EXAMPLES OF GLOBAL CONTRIBUTION TO INTEGRALS

EXAMPLE 1 THE MODIFIED BESSEL FUNCTION OF SECOND KIND OF ORDER 0 IS

$$K_0(z) = \int_1^\infty \frac{e^{-zs}}{\sqrt{s^2-1}}\, ds.$$

SHOW THAT $K_0(z) \sim -\log(z/2) - \gamma$ AS $z \to 0^+$ WHERE γ IS EULER'S CONSTANT.

SOLUTION PUTTING $z=0$, $\int_1^\infty \frac{1}{\sqrt{s^2-1}}\, ds$ IS DIVERGENT DUE TO UPPER ENDPOINT.

AS $s \to \infty$, $\frac{1}{\sqrt{s^2-1}} \sim \frac{1}{s}$ AND SO WE WRITE

$$K_0(z) = \int_1^\infty \left(\frac{e^{-zs}}{\sqrt{s^2-1}} - \frac{e^{-zs}}{s} \right) ds + \int_1^\infty \frac{e^{-zs}}{s}\, ds.$$

NOW LET $t = zs$ IN SECOND INTEGRAL: $\frac{ds}{s} = \frac{dt}{t}$ SO

$$K_0(z) = \underbrace{\int_z^\infty \frac{e^{-t}}{t}\, dt}_{E_1} + \int_1^\infty \left(\frac{e^{-zs}}{\sqrt{s^2-1}} - \frac{e^{-zs}}{s} \right) ds$$

NOW WE WRITE $K_0(z) = E_1(z) + \int_1^\infty e^{-zs} \left(\frac{1}{\sqrt{s^2-1}} - \frac{1}{s} \right) ds.$

BUT NOW

$$\frac{1}{(s^2-1)^{1/2}} - \frac{1}{s} = \frac{1}{s\,(1-1/s^2)^{1/2}} - \frac{1}{s} = \frac{1}{s}\left(1 + \frac{1}{2s^2} + \cdots\right) - \frac{1}{s} = O\left(1/s^3\right)$$

THUS THE INTEGRAL CONVERGES WHEN $s=0$. WE THEN HAVE USING $E_1 \sim -\log z - \gamma$ AS $z \to 0^+$ THAT

$$K_0(z) \sim -\log z - \gamma + \int_1^\infty \left(\frac{1}{\sqrt{s^2-1}} - \frac{1}{s} \right) ds + o(1) \quad \text{AS } z \to 0^+.$$

THEN

$$I \equiv \int_1^\infty \left(\frac{1}{\sqrt{s^2-1}} - \frac{1}{s} \right) ds = \lim_{R\to\infty} \int_1^R \left(\frac{1}{\sqrt{s^2-1}} - \frac{1}{s} \right) ds = \lim_{R\to\infty} \left(\cosh^{-1}(s) - \ln s \right)\Big|_1^R$$

$$= \lim_{R\to\infty} \left[\cosh^{-1}(R) - \ln R \right] = \lim_{R\to\infty} \left[\ln\left(R + \sqrt{R^2-1}\right) - \ln R \right]$$

$$I = \lim_{R\to\infty} \left[\ln\left[R + R\left(1 - 1/2R^2 + \cdots\right)\right] - \ln R \right] = \lim_{R\to\infty} \left[\ln R + \ln 2 - \ln R + O(1/R) \right] = \ln 2.$$

THUS $\quad K_0(z) \sim -\log z - \gamma + \log 2 \quad$ AS $\quad z \to 0^+$.

EXAMPLE 2 FIND THE FIRST TWO TERMS AS $X \to 0$ FOR

$$I = \int_0^\infty \frac{xt}{(x^2+t)^{3/2}(1+t)}\,dt = x\int_0^\infty \frac{t}{(x^2+t)^{3/2}(1+t)}\,dt. \quad \text{LET } I_0 \equiv \int_0^\infty \frac{t}{(x^2+t)^{3/2}(1+t)}\,dt$$

- WHEN $t \gg 1$ THE INTEGRAND I_0 IS $O\left(\frac{t}{t^{5/2}}\right)$ AND SO WE HAVE NO PROBLEM WITH UPPER ENDPOINT.
- WHEN $X \ll 0$ THE INTEGRAND IS $O\left(\frac{t}{t^{3/2}}\right)$ AND SO WE HAVE NO PROBLEM WITH LOWER ENDPOINT.

AS SUCH $I = O(X)$ AS $X \to 0$.

NOW WRITE

$$I = x\int_0^\infty \frac{t}{t^{3/2}(1+t)}\,dt + x\int_0^\infty\left(\frac{t}{(x^2+t)^{3/2}(1+t)} - \frac{t}{t^{3/2}(1+t)}\right)dt.$$

THUS

$$I = x\int_0^\infty \frac{1}{\sqrt{t}\,(1+t)}\,dt + x\int_0^\infty \frac{t}{(1+t)}\left(\frac{1}{(x^2+t)^{3/2}} - \frac{1}{t^{3/2}}\right)dt.$$

LET $t = u^2$ IN FIRST INTEGRAL TO GET

$$I = 2x\int_0^\infty \frac{1}{1+u^2}\,du + x\,J \qquad J \equiv \int_0^\infty \frac{t}{(1+t)}\left(\frac{1}{(x^2+t)^{3/2}} - \frac{1}{t^{3/2}}\right)dt.$$

WE GET $\displaystyle\int_0^\infty \frac{1}{1+u^2}\,du = \tan^{-1}u\Big|_0^\infty = \pi/2$.

SO $\quad I = x\pi + x\,J. \quad (*)$

NOW IN INTEGRAL FOR J WE ASK WHETHER THE CONTRIBUTION ARISES FROM WHERE $t = O(x^2)$ "LOCAL" OR FROM $t = O(1)$ "GLOBAL".

IF $t = O(1)$, THEN $(x^2+t)^{-3/2} - t^{-3/2} = O(x^2)$ AS $X \to 0^+$ SO $J = O(x^1)$

IF $t = O(x^2)$, THEN $(x^2+t)^{-3/2} - t^{-3/2} = O(x^{-3})$ AS $X \to 0^+$.

THUS IF $t = O(x^2) \rightarrow$

$$\underbrace{\frac{t}{(1+t)}}_{O(x^2)}\underbrace{\left(\frac{1}{(x^2+t)^{3/2}}-\frac{1}{t^{3/2}}\right)}_{O(x^{-3})}\underbrace{dt}_{O(x^2)} = O(x^2)\,O(x^{-3})\,O(x^2) \Rightarrow J = O(x).$$

THUS THE DOMINANT CONTRIBUTION COMES FROM THE LOCAL REGION WHERE $t = O(x^2)$. WE LET $t = x^2 u$ IN J TO GET

$$J = \int_0^\infty \frac{x^2 u}{(1+x^2 u)}\left(\frac{1}{(1+u)^{3/2}}-\frac{1}{u^{3/2}}\right)du \sim x^2\int_0^\infty\left(\frac{u}{(1+u)^{3/2}}-\frac{1}{u^{1/2}}\right)du.$$

NOW LET $t = 1+u$ SO

$$J \sim x\int_1^\infty\left(\frac{t-1}{t^{3/2}}-\frac{1}{(t-1)^{1/2}}\right)dt = x\lim_{R\to\infty}\left[\int_1^R\left(\frac{1}{t^{1/2}}-\frac{1}{t^{3/2}}-\frac{1}{(t-1)^{1/2}}\right)dt\right]$$

$$J \sim x\lim_{R\to\infty}\left[2t^{1/2}+2t^{-1/2}-2(t-1)^{1/2}\Big|_1^R\right] = x\left[-2+\lim_{R\to\infty}\left(2R^{1/2}-2(R-1)^{1/2}\right)\right]$$

$$J \sim x\left[-2+\lim_{R\to\infty}\left(2R^{1/2}-2R^{1/2}\left(1-\tfrac{1}{2R}+\cdots\right)\right)\right] \sim x^2(-2+0)$$

THUS $J \sim -2x$ AS $x \rightarrow 0$.

WE CONCLUDE THAT $I \sim x\pi - 2x^2$ AS $x \rightarrow 0^+$.

PROBLEM 3 DEFINE $I = \int_0^p \frac{\log(1+2y^2)}{y}\,dy$. WE WANT ASYMPTOTICS FOR I AS $p \rightarrow \infty$.

SOLUTION WE NEED TO DEAL WITH UPPER ENDPOINT AS THIS IS WHERE THE DIVERGENCE ARISES FROM.

LET $x = 2y^2$ SO $dx = 4y\,dy$ $\quad \frac{dx}{4y^2} = \frac{dy}{y} = \frac{dx}{2x}$.

THEN $$I = \frac{1}{2}\int_0^{2p^2} \frac{\log(1+X)}{X}\, dX$$

NOW SPLIT THE RANGE: $$I = \frac{1}{2}\int_0^{1} \frac{\log(1+X)}{X}\, dX + \frac{1}{2}\int_1^{2p^2} \frac{\log(1+X)}{X}\, dX.$$

NOW ADD AND SUBTRACT

$$I = \frac{1}{2}\int_0^{1} \frac{\log(1+X)}{X}\, dX + \frac{1}{2}\int_1^{2p^2} \left(\frac{\log(1+X)}{X} - \frac{\log X}{X} \right) dX + \frac{1}{2}\int_1^{2p^2} \frac{\log X}{X}\, dX.$$

THE LAST INTEGRAL CAN BE DONE BY MAKING $u = \log X$ SO $du = \frac{1}{X}\, dX$

THE SECOND INTEGRAL CONVERGES AS $p \to \infty$ SINCE

$$\frac{\log(X+1) - \log X}{X} \sim \frac{\log X + \log(1+1/X) - \log X}{X} = O(1/X^2)$$

AS $X \to +\infty$. THUS

$$I \sim \frac{1}{2}\int_0^{1} \frac{\log(1+X)}{X}\, dX + \frac{1}{4}\left(\log(2p^2)\right)^2 + \underbrace{\frac{1}{2}\int_1^{\infty} \left(\frac{\log(1+X) - \log X}{X} \right) dX}_{\equiv J}.$$

SO IN LAST INTEGRAL $J \equiv \frac{1}{2}\int_1^{\infty} \frac{\log(1+1/X)}{X}\, dX$. LET $u = 1/X$ SO $du = -\frac{1}{X^2}\, dX$

$$dX = -\frac{1}{u^2}\, du$$

$$J = \frac{1}{2}\int_1^{0} \frac{u \log(1+u)}{(-u^2)}\, du = \frac{1}{2}\int_0^{1} \frac{\log(1+u)}{u}\, du$$

THUS $$I \sim \frac{1}{4}\left(\log(2p^2)\right)^2 + \int_0^{1} \frac{\log(1+X)}{X}\, dX + o(1) \quad \text{AS } p \to \infty.$$

WE CAN USE COMPLEX VARIABLES TO CALCULATE $\int_0^{1} \frac{\log(1+X)}{X}\, dX = \pi^2/12$

THUS $$I \sim \frac{1}{4}\left(\log(2p^2)\right)^2 + \frac{\pi^2}{12} = (\log p)^2 + (\log 2)\log p + \frac{1}{4}(\log 2)^2 + \frac{\pi^2}{12}$$

AS $p \to \infty$.