Math 401: Quiz 2

Instructions: Open book and open notes. No collaboration or discussion of the problems with others. No posting of questions related to this quiz on Piazza or Chegg.

---

1. (10 points) Consider Laplace's equation for $u(r,\theta)$ in the quarter plane in 2-D when written in polar coordinates:

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0\,, \qquad 0 \le r < \infty\,, \quad 0 \le \theta \le \frac{\pi}{2}\,, \\ &u \text{ bounded as } r \to 0 \text{ and } r \to \infty\,, \\ &u(r,0) = f(r)\,, \quad u(r,\pi/2) = 0\,. \end{aligned} \tag{1}$$

We assume that $f(r)$ is bounded as $r \to \infty$ and that $f(0) = 0$.

(a) (1 point) What are the appropriate eigenfunctions $\Phi_n(\theta)$ in the θ direction?

(b) (2 points) By seeking an eigenfunction expansion for u in the form $u(r,\theta) = \sum_{n=1}^{\infty} c_n(r)\Phi_n(\theta)$, derive a 1-D boundary value problem (BVP) for $c_n(r)$.

(c) (3 points) Solve the 1-D BVP for $c_n(r)$ using a 1-D Green's function.

(d) (4 points) By summing the eigenfunction expansion for u show that $u(r,\theta) = \int_0^\infty f(\rho)S(\rho,r,\theta)\,d\rho$, where you are to determine an explicit formula for $S(\rho,r,\theta)$.

Solution:

(a) By separating variables, the eigenfunctions and eigenvalues in the θ direction satisfy

$$\Phi'' + \lambda\Phi = 0\,, \quad 0 \le \theta \le \pi/2\,; \qquad \Phi(0) = 0\,, \quad \Phi(\pi/2) = 0\,.$$

The eigenfunctions are $\Phi_n(\theta) = \sin(2n\theta)$ with $\lambda_n = 4n^2$. Thus, we expand $u(r,\theta)$ as

$$u(r,\theta) = \sum_{n=1}^{\infty} c_n(r)\sin(2n\theta)\,.$$

(b) By orthogonality we first identify that $c_n = 4\pi^{-1}\int_0^{\pi/2} u(r,\theta)\sin(2n\theta)\,d\theta$. Multiply the PDE for u by $4\pi^{-1}\sin(2n\theta)$ and integrate over $0 \le \theta \le \pi/2$. We obtain upon integrating by parts that

$$\begin{aligned} \frac{4}{\pi}\int_0^{\pi/2}\left(u_{rr} + r^{-1}u_r\right)\sin(2n\theta)\,d\theta + \frac{4}{\pi r^2}\int_0^{\pi/2} u_{\theta\theta}\sin(2n\theta)\,d\theta &= 0\,, \\ c_n'' + \frac{1}{r}c_n' + \frac{4}{\pi r^2}\left[u_\theta \sin(2n\theta) - 2nu\cos(2n\theta)\right]\big|_0^{\pi/2} - \frac{4n^2}{r^2}c_n &= 0\,, \\ c_n'' + \frac{1}{r}c_n' - \frac{4n^2}{r^2}c_n + \frac{8n}{\pi r^2}f(r) &= 0\,. \end{aligned}$$

Therefore, we have the 1-D BVP

$$c_n'' + \frac{1}{r}c_n' - \frac{4n^2}{r^2}c_n = -\frac{8n}{\pi r^2}f(r)\,, \quad 0 < r < \infty\,; \quad c_n \text{ bounded as } r \to 0 \text{ and } r \to \infty\,.$$

In Sturm-Liouville form this yields

$$\begin{aligned} &Lc_n \equiv [rc_n']' - \frac{4n^2}{r}c_n = \mathcal{F}(r)\,, \quad 0 < r < \infty\,; \qquad \mathcal{F}(r) \equiv -\frac{8n}{\pi r}f(r)\,, \\ &c_n \text{ bounded as } r \to 0 \quad \text{and as } r \to \infty\,. \end{aligned} \tag{2}$$

(c) Since L is a Sturm-Liouville operator, we can use Lagrange's identity in the usual way to get

$$c_n(r) = \int_0^1 G(\rho; r)\mathcal{F}(\rho)\, d\rho\,, \tag{3}$$

where $G(\rho; r)$ satisfies

$$LG \equiv [\rho G']' - \frac{4n^2}{\rho}G = \delta(\rho - r)\,, \quad 0 \le \rho \le \infty\,; \quad G \text{ bounded as } \rho \to 0 \text{ and as } \rho \to \infty\,. \tag{4}$$

The jump and continuity conditions are

$$G(r^+) - G(r^-) = 0\,, \qquad G'(r^+) - G'(r^-) = r^{-1}\,. \tag{5}$$

The homogeneous problem for G_H is Euler's equation with solutions $G_H = \text{span}\{\rho^{2n}, \rho^{-2n}\}$. We derive in the usual way that

$$G = \begin{cases} A\,(\rho/r)^{2n}\,, & \text{if } 0 \le \rho \le r \\ A\,(r/\rho)^{2n}\,, & \text{if } r \le \rho < \infty \end{cases}\,,$$

where by imposing the jump condition in (5) on the derivative of G we get $A = -1/(4n)$. We then write G compactly as

$$G = -\frac{1}{4n}\left(\frac{r_<}{r_>}\right)^{2n}\,, \quad \text{where} \quad r_< \equiv \min(r, \rho)\,, \quad r_> \equiv \max(r, \rho)\,. \tag{6}$$

Upon using (6) in (3) we get

$$c_n(r) = \frac{2}{\pi}\int_0^\infty \frac{f(\rho)}{\rho}\left(\frac{r_<}{r_>}\right)^{2n} d\rho\,. \tag{7}$$

(d) Substituting (7) into $u(r, \theta) = \sum_{n=1}^\infty c_n(r)\sin(2n\theta)$, we obtain that

$$u(r, \theta) = \frac{2}{\pi}\int_0^1 \frac{f(\rho)}{\rho} A\, d\rho\,, \qquad \text{where} \qquad A \equiv \sum_{n=1}^\infty \left(\frac{r_<}{r_>}\right)^{2n} \sin(2n\theta)\,. \tag{8}$$

We identify and calculate A as

$$A = \text{Im}\left(\sum_{n=1}^\infty z^n\right) = \text{Im}\left(\frac{z}{1-z}\right)\,, \quad \text{where} \quad z = \left(\frac{z_<}{z_>}\right)^2 e^{2i\theta}\,.$$

Multiplying top and bottom of the expression for A by the conjugate $\overline{1-z}$, we calculate that

$$A = \text{Im}\left(\frac{z}{1-z}\right) = \frac{1}{|1-z|^2}\text{Im}\left(z - |z|^2\right) = \frac{1}{|1-z|^2}\text{Im}(z)\,.$$

Putting in the expression for z and doing a little algebra, we get

$$A = \frac{r_<^2 r_>^2 \sin(2\theta)}{r_<^4 + r_>^4 - 2r_<^2 r_>^2 \cos(2\theta)} = \frac{r^2\rho^2\sin(2\theta)}{r^4 + \rho^4 - 2r^2\rho^2\cos(2\theta)}\,.$$

Finally, upon subsituting this expression for A into (8) we obtain that

$$u(r, \theta) = \frac{2r^2\sin(2\theta)}{\pi}\int_0^\infty \frac{\rho f(\rho)}{r^4 + \rho^4 - 2r^2\rho^2\cos(2\theta)}\, d\rho\,, \tag{9}$$

from which we can identify $S(\rho, r, \theta)$.

2. (10 points) In this problem we will solve the PDE in (1) of Question 1 by using a 2-D Green's function and the method of images.

   (a) (4 points) Let Ω be the quarter-plane defined by $\Omega \equiv \{ (\xi,\eta)\,|0 \le \xi < \infty\,,\ 0 \le \eta < \infty\}$, with boundary $\partial\Omega$. Let (x,y) be some given point strictly inside Ω. By using the method of images, determine the Green's function G satisfying

$$G_{\xi\xi} + G_{\eta\eta} = \delta(\xi - x)\delta(\eta - y)\,, \quad \text{in } \Omega\,; G = 0\,, \quad \text{on } \xi = 0\,; \qquad G = 0\,, \qquad \text{on } \eta = 0\,,\,. \tag{10}$$

   (b) (2 points) By using Green's second identity, write the solution u to the PDE (1) in Question 1 in terms of this 2-D Green's function.

   (c) (4 points) By calculating explicitly the terms in this formula for u in (b) show that you obtain exactly the same result as in part (d) of Question 1.

Solution:

(a) We first need to set up a clear notation. The delta singularity is fixed at $\mathbf{x} = (x,y)$ in the quarter-plane. A generic point in the quarter-plane is labelled by $\mathbf{x}' = (\xi,\eta)$. The free-space Green's function G_f is

$$G_f = \frac{1}{2\pi}\log|\mathbf{x}' - \mathbf{x}|\,. \tag{11}$$

By the method of images, we need three image charges located at $\mathbf{x}_1 = (x,-y)$, $\mathbf{x}_3 = (-x,-y)$, and $\mathbf{x}_2 = (-x,y)$. The overall Green's function that satisfies the boundary conditions is simply

$$G = \frac{1}{2\pi}\log|\mathbf{x}' - \mathbf{x}| - \frac{1}{2\pi}\log|\mathbf{x}' - \mathbf{x}_1| - \frac{1}{2\pi}\log|\mathbf{x}' - \mathbf{x}_2| + \frac{1}{2\pi}\log|\mathbf{x}' - \mathbf{x}_3|\,.$$

Writing this out in long hand gives

$$G = \frac{1}{4\pi}\log\left[(\xi - x)^2 + (\eta - y)^2\right] - \frac{1}{4\pi}\log\left[(\xi - x)^2 + (\eta + y)^2\right] \tag{12}$$

$$- \frac{1}{4\pi}\log\left[(\xi + x)^2 + (\eta - y)^2\right] + -\frac{1}{4\pi}\log\left[(\xi + x)^2 + (\eta + y)^2\right]\,. \tag{13}$$

(b) In this derivation, we write $\mathbf{x}'$ in cartesian coordinates as $\mathbf{x}' \equiv (\xi,\eta)$ with $\xi^2+\eta^2 \le 1$ and $\eta \ge 0$. We use Green's second identity, we write $\Delta u = u_{\xi\xi} + u_{\eta\eta}$, and integrate around the quarter-plane in the **counter-clockwise** direction and use decay at infinity. We then use $\Delta u = 0$, $\Delta G = \delta(\mathbf{x}' - \mathbf{x})$ and that $u = f(\xi)$ on the boundary segment $\eta = 0$ for $\xi \ge 0$. On this segment, the outward normal derivative $\partial_n G$ for G is $\partial_n G = -\partial_\eta G$ on $\eta = 0$. In this way we get,

$$\int_\Omega G\Delta u\, d\mathbf{x}' - \int_\Omega u\Delta G\, d\mathbf{x}' = -\int_0^\infty u\partial_n G\, d\xi\,, \tag{14}$$

$$-u(\mathbf{x}) = \int_0^\infty f(\xi) G_\eta|_{\eta=0}\, d\xi\,. \tag{15}$$

Then, by differentiating (12) we calculate after some algebra that

$$\begin{aligned}\frac{\partial G}{\partial\eta}|_{\eta=0} &= -\frac{1}{\pi}\frac{y}{(\xi - x)^2 + y^2} + \frac{1}{\pi}\frac{y}{(\xi + x)^2 + y^2}\\ &= \frac{y}{\pi}\frac{-4x\xi}{((\xi + x)^2 + y^2)\,((\xi - x)^2 + y^2)}\\ &= \frac{y}{\pi}\frac{-4x\xi}{(r^2 + \xi^2 + 2r\xi)\,(r^2 + \xi^2 - 2x\xi)}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{4xy\xi}{\pi} \frac{1}{\left(r^2+\xi^2\right)^2 - 4x^2\xi^2} \\
&= -\frac{4xy\xi}{\pi} \frac{1}{r^4+\xi^4+2r^2\xi^2 - 4x^2\xi^2} \\
&= -\frac{4xy\xi}{\pi} \frac{1}{r^4+\xi^4+2\xi^2(r^2 - 2r^2\cos^2\theta)} \\
&= -\frac{4xy\xi}{\pi} \frac{1}{r^4+\xi^4-2\xi^2r^2\cos(2\theta)} \\
&= -\frac{2r^2\xi\sin(2\theta)}{\pi\left(r^4+\xi^4-2\xi^2r^2\cos(2\theta)\right)} \,.
\end{aligned}
$$

Here $x = r\cos\theta$ and $y = r\sin\theta$ and we used the double-angle formulas for $\sin(2\theta)$ and $\cos(2\theta)$. Finally, putting the last expression of the above into (14) we obtain that

$$
u(r,\theta) = \frac{2r^2\sin(2\theta)}{\pi} \int_0^\infty \frac{\xi f(\xi)}{r^4+\xi^4-2r^2\xi^2\cos(2\theta)}\, d\xi \,.
$$

This is the same expression as in (9) except that the dummy variable is ξ rather than ρ.