# Math 401: Quiz 2

**Instructions: Open book and open notes. No collaboration or discussion of the problems with others. No posting of questions related to this quiz on Piazza or Chegg.**

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1. (10 points) Consider Laplace's equation for $u(r,\theta)$ in the quarter plane in 2-D when written in polar coordinates:

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0\,, \qquad 0 \le r < \infty\,, \quad 0 \le \theta \le \frac{\pi}{2}\,, \\ &u \text{ bounded as } \ r \to 0 \text{ and } r \to \infty\,, \\ &u(r,0) = f(r)\,, \ \ u(r,\pi/2) = 0\,. \end{aligned} \tag{1}$$

We assume that $f(r)$ is bounded as $r \to \infty$ and that $f(0) = 0$.

   (a) (1 point) What are the appropriate eigenfunctions $\Phi_n(\theta)$ in the $\theta$ direction?

   (b) (2 points) By seeking an eigenfunction expansion for $u$ in the form $u(r,\theta) = \sum_{n=1}^{\infty} c_n(r)\Phi_n(\theta)$, derive a 1-D boundary value problem (BVP) for $c_n(r)$.

   (c) (3 points) Solve the 1-D BVP for $c_n(r)$ using a 1-D Green's function.

   (d) (4 points) By summing the eigenfunction expansion for $u$ show that $u(r,\theta) = \int_0^{\infty} f(\rho)S(\rho,r,\theta)\,d\rho$, where you are to determine an explicit formula for $S(\rho,r,\theta)$.

2. (10 points) In this problem we will solve the PDE in (1) of Question 1 by using a 2-D Green's function and the method of images.

   (a) (4 points) Let $\Omega$ be the quarter-plane defined by $\Omega \equiv \{\,(\xi,\eta)\,|0 \le \xi < \infty\,,\ 0 \le \eta < \infty\}$, with boundary $\partial\Omega$. Let $(x,y)$ be some given point strictly inside $\Omega$. By using the method of images, determine the Green's function $G$ satisfying

$$G_{\xi\xi} + G_{\eta\eta} = \delta(\xi - x)\delta(\eta - y)\,, \quad \text{in } \Omega\,; \tag{10}$$

$$G = 0 \ \text{ on } \xi = 0\,; \qquad G = 0 \ \text{ on } \eta = 0\,. \tag{11}$$

   (b) (2 points) By using Green's second identity, write the solution $u$ to the PDE (1) in Question 1 in terms of this 2-D Green's function.

   (c) (4 points) By calculating explicitly the terms in this formula for $u$ in (b) show that you obtain exactly the same result as in part (d) of Question 1.