Math 401: Short Answer Questions

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1. (3 points) Let $a(x)$ be a continuous function on $0 \le x \le 1$. Consider the eigenvalues of

$$\phi'' + a(x)\phi' = -\lambda\phi\,, \quad 0 \le x \le 1\,; \qquad \phi(0) = \phi(1) = 0\,.$$

   (a) (1 point) Are the eigenvalues λ always real?

   (b) (1 point) Are the eigenvalues always positive?

   (c) (1 point) Write a variational principle to approximate the lowest eigenvalue.

2. (3 points) Let $\boldsymbol{a}(\boldsymbol{x}) = (a_1, a_2)^T$ be a continuous function for $\boldsymbol{x} \in \mathbb{R}^2$, and consider the 2-D eigenvalue problem in a bounded 2-D domain

$$\Delta\phi + \boldsymbol{a}\cdot\nabla\phi = -\lambda\phi\,, \quad \boldsymbol{x} \in \Omega\,; \qquad \phi = 0\,, \quad \boldsymbol{x} \in \partial\Omega\,. \tag{1}$$

   (a) (1 point) Identify the vector functions $\boldsymbol{a}(\boldsymbol{x})$ for which we can write this eigenvalue problem in the form

$$\nabla\cdot[p(\boldsymbol{x})\nabla\phi] = -\lambda r(\boldsymbol{x})\phi\,, \quad \boldsymbol{x} \in \Omega\,; \qquad \phi = 0\,, \quad \boldsymbol{x} \in \partial\Omega\,. \tag{2}$$

   where $p > 0$ and $r > 0$ in Ω.

   (b) (1 point) Prove that if (2) holds then any eigenvalue λ must be real and positive.

   (c) (1 point) Write down a specific function $\boldsymbol{a}$ for which we cannot transform (1) to (2).

3. (4 points) (a) (2 points) Let Ω be a sphere of radius L in 3-D centered at the origin. Find the value of the constant M for which the following problem has a solution:

$$\begin{aligned} &\Delta u = M\,, \quad \mathbf{x} \in \Omega\backslash\{\mathbf{0}\}\,; \qquad \partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \\ &u \sim \frac{5}{|\mathbf{x}|}\,, \quad \text{as} \quad \mathbf{x} \to \mathbf{0}\,. \end{aligned} \tag{3}$$

   (b) (2 points) Let Ω be a disk of radius L in 2-D centered at the origin. Find the value of the constant M for which the following problem has a solution:

$$\begin{aligned} &\Delta u = -M\,, \quad \mathbf{x} \in \Omega\backslash\{\mathbf{0}\}\,; \qquad \partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \\ &u \sim 2\log|\mathbf{x}|\,, \quad \text{as} \quad \mathbf{x} \to \mathbf{0}\,. \end{aligned} \tag{4}$$

4. (4 points) Let Ω be a quarter disk of radius L in 2-D, i.e. $\Omega = \{\mathbf{x} = (x,y)\,|x^2+y^2 \le L^2\,, \, x \ge 0\,,\ y \ge 0\}$. For $\xi \in \Omega$ fixed, let $G(\mathbf{x};\xi)$ satisfy

$$\Delta G = \delta(\mathbf{x} - \xi)\,, \quad \mathbf{x} \in \Omega\,; \qquad G = 0\,, \quad \mathbf{x} \in \partial\Omega\,. \tag{5}$$

   (a) (2 points) How many image points are needed to determine G. Where should the image points be located?

   (b) (2 points) Briefly describe in three sentences how one could alternatively determine G in an infinite series expansion (do not calculate the series!).

5. (4 points) Let Ω be the upper half space in 3-D, i.e. $\Omega = \{\mathbf{x} = (x,y,z)\,|z \ge 0\}$. Consider the PDE

$$\Delta u = 0\,, \quad \mathbf{x} \in \Omega\,; \qquad u_z = f(x,y)\,, \quad \text{on} \quad z = 0\,, \tag{6}$$

   where $f \to 0$ as $x^2 + y^2 \to \infty$ sufficiently fast.

   (a) (2 points) Without solving the PDE find the constant C so that $u \sim C/|\mathbf{x}|$ as $|\mathbf{x}| \to \infty$. (Hint: divergence theorem)

(b) (2 points) What is the Green's function that would be useful for solving for u.

6. (4 points) Let Ω be a bounded domain in 3-D, and consider the eigenvalue problem

$$\Delta\phi + \lambda\phi = 0\,, \quad \mathbf{x}\in\Omega\,; \qquad \phi = 0\,, \quad \mathbf{x}\in\partial\Omega\,. \tag{7}$$

Let λ_n be the n^{th} eigenvalue for this problem. Prove that $\lambda_n \to \infty$ as $n\to\infty$. (Hint: courant min-max and use a cube as the bounding domain).

7. (2 points) Let Ω be a bounded domain in 3-D, and consider the eigenvalue problem

$$\Delta\psi + \sigma\psi = 0\,, \quad \mathbf{x}\in\Omega\,; \qquad \partial_n\psi = 0\,, \quad \mathbf{x}\in\partial\Omega\,. \tag{8}$$

(a) (1 point) Calculate explicitly the first eigenvalue σ_1 of (8) and its eigenfunction normalized by $\int_\Omega \psi_1^2\, d\mathbf{x} = 1$.

(b) (1 point) Write a variational principle that can be used to calculate the second eigenvalue σ_2 of (8). (Hint: need to write the trial space in a clear way).

8. (2 points) Let Ω be a bounded domain in 3-D, and consider the two eigenvalue problems

$$\nabla\cdot[p(\mathbf{x}\nabla\phi] - q(x)\phi = -\lambda r(\mathbf{x})\phi\,, \quad \mathbf{x}\in\Omega\,; \qquad \phi = 0\,, \quad \mathbf{x}\in\partial\Omega\,. \tag{9}$$

where $p>0$, $r>0$ and $q\geq 0$ in Ω, and the corresponding problem with a non-flux condition

$$\nabla\cdot[p(\mathbf{x}\nabla\psi] - q(x)\psi = -\sigma r(\mathbf{x})\psi\,, \quad \mathbf{x}\in\Omega\,; \qquad \partial_n\psi = 0\,, \quad \mathbf{x}\in\partial\Omega\,. \tag{10}$$

Prove that for the smallest eigenvalues of (9) and (10) we must have $\sigma_1 \leq \lambda_1$.

9. (4 points) Now let Ω be the unit sphere in 3-D and consider the eigenvalue problem

$$\Delta\phi + \lambda\phi = 0\,, \quad \mathbf{x}\in\Omega\,; \qquad \phi = 0\,, \quad \mathbf{x}\in\partial\Omega\,. \tag{11}$$

Let λ_1 be the smallest eigenvalue of (11).

(a) (2 points) Suppose that on the upper hemisphere of $\partial\Omega$ we replace the condition $\phi = 0$ with $\partial_n\phi = 0$. Does the corresponding lowest eigenvalue decrease or increase in comparison with λ_1? Explain.

(b) (2 points) Suppose that we remove a small sphere Ω_δ from Ω and impose the additional condition that $\phi = 0$ on $\partial\Omega$. Does the corresponding lowest eigenvalue decrease or increase in comparison with λ_1? Explain.

10. (2 points) Consider the Green's function problem for $G(\mathbf{x},\xi,t)$ in $\mathbb{R}^2$ for the heat equation where there is bulk decay with $\mu>0$ constant,

$$G_t = D\Delta G - \mu G\,, \quad \mathbf{x}\in\mathbb{R}^2\,,\ t\geq 0\,, \qquad G(\mathbf{x},\xi,0) = \delta(\mathbf{x}-\xi)\,. \tag{12}$$

Here $D>0$. Determine an explicit formula for G by first writing $G = e^{-\sigma t}v$ for some convenient σ that you are to choose, and then writing down the solution for v.

11. (6 points) (Longer Question) With a smoke-stack at $x=0$ and a wind blowing in the positive x direction at the constant speed $V>0$, the steady-state diffusion of contaminent $c(x,y,z)$ in the lateral directions, but above the ground $z=0$, satisfies

$$\begin{aligned} &Vc_x = D\left(c_{yy} + c_{zz}\right)\,, \qquad x\geq 0\,, \quad 0\leq z<\infty\,, \qquad -\infty<y<\infty\,, \\ &c(0,y,z) = \delta(y, z-z_0)\,, \qquad c_z = 0\,, \quad \text{on} \quad z=0\,, \end{aligned} \tag{13}$$

where $D>0$ is a constant and $z_0>0$. The boundary condition $c_z = 0$ on $z=0$ models no flux of contaminent through the ground, and in our model we have neglected diffusion in the x direction.

(a) (2 points) Show that $\mathcal{D} \equiv D/V$ is an "effective" diffusivity. What are the units of $\mathcal{D}$? Which variable, i.e. x, y, z, is the "time-like" variable?

(b) (2 points) Write down the solution for c by finding the image point.

(c) (2 points) From your explicit solution, calculate the contaniment concentration on the ground $c(x, 0, 0)$ ahead of the smoke-stack. Where does it have a maximum?

12. (6 points) (Longer Question) Let $a(x)$ be a continuous function on $0 \leq x \leq 1$ and consider the BVP

$$Lu \equiv \phi'' - (a(x)\phi)' = f(x)\,, \quad 0 \leq x \leq 1\,; \qquad \phi'(0) = a(0)\phi(0)\,, \quad \phi'(1) = a(1)\phi(1)\,, \tag{14}$$

(a) (2 points) Determime the homogeneous adjoint problem.

(b) (2 points) Is there a nontrivial solution to the homogeneous adjoint problem?

(c) (2 points) What is the solvability condition for (14)?