MATH 550 HW #1

PROBLEM 1 (TRANSCENDENTAL EQUATIONS)

(i) SHOW THAT THE LARGE POSITIVE ROOTS OF $\tan x = x$ HAVE THE FORM

$$x = \mu + \frac{c_0}{\mu} + \frac{c_1}{\mu^3} + \dots \quad \text{WHEN } \mu = (n + 1/2)\pi \to +\infty,$$

FOR SOME c_0, c_1 TO BE FOUND. (HINT: YOU WILL NEED $\tan^{-1} y$ FOR $y \approx \pi/2$).

(ii) FOR $\varepsilon \to 0^+$ DERIVE A THREE-TERM EXPANSION FOR THE ROOT OF $\varepsilon x e^x = 1$.

(iii) THE ASYMPTOTIC EXPANSION OF THE BESSEL FUNCTION $J_\nu(x)$ AS $x \to \infty$ IS

$$J_\nu(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} \left[\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) - \frac{(4\nu^2 - 1)}{8x} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) \right]$$

FROM THIS EXPRESSION DERIVE A TWO-TERM APPROXIMATION FOR THE ROOTS OF $J_\nu(x) = 0$ FOR $x \gg 1$.

PROBLEM 2 (SIMPLE POLYNOMIALS)

(i) DERIVE A THREE-TERM EXPANSION FOR EACH OF THE ROOTS OF $x^2 + (\varepsilon - 4)x + (3 + 2\varepsilon) = 0$ AS $\varepsilon \to 0^+$.

(ii) DERIVE A THREE TERM EXPANSION FOR THE 2 ROOTS OF $(x-2)^2(x-3) + \varepsilon = 0$ FOR WHICH $x \to 2$ AS $\varepsilon \to 0^+$.

PROBLEM 3 (EIGENVALUE PERTURBATION THEORY)

LET A AND B be the 20×20 MATRICES

$$A = \begin{pmatrix} 20 & 20 & & & \\ & 19 & \ddots & & 0 \\ & & 18 & \ddots & \\ & 0 & & \ddots & 20 \\ & & & & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & \cdots & & 0 \\ 0 & & & \vdots \\ \vdots & & 0 & \vdots \\ 0 & & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

CONSIDER THE EIGENVALUE PROBLEM

$$A\underline{x} + \varepsilon B\underline{x} = \lambda \underline{x} \qquad \text{FOR} \quad \varepsilon = 0 \quad \lambda = \lambda_n = n$$

$$n = 1, \ldots 20.$$

(i) USING Eigenvalue perturbation theory show that for $\varepsilon \ll 1$

$$\lambda_n = n + \varepsilon \frac{\underline{X}_0^{*T} B \underline{X}_0}{\underline{X}_0^{*T} \underline{X}_0} + \ldots \quad \text{where} \quad A^T \underline{X}_0^* = n \underline{X}_0^*.$$

BY calculating $\underline{X}_0, \underline{X}_0^*$ explicitly show that

$$(*) \qquad \lambda_n = n + \frac{\varepsilon\,(20^{19})(-1)^n}{(20-n)!\,(n-1)!} + \ldots$$

(ii) WHICH ROOT IS THE MOST sensitive to perturbations? (i.e. MAXIMIZE coefficient IN $(*)$). HOW SMALL DOES ε HAVE TO BE IN ORDER THAT THIS MOST SENSITIVE ROOT ONLY HAS A 2% PERTURBATION TO IT'S VALUE?

(iii) NOW CONSIDER $A\underset{\sim}{x} + \varepsilon B \underset{\sim}{x} = \underset{\sim}{b}$ WITH A AND B AS ABOVE AND $\underset{\sim}{b}$ ARBITRARY. SINCE B HAS RANK ONE IT CAN BE WRITTEN AS $B = \underset{\sim}{u}\underset{\sim}{v}^T$ FOR SOME VECTORS $\underset{\sim}{u}$ AND $\underset{\sim}{v}$.

FOR $\varepsilon \ll 1$ DERIVE THAT

$$x = \left(A^{-1} - \frac{A^{-1}\varepsilon B A^{-1}}{1 + \varepsilon \underset{\sim}{v}^T A^{-1} \underset{\sim}{u}}\right) \underset{\sim}{b} \qquad \text{(SHERMAN - WOODBURY - MORRISON FORMULA)}$$

$$x \sim A^{-1}\left[I - \varepsilon B A^{-1}\right] \underset{\sim}{b} \qquad \text{FOR } \varepsilon \ll 1$$

PROVIDED THAT $\varepsilon \underset{\sim}{v}^T A^{-1} \underset{\sim}{u} \ll 1$.

BY CALCULATING A^{-1} SHOW THAT WE REQUIRE

$$\varepsilon \underset{\sim}{v}^T A^{-1} \underset{\sim}{u} = \frac{\varepsilon\,(-20)^{19}}{20!} \ll 1$$

PROBLEM 4 CONSIDER A PROJECTILE THROWN UPWARDS UNDER THE EFFECT OF THE INVERSE SQUARE LAW. IF $X \geq 0$ MEASURES DISTANCE ABOVE GROUND, AND R IS RADIUS OF EARTH, THEN

$$\frac{d^2 x}{dt^2} = -gR^2/(x+R)^2 \quad ; \quad x(0) = 0, \quad \frac{dx}{dt}(0) = V_0 .$$

ASSUME V_0 IS SMALL SO THAT EFFECT OF INVERSE SQUARE LAW IS "WEAK".

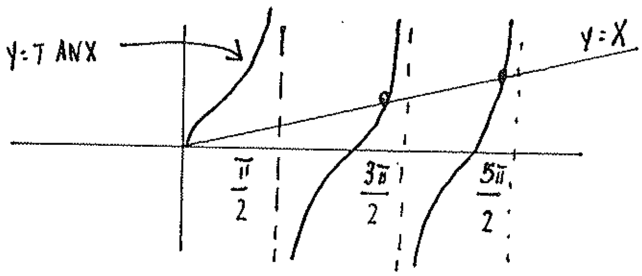
(i) BY NON-DIMENSIONALIZING, FIND A SMALL PARAMETER ε FOR THIS SITUATION.

(ii) FOR $\varepsilon \to 0$, SOLVE THE PROBLEM TO OBTAIN A TWO-TERM APPROXIMATION FOR THE TIME TO REACH MAXIMUM ALTITUDE.

M550 HW 1 SOLUTIONS

SOLUTION 1

(i) FIND LARGE POSITIVE ROOTS OF $\tan x = x$.



TO A FIRST APPROXIMATION

$$x_n \approx (n + 1/2)\pi \quad \text{AS} \quad n \to \infty.$$

NOW $x = n\pi + \tan^{-1}(x)$ WHERE $\tan^{-1} x$ IS THE PRINCIPAL BRANCH OF $\tan^{-1}$. THUS, WE CONSIDER THE ITERATION SCHEME

$$x_{m+1} = n\pi + \tan^{-1}(x_m) \qquad m = 0, 1, 2, \ldots \qquad x_0 = (n + 1/2)\pi.$$

WE RECALL THAT $\tan^{-1} z \sim \frac{\pi}{2} - \frac{1}{z} + \frac{1}{3z^3} + \ldots$ AS $z \to +\infty$.

NOW $x_1 = n\pi + \tan^{-1}(x_0) = (n + 1/2)\pi + \ldots \qquad \tan^{-1}(\infty) = \pi/2.$

NOW $x_2 = n\pi + \tan^{-1}(x_1) = n\pi + \left(\frac{\pi}{2} - \frac{1}{(n+1/2)\pi} + \ldots\right) = \mu - 1/\mu$

WITH $\mu = (n + 1/2)\pi$.

$$x_3 = n\pi + \tan^{-1}(x_2) = n\pi + \frac{\pi}{2} - \frac{1}{\mu - 1/\mu} + \frac{1}{3\mu^3} + \ldots$$

SO $x_3 = \mu - \frac{1}{\mu(1 - 1/\mu^2)} + \frac{1}{3\mu^3} + \ldots = \mu - \frac{1}{\mu}\left(1 + \frac{1}{\mu^2}\right) + \frac{1}{3\mu^3} + \ldots$

HENCE, $x_3 = \mu - \frac{1}{\mu} - \frac{2}{3\mu^3} + \ldots$

THUS $\underline{x \sim \mu - \frac{1}{\mu} - \frac{2}{3\mu^3} + \ldots} \qquad \mu = (n + 1/2)\pi \to \infty.$

(ii) FOR $\varepsilon \to 0^+$ FIND A THREE TERM EXPANSION FOR THE ROOTS

OF $\varepsilon x e^x = 1$.

WE WRITE $\quad \varepsilon e^x = \dfrac{1}{x} \qquad \ln \varepsilon + x = -\ln x.$

SO $\quad x = -\ln x - \ln \varepsilon. \qquad \varepsilon \to 0^+.$

WE SET UP ITERATION SCHEME

$$x_{n+1} = -\ln \varepsilon - \ln x_n \ , \quad n = 0, 1, 2, 3, \ldots$$

NOW $\quad x_0 = -\ln \varepsilon$. WE CALCULATE,

$$x_1 = -\ln \varepsilon - \ln[-\ln \varepsilon]$$

$$x_2 = -\ln \varepsilon - \ln[-\ln \varepsilon - \ln(-\ln \varepsilon)]$$

$$x_2 = -\ln \varepsilon - \ln\left[-\ln \varepsilon \left(1 + \frac{\ln(-\ln \varepsilon)}{\ln \varepsilon}\right)\right]$$

$$x_2 = -\ln \varepsilon - \ln(-\ln \varepsilon) - \ln\left(1 + \frac{\ln(-\ln \varepsilon)}{\ln \varepsilon}\right)$$

NOW $\quad \dfrac{\ln(-\ln \varepsilon)}{\ln \varepsilon} \ll 1$ AS $\varepsilon \to 0^+$ AND $\ln(1+h) \sim h$ AS $h \to 0$.

HENCE, $\quad x_2 \sim -\ln \varepsilon - \ln(-\ln \varepsilon) - \dfrac{\ln(-\ln \varepsilon)}{\ln \varepsilon} \quad$ AS $\varepsilon \to 0$

OR EQUIVALENTLY,

$$x \sim \ln\left(\frac{1}{\varepsilon}\right) + \ln\left(\ln(1/\varepsilon)\right) + \frac{\ln(\ln(1/\varepsilon))}{\ln(1/\varepsilon)} + \ldots$$

(iii) THE ZEROES SATISFY FOR $X \gg 1$:

$$\cos\left(X - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) = \frac{(4\nu^2-1)}{8X}\sin\left(X - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

HENCE $$\tan\left(X - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) = \frac{8X}{4\nu^2-1}.$$

SO $$X - \frac{\nu\pi}{2} - \frac{\pi}{4} = n\pi + \tan^{-1}\left(\frac{8X}{4\nu^2-1}\right)$$ n = integer
$\tan^{-1}$ = PRINCIPAL BRANCH.

BUT (*) $$\tan^{-1} Z = \begin{cases} \pi/2 - 1/Z + \cdots & Z \to +\infty \\ -\pi/2 - 1/Z + \cdots & Z \to -\infty \end{cases}$$

NOW CONSIDER ITERATION SCHEME:

$$X_{m+1} = \frac{\nu\pi}{2} + \left(n + \frac{1}{4}\right)\pi + \tan^{-1}\left(\frac{8X_m}{4\nu^2-1}\right), \quad m = 0, 1, 2, \ldots$$

WITH $$X_0 = \left(n + \frac{\nu}{2} + \frac{1}{4}\right)\pi.$$

- NOW SUPPOSE $4\nu^2 - 1 > 0$. THEN,

$$X_1 = \left(n + \frac{\nu}{2} + \frac{1}{4}\right)\pi + \frac{\pi}{2} = \left(n + \frac{\nu}{2} + \frac{3}{4}\right)\pi$$

$$X_2 = \left(n + \frac{\nu}{2} + \frac{1}{4}\right)\pi + \left[\frac{\pi}{2} - \frac{(4\nu^2-1)}{8X_0} + \cdots\right]$$

SO $$X_2 = \left(n + \frac{\nu}{2} + \frac{3}{4}\right)\pi - \frac{(4\nu^2-1)}{8\left[n + \frac{\nu}{2} + \frac{3}{4}\right]\pi} + \cdots \quad n \to \infty.$$

- NOW SUPPOSE $4\nu^2 - 1 < 0$. THEN

$$X_1 = \left(n + \frac{\nu}{2} + \frac{1}{4}\right)\pi - \frac{\pi}{2} = \left(n + \frac{\nu}{2} - \frac{1}{4}\right)\pi$$

WE OBTAIN, $$X_2 = \left(n + \frac{\nu}{2} - \frac{1}{4}\right)\pi - \frac{(4\nu^2-1)}{8\left[n + \frac{\nu}{2} - \frac{1}{4}\right]\pi} + \cdots \quad n \to \infty.$$

SOLUTION 2

(i) $x^2 + (\varepsilon - 4)x + (3 + 2\varepsilon) = 0.$

FOR $\varepsilon = 0 \longrightarrow x^2 - 4x + 3 = (x-3)(x-1) = 0. \longrightarrow x = 1, 3.$ (simple ROOT)

- WE LET $x = 1 + a_1 \varepsilon + a_2 \varepsilon^2 + \dots$

$$(1 + a_1 \varepsilon + a_2 \varepsilon^2 + \dots)^2 + (\varepsilon - 4)(1 + a_1 \varepsilon + a_2 \varepsilon^2) + (3 + 2\varepsilon) = 0$$

$$\longrightarrow 1 + 2a_1 \varepsilon + \varepsilon^2 (2a_2 + a_1^2) - 4 + \varepsilon(1 - 4a_1) + \varepsilon^2 (a_1 - 4a_2) + (3 + 2\varepsilon) = 0$$

HENCE, $\varepsilon(2a_1 + 1 - 4a_1 + 2) + \varepsilon^2 (2a_2 + a_1^2 + a_1 - 4a_2) + \dots = 0.$

HENCE $-2a_1 + 3 = 0 \longrightarrow a_1 = 3/2$

$-2a_2 + a_1^2 + a_1 = 0 \longrightarrow -2a_2 = -\frac{9}{4} - \frac{3}{2} = -\frac{15}{4}.$

SO $a_1 = 3/2$, $a_2 = 15/8 \longrightarrow \underline{x = 1 + \frac{3\varepsilon}{2} + \frac{15}{8}\varepsilon^2 + \dots}$

- WE LET $x = 3 + a_1 \varepsilon + a_2 \varepsilon^2 + \dots$ SO,

$$(3 + a_1 \varepsilon + a_2 \varepsilon^2)^2 + (\varepsilon - 4)(3 + a_1 \varepsilon + a_2 \varepsilon^2 + \dots) + (3 + 2\varepsilon) = 0$$

$$\longrightarrow 9 - 12 + 6a_1 \varepsilon + 6a_2 \varepsilon^2 + a_1^2 \varepsilon^2 + 3\varepsilon + a_1 \varepsilon^2 - 4a_1 \varepsilon - 4a_2 \varepsilon^2 + 3 + 2\varepsilon = 0$$

SO $\varepsilon(6a_1 + 3 - 4a_1 + 2) + \varepsilon^2 (6a_2 + a_1^2 + a_1 - 4a_2) = 0$

$\longrightarrow 2a_1 + 5 = 0 \longrightarrow a_1 = -5/2$

$2a_2 = -a_1^2 - a_1 = -\frac{25}{4} + \frac{5}{2} = -\frac{15}{4} \longrightarrow a_2 = -15/8$

SO $\underline{x = 3 - \frac{5\varepsilon}{2} - \frac{15}{8}\varepsilon^2 + \dots}$

(ii) $(x-2)^2(x-3)+\varepsilon=0$ WITH $x\to 2$ AS $\varepsilon\to 0$.

DEFINE $p(x)=(x-2)^2(x-3)$.

THEN $p(2)=p'(2)=0$

NOW Let $x=2+a_0\varepsilon^{1/2}+a_1\varepsilon+\dots$

NOW $p(2+a_0\varepsilon^{1/2}+a_1\varepsilon)+\varepsilon=0$

$$p(2)+\frac{p''(2)}{2!}\left(a_0\varepsilon^{1/2}+a_1\varepsilon\right)^2+\frac{p'''(2)}{3!}\left(a_0\varepsilon^{1/2}+\dots\right)^3+\varepsilon=0$$
$$+\dots$$

SO $$\frac{p''(2)}{2}\left[a_0^2\varepsilon+2a_0a_1\varepsilon^{3/2}\right]+\frac{p'''(2)}{6}\left(a_0^3\varepsilon^{3/2}+\dots\right)+\varepsilon=0$$

SO $$\varepsilon\left[\frac{p''(2)}{2}a_0^2+1\right]+\varepsilon^{3/2}\left[a_0a_1p''(2)+\frac{a_0^3}{6}p'''(2)\right]+\dots=0.$$

SO $$a_0^2=-2/p''(2) \qquad a_1p''(2)=-\frac{a_0^2}{6}p'''(2)=\frac{1}{3}\frac{p'''(2)}{p''(2)}.$$

SO $$a_0^2=-2/p''(2), \quad a_1=\frac{1}{3}\frac{p'''(2)}{[p''(2)]^2}.$$

NOW $p'(x)=(x-2)^2+2(x-2)(x-3)$

$p''(x)=2(x-2)+2(x-3)+2(x-2)\longrightarrow p''(2)=-2$

$p'''(x)=6\longrightarrow p'''(2)=6.$

SO $$a_0^2=1, \quad a_1=\frac{1}{3}\frac{6}{4}=\frac{1}{2}\longrightarrow a_0=\pm 1$$

SO $$x\sim 2\pm\sqrt{\varepsilon}+\varepsilon/2 \quad \text{AS} \quad \varepsilon\to 0.$$

SOLUTION 3 LET $A = \begin{pmatrix} 20 & 20 & & 0 \\ & 19 & \ddots & \\ & & \ddots & 20 \\ 0 & & & 1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & & \\ \vdots & & \\ 1 & 0 \cdots & 0 \end{pmatrix}$

WE WRITE THE EIGENVALUE PROBLEM

$$A\underline{x} + \varepsilon B \underline{x} = \lambda \underline{x}$$

WHEN $\varepsilon = 0$, $\lambda = \lambda_0 = n$, $n = 1, \ldots, 20$.

(i) WE FIX THE EXPANSION AS

$$\underline{x} = \underline{x}_0 + \varepsilon \underline{x}_1 + \cdots \qquad \text{WITH} \quad A\underline{x}_0 = n\underline{x}_0 .$$
$$\lambda = n + \varepsilon \lambda_1 + \cdots$$

THEN $\varepsilon A \underline{x}_1 + \varepsilon B \underline{x}_0 = \varepsilon n \underline{x}_1 + \varepsilon \lambda_1 \underline{x}_0 + \cdots$

HENCE, (*) $(A - nI)\underline{x}_1 = -B\underline{x}_0 + \lambda_1 \underline{x}_0$.

NOW LET $\underline{\phi}_0$ DENOTE THE NONTRIVIAL SOLUTION OF

$$(A - nI)^T \underline{\phi}_0 = 0 \longrightarrow A^T \underline{\phi}_0 = n \underline{\phi}_0 .$$

THEN, THE SOLVABILITY CONDITION FOR (*) IS THAT

$$-\underline{\phi}_0^T B \underline{x}_0 + \lambda_1 \underline{\phi}_0^T \underline{x}_0 = 0$$

HENCE
$$\lambda_1 = \frac{\underline{\phi}_0^T B \underline{x}_0}{\underline{\phi}_0^T \underline{x}_0} . \qquad (1)$$

WE NOW CALCULATE $\underline{\phi}_0$ AND $\underline{x}_0$ EXPLICITLY.

IT IS SIMPLE TO SHOW FOR THE UNPERTURBED VALUE $\lambda_0 = n$ THAT

$$\underline{X}_0 = \begin{pmatrix} \frac{(-20)^{20-n}}{(20-n)!} \\ \vdots \\ \frac{(-20)^2}{2!} \\ -20 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} 20-n \\ \\ \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n-1 \text{ zeroes} \end{matrix} \qquad \underline{\phi}_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 20 \\ (20)^2/2! \\ \vdots \\ \frac{(20)^{n-1}}{(n-1)!} \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} 20-n \text{ zeroes} \\ \\ \\ \\ \\ \\ \end{matrix}$$

THEN

$$B\underline{X}_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-20)^{20-n}/(20-n)! \end{pmatrix} \qquad \underline{\phi}_0^T B \underline{X}_0 = + \frac{(-20)^{20-n}(20)^{n-1}}{(20-n)!\,(n-1)!}$$

SO

$$\underline{\phi}_0^T B \underline{X}_0 = \frac{(-1)^n (20)^{19}}{(20-n)!\,(n-1)!}$$

NOW WE CALCULATE $\underline{\phi}_0^T \underline{X}_0 = 1$.

HENCE FROM (1)

$$\lambda_1 = \frac{\underline{\phi}_0^T B \underline{X}_0}{\underline{\phi}_0^T \underline{X}_0} = \frac{(-1)^n (20)^{19}}{(20-n)!\,(n-1)!}$$

THEN FOR $\varepsilon \to 0^+$, THE PERTURBATION TO $\lambda_0 = n$ IS

$$\lambda \sim n + \frac{\varepsilon (-1)^n (20)^{19}}{(20-n)!\,(n-1)!} + \cdots \quad \text{AS} \quad \varepsilon \to 0.$$

(ii) WE CALCULATE THE COEFFICIENTS

$$a_n = \frac{(-1)^n \, 20^{19}}{(20-n)!\,(n-1)!}$$

WE OBTAIN FROM A NUMERICAL EVALUATION THAT THE LARGEST OF THESE IS

$$a_{10} = .39815 \times 10^{13} = -a_{11} .$$

THUS $\quad \varepsilon a_{10} << 1$ IS NEEDED.

FOR A 2% PERTURBATION,

$$\lambda_{10} = 10 \left(1 + \frac{\varepsilon a_{10}}{10} \right)$$

WANT $\quad \dfrac{\varepsilon a_{10}}{10} = .02$

SO $\quad \varepsilon = \dfrac{.20}{a_{10}} = \dfrac{.20}{.39815 \times 10^{13}} \cong .50 \times 10^{-13} .$

(iii) <u>LEMMA</u> (SHERMAN-MORRISON)

LET $\underset{\sim}{u}, \underset{\sim}{v}$ BE TWO VECTORS AND A IS A NONSINGULAR SQUARE MATRIX. THEN

$$(A + \underset{\sim}{u}\underset{\sim}{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T A^{-1}}{1 + \underset{\sim}{v}^T A^{-1}\underset{\sim}{u}} .$$

<u>PROOF</u> IF $(I + X)$ IS INVERTIBLE (X IS $N \times N$), THEN

$$(1) \qquad (I + X)^{-1} = I - X + X^2 - X^3 + \ldots = \sum_{K=0}^{\infty} (-1)^K X^K$$

THEN $$(A + \underset{\sim}{u}\underset{\sim}{v}^T)^{-1} = \left(A\left[I + A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T\right]\right)^{-1} = (I + A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T)^{-1}A^{-1}.$$

NOW USING (1) WE OBTAIN,

$$(A + \underset{\sim}{u}\underset{\sim}{v}^T)^{-1} = \big(I - A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T + A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T - A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T + \ldots \big) A^{-1}$$

$$(A + \underset{\sim}{u}\underset{\sim}{v}^T)^{-1} = A^{-1} - A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T A^{-1} + A^{-1}\underset{\sim}{u}(\underset{\sim}{v}^T A^{-1}\underset{\sim}{u})\underset{\sim}{v}^T A^{-1} - A^{-1}\underset{\sim}{u}(\underset{\sim}{v}^T A^{-1}\underset{\sim}{u})(\underset{\sim}{v}^T A^{-1}\underset{\sim}{u})\underset{\sim}{v}^T A^{-1} + \ldots$$

NOW DEFINE $\sigma = \underset{\sim}{v}^T A^{-1}\underset{\sim}{u}$, WHICH IS A SCALAR.

THEN $$(A + \underset{\sim}{u}\underset{\sim}{v}^T)^{-1} = A^{-1} - A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T A^{-1}\left(1 - \sigma + \sigma^2 - \sigma^3 + \sigma^4 \ldots\right)$$

SUMMING THE SERIES ASSUMING THAT $|\sigma| < 1$ GIVES,

$$(A + \underset{\sim}{u}\underset{\sim}{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T A^{-1}}{1 + \underset{\sim}{v}^T A^{-1}\underset{\sim}{u}} .$$

NOW REPLACING $\underset{\sim}{u}$ BY $\varepsilon\underset{\sim}{u}$ WE CLAIM THAT IF $B = \underset{\sim}{u}\underset{\sim}{v}^T$ (i.e. A RANK 1 MATRIX) THEN

$$(A + \varepsilon\underset{\sim}{u}\underset{\sim}{v}^T)^{-1} = A^{-1} - \frac{\varepsilon A^{-1}\underset{\sim}{u}\underset{\sim}{v}^T A^{-1}}{1 + \varepsilon\underset{\sim}{v}^T A^{-1}\underset{\sim}{u}} .$$

FOR OUR PROBLEM $u = (0, \dots, 0, 1)^T$, $v = (1, 0, \dots, 0)^T$.

THEN $\underline{u}\,\underline{v}^T = B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & \\ 1 & 0 & \cdots & 0 \end{pmatrix}$

NOW FOR $A\underline{x} + \varepsilon B \underline{x} = \underline{b}$ WE GET

$$(A + \varepsilon \underline{u}\,\underline{v}^T)\,\underline{x} = \underline{b}$$

SO $$\underline{x} = (A + \varepsilon \underline{u}\,\underline{v}^T)^{-1}\underline{b} = \left(A^{-1} - \frac{\varepsilon A^{-1}\underline{u}\,\underline{v}^T A^{-1}}{1 + \varepsilon \underline{v}^T A^{-1}\underline{u}} \right)\underline{b} \qquad (1)$$

BY THE LEMMA.

THIS IS VALID PROVIDED THAT $|\varepsilon \underline{v}^T A^{-1}\underline{u}| < 1$. THEN FOR $\varepsilon \ll 1$

$$(2) \qquad \underline{x} \sim A^{-1}(I - \varepsilon \underline{u}\,\underline{v}^T A^{-1})\,\underline{b}.$$

WE REQUIRE $\varepsilon \underline{v}^T A^{-1}\underline{u} < 1$ FOR INVERSION AND $\varepsilon \underline{v}^T A^{-1}\underline{u} \ll 1$ FOR (2) TO BE A GOOD APPROXIMATION FOR THE EXACT FORMULA (1).

WE READILY CALCULATE

$$A^{-1} = \begin{pmatrix} \frac{19!}{20!} & (-20)\frac{18!}{20!} & \cdots & \boxed{(-20)^{19}\frac{0!}{20!}} \\ & \frac{18!}{19!} & (-20)\frac{17!}{19!} \quad \cdots & (-20)^{18}\frac{0!}{19!} \\ & & \ddots \quad \frac{17!}{18!} & \\ & & & \ddots \quad \frac{0!}{1!} \end{pmatrix} \quad \leftarrow A_{IN}$$

SO $$\underline{v}^T A^{-1}\underline{u} = (A)_{IN} = \frac{(-20)^{19}}{20!}$$

HENCE WE REQUIRE $$\frac{\varepsilon(-20)^{19}}{20!} \ll 1$$

THIS REQUIRES

$2.155 \times 10^6\, \varepsilon \ll 1$

i.e. $\varepsilon \approx 10^{-7}$.

PROBLEM 4 $\quad X'' = \dfrac{-gR^2}{(X+R)^2}, \quad X(0)=0, \quad X'(0)=V_0.$

(i) WITH NO INVERSE SQUARE LAW $X'' = -g$ SO $X = -\dfrac{gt^2}{2} + V_0 t$

THEN $X' = 0$ WHEN $-gt + V_0 = 0 \rightarrow t = T = V_0/g$.

THE MAX HEIGHT $H = -\dfrac{g}{2}\left(\dfrac{V_0}{g}\right)^2 + V_0^2/g = V_0^2/2g$.

WE ASSUME THAT $\varepsilon \equiv H/R \ll 1$ SO THAT MAX HEIGHT IS SMALL COMPARED TO EARTH'S RADIUS.

NOW PUT $X = HU$, $t = T\tau$. THEN $\quad X'(0) = \dot{U}(0)\dfrac{H}{T} = \dot{U}(0)\dfrac{V_0}{2}$.

THEN, $\dfrac{H}{T^2}\ddot{U} = \dfrac{-g}{\left(1+\frac{H}{R}U\right)^2}$ $\quad$ BUT $\dfrac{H}{T^2} = \dfrac{V_0^2/2g}{(V_0/g)^2} = \dfrac{g}{2}$. $\quad$ HERE $\dot{} = \dfrac{d}{d\tau}$.

THUS

$$(*)\left\{\begin{array}{l} \ddot{U} = \dfrac{-2}{(1+\varepsilon U)^2} \\ \dot{U}(0) = 2, \quad U(0) = 0 \end{array}\right.$$

SO $\varepsilon = \dfrac{H}{R} = \dfrac{V_0^2/2g}{R} \ll 1$

(ii)

NOW EXPAND AS $U = U_0 + \varepsilon U_1 + \ldots$

THEN USING $(1+\varepsilon U)^{-2} \sim 1 - 2\varepsilon U + \ldots$ WE OBTAIN

$$\ddot{U}_0 + \varepsilon \ddot{U}_1 = -2\left(1 - 2\varepsilon U_0 + \ldots\right)$$

$$\dot{U}_0(0) + \varepsilon \dot{U}_1(0) + \ldots = 2, \qquad U_0(0) + \varepsilon U_1(0) + \ldots = 0.$$

THUS $\quad \ddot{U}_0 = -2\,; \quad U_0(0) = 0, \quad \dot{U}_0(0) = 2 \longrightarrow \quad U_0 = -t^2 + 2t$

$$\ddot{U}_1 = 4U_0\,; \quad U_1(0) = \dot{U}_1(0) = 0.$$

THUS $\quad \ddot{u}_1 = 8t - 4t^2 \quad , \qquad u_1(0) = \dot{u}_1(0) = 0.$

SO $\quad u_1 = \frac{4}{3}t^3 - \frac{1}{3}t^4.$

THUS FOR $\varepsilon \ll 1$,

$$u \sim (-t^2 + 2t) + \varepsilon\left[\frac{4}{3}t^3 - \frac{1}{3}t^4\right]$$

NOW FIND t_{MAX} FOR WHICH $u' = 0$.

$$u' = (-2t + 2) + \varepsilon\left[4t^2 - \frac{4}{3}t^3\right]$$

EXPAND $\quad t_{MAX} = 1 + \varepsilon t_1 + \ldots$

$$0 = 2 - 2(1 + \varepsilon t_1) + \varepsilon\left[4 - 4/3\right] + \ldots$$

$$0 = -2\varepsilon t_1 + \varepsilon\left(8/3\right)$$

THUS $\quad t_1 = 4/3.$

WE CONCLUDE THAT $\quad t_{MAX} = 1 + 4\varepsilon/3 + \ldots$

THUS IN ORIGINAL TIME $\quad t_{MAX} = \frac{V_0}{g}\left(1 + \frac{4\varepsilon}{3} + \ldots\right) \quad , \quad \varepsilon = \frac{V_0^2/2g}{R} \ll 1.$

THE MAXIMUM HEIGHT REACHED IS

$$u_{MAX} = -(1 + \varepsilon t_1)^2 + 2(1 + \varepsilon t_1) + \varepsilon\left(4/3 - 1/3\right) = -(1 + 2\varepsilon t_1) + 2 + 2\varepsilon t_1 + \varepsilon.$$

THUS $\quad u_{MAX} = 1 + \varepsilon = 1 + \frac{V_0^2/2g}{R}.$

IN TERMS OF PHYSICAL DISTANCE $\quad X_{MAX} = \frac{V_0^2}{2g}\left(1 + \frac{V_0^2/2g}{R} + \ldots\right) \qquad (*)$

REMARK WE CAN ALSO FIND THE MAXIMUM HEIGHT OBTAINED THROUGH AN ENERGY INTEGRAL.

$$X'' = -\frac{gR^2}{(X+R)^2}$$

MULTIPLY BY X' AND INTEGRATE

$$\frac{1}{2} X'^2 = \frac{gR^2}{X+R} + C$$

BUT $X'(0) = V_0$, $X(0) = 0 \longrightarrow \quad C = \frac{1}{2} V_0^2 - gR.$

THUS $\quad \frac{1}{2} X'^2 = \frac{gR^2}{X+R} + \frac{1}{2} V_0^2 - gR$

NOW AT MAX. POSITION WE HAVE $X'(t_{MAX}) = 0.$

THUS $\quad \frac{gR^2}{X_m + R} - gR + \frac{1}{2} V_0^2 = 0 \quad$ WHERE $\quad X_m = X(t_{MAX})$

SO $\quad -\frac{X_m gR}{X_m + R} + \frac{1}{2} V_0^2 = 0.$

THUS $\quad \frac{X_m}{X_m + R} = \frac{V_0^2}{2gR} \equiv f$

SO $\quad (1-f) X_m = fR$

THUS $\quad X_m = \frac{fR}{1-f} \sim fR(1+f) \quad$ FOR $\quad f \ll 1$

THE MAX HEIGHT IS $\quad X_m = \frac{fR}{1-f} \sim fR(1+f) \quad$ WHEN $\quad f \ll 1$

THIS IS $\quad X_m = R\left(\frac{V_0^2}{g2R}\right)\left(1 + \frac{V_0^2}{2gR} + \dots\right)$ IN AGREEMENT WITH (*) ON PREVIOUS PAGE.