u(\underline{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G\Big|_{z'=0} f(x', y')\, dx'\, dy'$$

NOW $$G\Big|_{z'=0} = -\frac{1}{2\pi [(x'-x)^2 + (y'-y)^2 + z^2]}$$ FROM PART (i).

WE CONCLUDE THAT $$u(\underline{x}) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x', y')}{[(x'-x)^2 + (y'-y)^2 + z^2]^{1/2}}\, dx'\, dy'.$$

(iii) NOW LET $\underline{x}' = (x', y', 0)$ AND $\underline{x} = (x, y, z)$.

FOR $|\underline{x}| \to \infty$ WE USE $|\underline{x}' - \underline{x}|^2 = |\underline{x}|^2 + |\underline{x}'|^2 - 2\underline{x}\cdot\underline{x}'$

$$|\underline{x}' - \underline{x}|^2 = |\underline{x}|^2 \left(1 - \frac{2\underline{x}\cdot\underline{x}'}{|\underline{x}|^2} + \cdots\right) \quad \text{FOR } |\underline{x}| \text{ LARGE.}$$

NOW $$|\underline{x}' - \underline{x}| \cong |\underline{x}| \left(1 - 2\frac{\underline{x}\cdot\underline{x}'}{|\underline{x}|^2}\right)^{1/2}$$

SO $$\frac{1}{|\underline{x}' - \underline{x}|} \cong \frac{1}{|\underline{x}|}\left(1 - \frac{2\underline{x}\cdot\underline{x}'}{|\underline{x}|^2}\right)^{-1/2} \cong \frac{1}{|\underline{x}|} + \frac{\underline{x}\cdot\underline{x}'}{|\underline{x}|^3} + \cdots$$

USING $(1-h)^{-1/2} \sim 1 + h/2$.

NOW THIS GIVES

$$u(\underline{x}) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x', y')}{[(x'-x)^2 + (y'-y)^2 + z^2]^{1/2}}\, dx'\, dy' \cong -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{|\underline{x}|} + \frac{\underline{x}\cdot\underline{x}'}{|\underline{x}|^3}\right) f(x', y')\, dx'\, dy'$$

THIS GIVES THAT

$$u(\underline{x}) \sim \left(-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\, dx'\, dy'\right)\frac{1}{|\underline{x}|} + \frac{\underline{p}\cdot\underline{x}}{|\underline{x}|^3} + \cdots$$

WHERE WE IDENTIFY

$$C = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\, dx'\, dy'$$

$$\underline{p} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{x}' f(x', y')\, dx'\, dy'$$

WHERE $\underline{x}' = (x', y', 0)$.

<u>REMARK</u> WE COULD ALSO FIND C BY DIVERGENCE THEOREM. INTEGRATING OVER A HEMI-SPHERE Ω_R WE OBTAIN

$$\lim_{R\to\infty}\int_{\Omega_R} \Delta u\, d\underline{x} = \lim_{R\to\infty}\int_{\partial\Omega_R} \nabla u\cdot\hat{n}\, ds + \lim_{R\to\infty}\int_{-R}^{R}\int_{-R}^{R}\left(-\frac{\partial u}{\partial z'}\right)\Big|_{z'=0} dx'\, dy'$$

BUT $\Delta u = 0$ AND $\nabla u\cdot\hat{n} = \partial u/\partial r$ WITH $u \sim C/r$ AS $r \to \infty$.

HENCE $$0 = 2\pi\left(-C/R^2\right)R^2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(-f(x', y')\right)dx'\, dy'$$

WE CONCLUDE THAT $$C = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\, dx'\, dy'.$$

PROBLEM 3

CONSIDER $$U_{xx} + U_{yy} + U_{zz} = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0) \quad \text{IN } 0 < z < \pi,\ -\infty < x, y < \infty$$
$$U = 0 \text{ ON } z = 0, \pi \quad \text{AND} \quad U \to 0 \text{ AS } (x^2+y^2+z^2)^{1/2} \to \infty.$$

(i) WE EXPAND $$U(x,y,z) = \sum_{n=1}^{\infty} C_n(x,y) \sin(nz)$$

AS $\{\sin(nz)\}$ ARE THE EIGENFUNCTIONS IN THE z DIRECTION.

BY ORTHOGONALITY $$C_n = \frac{2}{\pi}\int_0^{\pi} U \sin(nz)\,dz.$$

MULTIPLY THE PDE BY $\sin(nz)$ AND INTEGRATE.

$$C_{n\,xx} + C_{n\,yy} + \frac{2}{\pi}\int_0^{\pi} U_{zz}\sin(nz)\,dz = \frac{2}{\pi}\,\delta(x-x_0)\,\delta(y-y_0)\sin(nz_0)$$

INTEGRATING BY PARTS

$$C_{n\,xx} + C_{n\,yy} + \frac{2}{\pi}\underbrace{\Big[U_z \sin(nz) - nU\cos(nz)\Big]\Big|_0^{\pi}}_{=0} - \frac{2}{\pi}n^2\int_0^{\pi} U\sin(nz)\,dz = \frac{2}{\pi}\,\delta(x-x_0)\,\delta(y-y_0)\cdot\sin(nz_0)$$

THEREFORE $$\Delta_0 C_n - n^2 C_n = \frac{2}{\pi}\,\delta(\underline{x}-\underline{x}_0)\sin(nz_0) \qquad (1)$$

IS THE PDE FOR $C_n(x,y)$. HERE $\Delta_0 \equiv \partial_{xx} + \partial_{yy}$, $\underline{x} = (x,y)$, $\underline{x}_0 = (x_0,y_0)$

(ii) NOW RECALL THAT $\Delta v^F - k^2 v^F = \delta(\underline{x}-\underline{x}_0)$ IN $\mathbb{R}^2$

HAS SOLUTION $$v^F = -\frac{1}{2\pi}K_0(k|\underline{x}-\underline{x}_0|).$$

COMPARING WITH (1) WE CONCLUDE THAT

$$C_n(\underline{x}) = \frac{2}{\pi}\sin(nz_0)\,K_0(n|\underline{x}-\underline{x}_0|)\left(-\frac{1}{2\pi}\right) = -\frac{1}{\pi^2}K_0(n|\underline{x}-\underline{x}_0|)\sin(nz_0).$$

WHERE $K_0(z)$ IS THE MODIFIED BESSEL FUNCTION

PROBLEM 4

(i) $\delta(x^2-5x+4) = \delta((x-4)(x-1))$.

THUS CONCENTRATION POINTS ARE $x=1$, $x=4$.

RECALL $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$

SO THAT IF $f(x) = x^2-5x+4$ THEN $f'(1) = 2-5 = -3$, $f'(4) = (2x-5)\big|_{x=4} = 3$.

THUS $\delta(x^2-5x+4) = \frac{1}{3}\delta(x-4) + \frac{1}{3}\delta(x-1)$.

(ii) DEFINE $f_\sigma(x) = \frac{\sigma}{(x-1)^2+\sigma^2}$. THEN $\lim_{\substack{\sigma\to 0\\ x\neq 1}} f_\sigma(x) = 0$, $\lim_{\substack{\sigma\to 0\\ x=1}} f_\sigma(x) = \infty$.

NOW WE INTEGRATE $\int_{-\infty}^{\infty} \frac{\sigma}{(x-1)^2+\sigma^2}\,dx = \frac{\sigma^2}{\sigma^2}\int_{-\infty}^{\infty}\frac{1}{y^2+1}\,dy = \arctan y\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$.

let $x-1 = \sigma y$

WE CONCLUDE THAT $\lim_{\substack{\sigma\to 0\\ x\neq 1}} f_\sigma(x) = \pi\,\delta(x-1)$.

(iii) NOW $u_{rr} + \frac{2}{r}u_r - u = f(r)$ ON $1\le r\le 2$, $u(1) = u(2) = 0$.

MULTIPLY BY r^2 TO OBTAIN

$$r^2 u_{rr} + 2u_r - r^2 u = r^2 f(r)$$

$$\rightarrow \quad (r^2 u_r)_r - r^2 u = r^2 f(r)$$

$Lu \equiv [r^2 u_r]_r - r^2 u$ IS SELF-ADJOINT.

(iv) RECALL IF $\Delta u = \delta(\underline{x}-\underline{x}_0)$ IN 2-D, THEN $u \sim \frac{1}{2\pi}\log|\underline{x}-\underline{x}_0|$.

NOW TAKE GRADIENT. IT COMMUTES WITH LAPLACIAN SO THAT

$$\Delta(\nabla u) = \nabla_x\,\delta(\underline{x}-\underline{x}_0).$$

NOW DOT PRODUCT WITH $\underline{p}$ $\quad \Delta(\nabla u)\cdot\underline{p} = \nabla_x\,\delta(\underline{x}-\underline{x}_0)\cdot\underline{p}$.

SINCE p CONSTANT VECTOR $\quad \Delta(\nabla_x u\cdot\underline{p}) = \nabla_x\,\delta(\underline{x}-\underline{x}_0)\cdot\underline{p}$.

THUS $v = \nabla_{\underline{x}} u\cdot\underline{p} = \frac{1}{2\pi}\nabla_{\underline{x}}\log|\underline{x}-\underline{x}_0|\cdot\underline{p} = \frac{1}{2\pi}\frac{(\underline{x}-\underline{x}_0)\cdot\underline{p}}{|\underline{x}-\underline{x}_0|^2}$.

(NOTE: $\nabla_x \log|\underline{x}-\underline{x}_0| = \frac{1}{2}\nabla_x \log((x-x_0)^2+(y-y_0)^2) = \frac{1}{2\pi}\frac{(x-x_0,\,y-y_0)}{|\underline{x}-\underline{x}_0|^2} = \frac{\underline{x}-\underline{x}_0}{|\underline{x}-\underline{x}_0|^2}$.