ASYMPTOTIC EXPANSIONS

THE O, o, $\sim$ SYMBOLS

(i) $f(z) = O(g(z))$ AS $z \to z_0$, MEANS f/g BOUNDED AS $z \to z_0$.

(ii) $f(z) = o(g(z))$ AS $z \to z_0$, MEANS $f/g \to 0$ AS $z \to z_0$

(iii) $f(z) \sim g(z)$ AS $z \to z_0$, MEANS $f/g \to 1$ AS $z \to z_0$.

THIS IS EQUIVALENT TO $f(z) = g(z)\left[1 + o(1)\right]$ AS $z \to z_0$.

EXAMPLES

(i) $\sin z = z + o(z)$, $\sin z = z + O(z^3)$, $\sin z \sim z - z^3/6$ AS $z \to 0$

(ii) IF $f(x) = x^2 + x\log x$, $f(x) \sim x^2$ AS $x \to \infty$, $f(x) \sim x \log x$ AS $x \to 0$.

(iii) $e^{-x} = o(x^{-n})$ AS $x \to \infty$ FOR ANY $n > 0$, $e^{-1/x} \sim 1$ AS $x \to \infty$.

(iv) $n! \sim (2\pi n)^{1/2}\, n^n \exp(-n)\left[1 + o(1)\right]$ AS $n \to \infty$.

AN ASYMPTOTIC POWER SERIES

LET $f(z)$ BE DEFINED IN A SET Ω (WHICH INCLUDES $z \to \infty$).
A POWER SERIES $\sum_{m=0}^{\infty} a_m z^{-m}$, convergent OR divergent, is called
AN Asymptotic power series of $f(z)$ IF FOR EACH FIXED $N \geq 0$
WE HAVE

$$f(z) = \sum_{n=0}^{N} a_n z^{-n} + O\left(z^{-(N+1)}\right) \quad \text{AS } z \to \infty.$$

THIS CAN ALSO BE WRITTEN AS $f(z) \sim \sum_{n=0}^{N} a_n z^{-n}$

THE KEY POINT IS THAT SUCH SERIES MAY NOT CONVERGE.

LET $S_N(Z) = \sum_{n=0}^{N} a_n Z^{-n}$. THEN,

$$|F(z) - S_N(z)| \longrightarrow 0 \quad \text{FOR } \underline{\text{FIXED}}\ N \text{ AS } Z \to \infty$$

<u>BUT</u> $$|F(z) - S_N(z)| \not\longrightarrow 0 \quad \text{FOR FIXED } Z \text{ AS } N \to \infty.$$

WE NOW ILLUSTRATE THE USE OF THESE SERIES.

CONSIDER THE EXPONENTIAL INTEGRAL

$$E_i(X) = \int_X^{\infty} e^{-t}\, t^{-1}\, dt.$$

WE WANT $E_i(X)$ FOR $X >> 1$. WE DEFINE $E_i^m(X) = \int_X^{\infty} e^{-t}\, t^{-m}\, dt.$

WE THEN INTEGRATE BY PARTS. $u = t^{-m}$, $du = -m t^{-m-1} dt$, $dv = e^{-t} dt$, $v = -e^{-t}$.

THIS YIELDS,

$$E_i^m(X) = -t^{-m} e^{-t} \Big|_X^{\infty} - m \int_X^{\infty} t^{-(m+1)} e^{-t}\, dt.$$

THUS $$E_i^m(X) = X^{-m} e^{-X} - m\, E_i^{m+1}(X) \quad \text{RECURRENCE Relation.}$$

WE WANT $E_i^1 \equiv E_i$. SO

$$E_i(X) = X^{-1} e^{-X} - E_i^2(X) = X^{-1} e^{-X} - (X^{-2} e^{-X} - 2 E_i^3(X))$$

$$E_i(X) = X^{-1} e^{-X} - X^{-2} e^{-X} + 2 E_i^3(X) = X^{-1} e^{-X} - X^{-2} e^{-X} + 2X^{-3} e^{-X} - 6 E_i^4(X)$$

CONTINUING ON WE OBTAIN

$$E_i(X) = e^{-X} \sum_{m=1}^{n} (-1)^{m+1} \frac{(m-1)!}{X^m} + (-1)^n n! \int_X^{\infty} e^{-t}\, t^{-(n+1)}\, dt.$$

THIS FORMULA IS EXACT. NOW DEFINE $S_n(X)$ AND THE REMAINDER $r_m(X)$ BY

$$S_m(X) = e^{-X} \sum_{m=1}^{n} (-1)^{m+1} \frac{(m-1)!}{X^m} \qquad r_n(X) = (-1)^n n! \int_X^{\infty} e^{-t}\, t^{-(n+1)}\, dt.$$

THUS, WE WRITE $$E_i(X) = S_n(X) + r_n(X)$$

<u>REMARKS</u> (i) FOR FIXED X, THE SERIES $S_n(X)$ DIVERGES AS $n \to \infty$. THUS, SINCE $E_i(X)$ IS FINITE, WE MUST HAVE $r_n(X) \to \infty$ FOR FIXED X AS $n \to \infty$.

(ii) NOTE THAT $|r_n(x)| \leq n! \int_x^\infty |t^{-(n+1)} e^{-t}| \, dt \leq \dfrac{n!}{x^{n+1}} \int_x^\infty e^{-t} \, dt \leq \dfrac{n! \, e^{-x}}{x^{n+1}}$

THUS $r_n(x) = O\left(x^{-n+1} e^{-x} \right)$ AND

$$(*) \quad E_i(x) = e^{-x} \sum_{m=1}^{n} (-1)^{m+1} \frac{(m-1)!}{x^m} + O\left(e^{-x} x^{-(n+1)} \right).$$

THEREFORE,

$$E_i(x) \sim e^{-x} \sum_{m=1}^{n} (-1)^{m+1} \frac{(m-1)!}{x^m}$$

IF X IS LARGE ENOUGH AND <u>n IS FIXED</u>, THEN $S_n(X)$ GIVE. A GOOD APPROXIMATION TO $E_i(x)$

<u>OPTIMAL NUMBER OF TERMS</u> :

FOR CONVENIENCE WE DEFINE $\tilde{E}_i(X)$ BY

$$\tilde{E}_i(x) = x e^x E_i(x)$$

THEN FROM (*) WE HAVE

$$\tilde{E}_i(x) = \sum_{m=1}^{n} (-1)^{m+1} \frac{(m-1)!}{x^{m-1}} + O\left(x^{-n} \right)$$

RE-WRITING GIVES

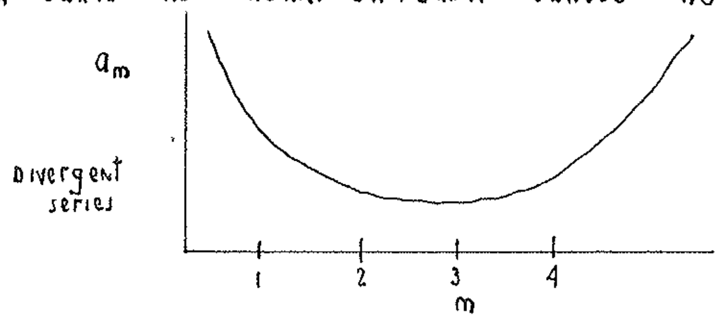
$$\tilde{E}_i(x) = \sum_{m=0}^{n-1} (-1)^m \frac{m!}{x^m} + O\left(x^{-n} \right)$$

LET $S_n = \displaystyle\sum_{m=0}^{n-1} (-1)^m \frac{m!}{x^m}$. THEN WE GET THE TABLE

X	$\tilde{E}_i(X)$	S_1	S_2	S_3	S_4	S_5	S_6	
5	.85212	1	.8000	.8800	.8352	.8736	.8352	.
10	.91563	1	.9000	.9200	.9140	.9164	.9152	.
100	.99019	1	.9900	.9902	.99019			

WHEN $X = 10$ THE OPTIMAL NUMBER OF TERMS IS 10 WITH $S = .91544$

NOW IF WE PLOT $a_m = m!/X^m$ AS A FUNCTION OF m FOR FIXED X WE (A CURVE AS SHOWN. DIFFERENT CURVES HOLD FOR DIFFERENT X.



IF WE TRUNCATE THE SERIES AT $N-1$ THEN THE ERROR COMMITED IS $O(X^{-N})$. IN PARTICULAR, THE REMAINDER r_N SATISFIE.

$$|r_N| \leq \frac{N!}{X^N} = a_N$$

WE WANT TO CHOOSE N SO THAT THE REMAINDER IS AS SMALL AS POSSIBLE THUS CHOOSE N SO THAT $a_{N-1} = a_N$ (OR the nearest integer).

THUS $(N-1)!\, X^{-(N-1)} = N!\, X^{-N} \Rightarrow N = X$. (OR nearest integer)

THUS THE OPTIMAL NUMBER OF TERMS IS $N = X$. FOR $X = 10$ THIS GIVES $N = 10$.

NOTICE, HOWEVER, THAT WHEN $X = 100$ WE HAVE $S_3 = .9902$ APPROXIMATING $\tilde{E}_i(100) = .99019$. WHEN $X = 10$, $S_{10} = .91544$ WHICH IS TO BE COMPARED WITH $\hat{E}_i(10) = .91563$.

REMARK (i) TO CHOOSE THE OPTIMAL NUMBER OF TERMS FOR FIXED X WE TRY TO MAKE THE REMAINDER TERM r_n AS SMALL AS POSSIBLE. THIS TYPICALLY MEANS THAT WE CHOOSE n SO THAT THE COEFFICIENTS a_m ARE AS SMALL AS POSSIBLE.

(ii) GIVEN THAT WE HAVE FOUND THE OPTIMAL NUMBER OF TERMS, THAT IS THE BEST THAT WE CAN DO WITH APPROXIMATING THE INTEGRAL. WE CANNOT GET A BETTER APPROXIMATION.

NATSON'S LEMMA

LET $F(s) = \int_0^\infty e^{-st} f(t)\,dt$ $\quad$ s COMPLEX

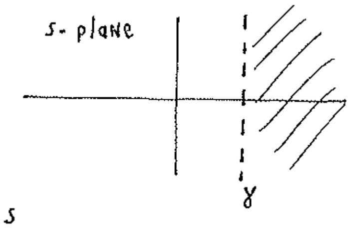
ND ASSUME THAT

(i) $\quad f(t) = O(e^{\gamma t}) \quad t \to \infty \quad f$ BOUNDED FOR $0 < t < \infty$

(ii) $\quad f(t) \sim \sum_{n=0}^{\infty} a_n t^{\alpha_n}$ AS $t \to 0 \quad -1 < \alpha_0 < \alpha_1 < \ldots$ (NOTICE f CAN BLOW-UP AS $t \to 0$)

IEN $\quad F(s) \sim \sum_{n=0}^{\infty} a_n \dfrac{\Gamma(\alpha_n + 1)}{s^{\alpha_n+1}}$ AS $s \to \infty$ ALONG ANY PATH IN THE HALF PLANE $|\arg(s-\gamma)| < \pi/2$.

THE RESULT SHOWS THAT THE SMALL t BEHAVIOR OF $f(t)$ DETERMINES THE ARGE s BEHAVIOR OF $F(s)$.



PROOF OF WATSON'S LEMMA

WE FIRST CONSIDER s REAL AND DECOMPOSE $F(s)$ AS

$$F(s) = \int_0^R e^{-st} f(t)\,dt + \int_R^\infty e^{-st} f(t) = F_1(s) + F_2(s) \quad \text{FOR ANY } R > 0.$$

)W FROM (i) WE HAVE $|f(t)| \le K e^{\gamma t}$ FOR $t \ge R$. THUS

$$|F_2(s)| \le \int_R^\infty |f(t)| e^{-st}\,dt \le \frac{K}{(s-\gamma)} e^{-(s-\gamma)R} = o(e^{-(s-\gamma)R}) \quad s \to \infty.$$

IUS $\quad |F_2(s)| = o(e^{(s-\gamma)R})$ AS $s \to \infty$. (need $s > \gamma$), AND SO

$$(**) \quad F(s) = \underbrace{\int_0^R e^{-st} f(t)\,dt}_{F_1(s)} + o(e^{-(s-\gamma)R})$$

W FOR EACH POSITIVE INTEGER N WE SET

$$f(t) = \sum_{m=0}^{N} a_m t^{\alpha_m} + P_N(t)$$

ERE WE KNOW THAT $P_N(t) = O(t^{\alpha_{N+1}})$ AS $t \to 0$. SINCE THE RANGE INTEGRATION IS FINITE AND SINCE N IS FINITE WE CAN INTEGRATE TERM BY RM

$$F_1(s) = \sum_{m=0}^{N} a_m \int_0^R t^{\alpha_m} e^{-st}\,dt + \int_0^R P_N(t) e^{-st}\,dt$$

NE NOW WRITE
$$\int_0^R t^{d_m} e^{-st}\,dt = \int_0^\infty t^{d_m} e^{-st}\,dt - \int_R^\infty t^{d_m} e^{-st}\,dt.$$

HUS
$$\int_0^R t^{d_m} e^{-st}\,dt = \frac{\Gamma(d_m+1)}{s^{d_m+1}} + O(e^{-sR}) \qquad s \to \infty.$$

HEREFORE, WE HAVE
$$(*) \quad F_1(s) = \sum_{m=0}^{N} a_m \frac{\Gamma(d_m+1)}{s^{d_m+1}} + \int_0^R e^{-st} P_N(t)\,dt + O(e^{-sR}).$$

INALLY, SINCE $P_N(t) = O(t^{d_{N+1}})$ AS $t \to 0$ WE HAVE $\int_0^R P_N(t) e^{-st}\,dt = O\left(1/s^{d_{N+1}+1}\right.$

AS $s \to \infty$

HUS SINCE $(*)$ HOLDS FOR EACH N WE HAVE
$$F(s) \sim \sum_{m=0}^{\infty} a_m \frac{\Gamma(d_m+1)}{s^{d_m+1}} \qquad \text{AS} \quad s \to \infty.$$

IF s IS COMPLEX THEN FROM $(**)$ ON THE PREVIOUS PAGE WE NEED HAT $\mathrm{Re}(s) > \gamma$ SO THAT THE "ERROR" TERM $O(e^{-(s-\gamma)R})$ CAN BE NEGLECTED S $s \to \infty$. THUS THE RESULT FOR $F(s)$ APPLIES WHEN $|\arg(s-\gamma)| < \pi/2$ AND $\to \infty$.

<u>XAMPLE 1</u> USE WATSON'S LEMMA TO FIND THE BEHAVIOR OF $\mathrm{erfc}(x)$ AS $x \to \infty$.

CALL THAT
$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt$$

OW LET $t = x u^{1/2}$. THEN
$$\mathrm{erfc}(x) = \frac{x}{\sqrt{\pi}} \int_1^\infty \frac{e^{-x^2 u}}{u^{1/2}}\,du$$

W LET $u = t+1$. THEN
$$\mathrm{erfc}(x) = \frac{x e^{-x^2}}{\sqrt{\pi}} \int_0^\infty \frac{e^{-x^2 t}}{(t+1)^{1/2}}\,dt$$

Y THE BINOMIAL THEOREM WE KNOW
$$(1+t)^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \frac{1\cdot 3\cdot 5 \cdots (2k+1)}{2^k\, k!}\, t^k \qquad |t|<1.$$

THUS BY WATSON'S LEMMA

$$\operatorname{erfc}(x) \sim \frac{x e^{-x^2}}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdots (2k+1)}{2^k \; x^{2k+2}} \qquad x \to \infty$$

FOR $|\arg(x)| < \pi/4$.

INTEGRATION BY PARTS: FURTHER EXAMPLES: INTEGRATION BY PARTS VS WATSON'S LEMMA

EXAMPLE FIND THE BEHAVIOR OF $I(X) = \int_0^X t^{-1/2} e^{-t}\, dt$ AS $X \to \infty$.

WE WRITE

$$I(X) = \int_0^\infty t^{-1/2} e^{-t}\, dt - \int_X^\infty t^{-1/2} e^{-t}\, dt = \Gamma(1/2) - \int_X^\infty t^{-1/2} e^{-t}\, dt.$$

LET $t = XU$. THEN $I(X) = \Gamma(1/2) - X^{1/2} \int_1^\infty e^{-XU} U^{-1/2}\, dU$

THUS

$$I(X) = \Gamma(1/2) - X^{1/2} e^{-X} \int_0^\infty \frac{e^{-Xt}}{(1+t)^{1/2}}\, dt$$

NOW $(1+t)^{-1/2} = \sum_{K=0}^{\infty} (-1)^K \frac{1\cdot 3 \cdots (2K+1)}{2^K K!} t^K$

THUS USING WATSONS LEMMA

$$I(X) \sim \Gamma(1/2) - X^{1/2} e^{-X} \sum_{K=0}^{\infty} (-1)^K \frac{1\cdot 3 \cdots (2K+1)}{2^K X^K}$$

NOW TRY INTEGRATING BY PARTS. LET

$$E_m(X) = \int_X^\infty t^{-1/2+m} e^{-t}\, dt = -e^{-t} t^{m-1/2} \Big|_X^\infty + (m - 1/2) \int_X^\infty e^{-t} t^{m-3/2}\, dt$$

THUS

$$E_m(X) = X^{m-1/2} e^{-X} + (m - 1/2) E_{m-1}(X)$$

WE WANT $E_0(X)$:

$$E_0(X) = X^{-1/2} e^{-X} - \frac{1}{2} E_{-1}(X) = X^{-1/2} e^{-X} - \frac{1}{2} X^{-3/2} e^{-X} + \frac{3}{4} E_{-2}(X)$$

$$E_0(X) = X^{-1/2} e^{-X} - \frac{1}{2} X^{-3/2} e^{-X} + \frac{3}{4} X^{-5/2} e^{-X} - \frac{3\cdot 5}{2^3} X^{-7/2} e^{-X} + \cdots \frac{3\cdot 5\cdot 7}{2^4} X^{-9/2} e^{-X} + \cdots$$

THEN

$$I(X) \sim \Gamma(1/2) - X^{-1/2} e^{-X} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2X)^n} (1\cdot 3 \cdots (2n+1)) \right]$$

REMARK IF WE DID INTEGRATION BY PARTS DIRECTLY ON

$$I(X) = \int_0^X t^{-1/2} e^{-t}\, dt = -e^{-t} t^{-1/2} \Big|_0^X - \frac{1}{2} \int_0^X e^{-t} t^{-3/2}\, dt$$

WE PRODUCE A DIVERGENT INTEGRAL. THIS PROCEDURE IS CLEARLY INCORRECT.

EXAMPLE CONSIDER THE STIELTJES INTEGRAL

$$I(X) = \int_0^\infty \frac{e^{-t}}{1+Xt}\, dt \qquad \text{FOR} \quad X \to 0^+$$

THEN

$$I(X) = \frac{1}{X} \int_0^\infty \frac{e^{-\lambda U}}{1+U}\, dU \qquad \lambda = 1/X \to \infty \quad \text{AS} \quad X \to 0.$$

THUS $(1+U)^{-1} = \sum_{n=0}^{\infty} (-1)^n U^n$ GIVES $I(X) \sim \frac{1}{X} \sum_{n=0}^{\infty} (-1)^n \int_0^\infty e^{-\lambda U} U^n\, dU$

AND SO $I(X) \sim \sum_{n=0}^{\infty} (-1)^n n!\, X^n$ AS $X \to 0$.

EXAMPLE CONSIDER $I(X) = \int_0^X e^{t^2}\,dt$ AS $X \to \infty$

INTEGRATE BY PARTS DIRECTLY GIVES

$$\int_0^X e^{t^2}dt = \frac{1}{2t} e^{t^2}\Big|_0^X + \frac{1}{2}\int_0^X t^{-2} e^{t^2} dt \qquad \text{WHICH IS WRONG.}$$

FURTHERMORE WE CANNOT WRITE $\int_0^X e^{t^2}dt = \int_0^\infty e^{t^2}dt - \int_X^\infty e^{t^2}dt.$

INSTEAD WRITE $I(X) = \int_0^a e^{t^2}dt + \int_a^X e^{t^2}dt.$

LET $I_m(X) = \int_a^X \frac{t\,e^{t^2}}{t^{m+1}}dt = \frac{1}{2} t^{-(m+1)} e^{t^2}\Big|_a^X + \frac{(m+1)}{2}\int_a^X e^{t^2} t^{-(m+2)} dt$

SO $I_m(X) = \frac{1}{2} X^{-(m+1)} e^{X^2} - \frac{1}{2} a^{-(m+1)} e^{a^2} + \frac{(m+1)}{2} I_{m+2}(X)$

CLEARLY $I_m(X) = \frac{1}{2}\frac{e^X}{X}\left(1 + \frac{1}{2X^2} + \frac{1\cdot 3}{2^2 X^4} + \frac{1\cdot 3\cdot 5}{2^3 X^6} + \cdots\right) + O(1) \quad X \to \infty$

FINALLY,

$$I_m(X) \sim \frac{1}{2}\frac{e^X}{X}\left(1 + \sum_{K=1}^{\infty} \frac{1\cdot 3\cdots(2K+1)}{(2X^2)^K}\right) \qquad X \to \infty.$$

THE BREAKPOINT a PLAYS NO ROLE IN THE ASYMPTOTICS OF $I(X)$.

WE NOW CONSIDER A MORE GENERAL INTEGRAL

$$I(X) = \int_a^b f(t)\, e^{X\phi(t)} dt$$

$I(X)$ IS DETERMINED BY THE VALUE OF t FOR WHICH $\phi(t)$ IS MAXIMUM.

SUPPOSE $\phi(t)$, $\phi'(t)$ AND $f(t)$ ARE CONTINUOUS. TRY INTEGRATING BY PARTS ONCE

$$(*) \qquad I(X) = \frac{1}{X}\frac{f(t)}{\phi'(t)} e^{X\phi(t)}\Big|_a^b - \frac{1}{X}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right) e^{X\phi(t)} dt$$

CASE 1 SUPPOSE THAT $\phi'(t) \neq 0$ ON $a \le t \le b$. THEN THE INTEGRAL IN $(*)$ EXISTS FOR $a \le t \le b$. THEN

$$I(X) \sim \frac{1}{X}\frac{f(b)}{\phi'(b)} e^{X\phi(b)} \qquad \text{WHEN} \quad \phi'(t) > 0 \quad \text{ON} \quad [a,b]$$

$$I(X) \sim -\frac{1}{X}\frac{f(a)}{\phi'(a)} e^{X\phi(a)} \qquad \text{WHEN} \quad \phi'(t) < 0 \quad \text{ON} \quad [a,b]$$

IN CERTAIN CASES FURTHER INTEGRATION BY PARTS ARE NEEDED, AND CAN BE DONE WITHOUT PRODUCING DIVERGENT INTEGRALS.

CASE 2 INTEGRATION BY PARTS CEASES TO BE VALID WHEN A DIVERGENT INTEGRAL IS PRODUCED. IN PARTICULAR IF $\phi'(t^*) = 0$ IN $t^* \in [a,b]$, THEN (*) IS NOT VALID. WE WILL TREAT THIS CASE LATER.

REMARK WHEN WE ARE ALLOWED TO INTEGRATE TERM BY TERM REPEATEDLY WE WILL GENERATE A SERIES IN $1/x$.

$$I(x) \sim e^{x\phi(t^*)} \sum_{n=1}^{\infty} a_n x^{-n} \qquad x \to \infty \qquad t^* = a \quad \text{OR} \quad t^* = b.$$

EXAMPLE CONSIDER $I(x) = \int_1^2 e^{x\cosh t}\, dt$ AS $x \to \infty$. NOTE $\phi'(t) = \sinh t \neq 0$ FOR $t \in [1,2]$

THEN
$$I(x) = \frac{1}{x} \frac{e^{x\cosh t}}{\sinh t}\Big|_1^2 - \frac{1}{x}\int_1^2 \frac{d}{dt}\left(\frac{1}{\sinh t}\right) e^{x\cosh t}\, dt$$

THUS
$$I(x) \sim \frac{1}{x} \frac{e^{x\cosh 2}}{\sinh 2} \qquad \text{AS} \quad x \to \infty.$$

EXAMPLE NOW CONSIDER $I(x) = \int_0^\infty \log[1+t]\, e^{-x\sinh^2 t}\, dt$. TRY INTEGRATING BY PARTS ONCE.

THEN
$$I(x) = -\frac{1}{x} \frac{\log(1+t)}{\sinh(2t)} e^{-x\sinh^2 t}\Big|_0^\infty + \frac{1}{x}\int_0^\infty \frac{d}{dt}\left(\frac{\log(1+t)}{\sinh(2t)}\right) e^{-x\sinh^2 t}\, dt$$

SINCE $\dfrac{\log(1+t)}{\sinh(2t)} \longrightarrow \dfrac{1}{2}$ AS $t \to 0$ THEN

$$I(x) \sim \frac{1}{2x} \qquad \text{AS} \quad x \to \infty.$$ HOWEVER A further integration BY PARTS IS NOT JUSTIFIED. IN FACT IT CAN BE SHOWN THAT

$$\int_0^\infty \frac{d}{dt}\left(\frac{\log(1+t)}{\sinh 2t}\right) e^{-x\sinh^2 t}\, dt = O(x^{-1/2}) \qquad x \to \infty.$$

EXAMPLE $I(x) = \int_0^\infty e^{-xt^2}\, dt$. WE KNOW THAT $I(x) = \frac{1}{2}(\pi/x)^{1/2}$ exactly!

AN integration by parts IS NOT permissible since

$$I(x) = \frac{e^{-xt^2}}{(-2xt)}\Big|_0^\infty - \int_0^\infty \frac{1}{2xt^2} e^{-xt^2}\, dt = \infty - \infty.$$

WE WILL SHOW LATER THAT $\int_0^\infty e^{-x\sinh^2 t}\, dt \sim \frac{1}{2}(\pi/x)^{1/2} \quad x \to \infty$

NOTICE THAT $\sinh^2 t \sim t^2$ AS $t \to 0$.

EXAMPLE INTEGRATION BY PARTS ALSO FAILS FOR

$$I(\lambda) = \int_0^\infty \log x \; e^{-\lambda x}\, dx$$

LET $t = \lambda x$. THEN $I(\lambda) = -\frac{\log \lambda}{\lambda} \int_0^\infty e^{-t}\, dt + \frac{1}{\lambda}\int_0^\infty (\log t)\, e^{-t}$

THUS FOR ANY $\lambda > 0$ $\quad I(\lambda) = -\frac{\log \lambda}{\lambda} + \frac{1}{\lambda}\int_0^\infty (\log t) e^{-t}\, dt$

THUS FOR $\lambda \to \infty$ THE SECOND TERM IN THE EXPANSION OF $I(\lambda)$ IS A GLOBAL CONTRIBUTION. UP TO NOW, INTEGRATION BY PARTS AND WATSON'S LEMMA HAVE GIVEN ASYMPTOTIC BEHAVIORS OF INTEGRALS IN TERMS OF LOCAL CONTRIBUTIONS FROM ENDPOINTS.

IN FACT $\gamma = -\int_0^\infty (\log x)\, e^{-x}\, dx$ WHERE γ IS EULERS CONSTANT. THE STANDARD DEFINITION FOR γ IS $\gamma = \lim_{n \to \infty} \left[1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n \right]$ $\quad \gamma \approx .57792.$

THUS $\quad I(\lambda) = -\frac{\log \lambda}{\lambda} - \frac{\gamma}{\lambda}$ $\quad$ WE SHOW THAT γ IS INDEED EQUAL TO $-\int_0^\infty (\log x)\, e^{-x}\, dx$ BELOW.

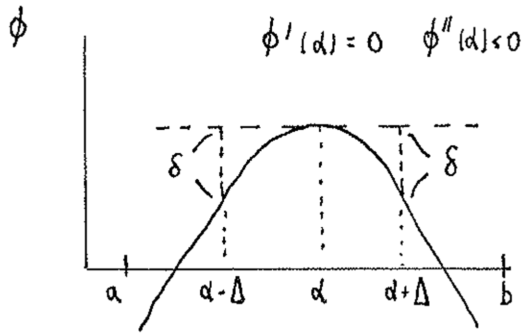
LAPLACE'S METHOD

CONSIDER (*) $I(X) = \int_a^b f(t)\, e^{X\phi(t)}\, dt \qquad X \to \infty \qquad f, \phi, \phi'$ CONTINUOUS

THE DOMINANT CONTRIBUTION TO $I(X)$ ARISES FROM THOSE REGIONS WHERE $\phi(t)$ IS A MAXIMUM. NEAR THESE REGIONS A LOCAL EXPANSION IS USED FOR f AND ϕ AND THE RESULTING INTEGRAL IS EVALUATED. IN A SMALL NEIGHBOURHOOD OF THE MAXIMUM WE CAN CONVERT (*) INTO A SUITABLE FORM FOR WATSONS LEMMA.

CASE I

SUPPOSE THAT $a < \alpha < b$ AND $\phi(\alpha) > \phi(t)$ FOR $t \in (a,b)$ AND THAT ϕ IS SMOOTH NEAR α. THEN WE HAVE AN INTERIOR MAXIMUM AS SHOWN. ASSUME ALSO THAT $f(\alpha) \neq 0$.



WE WRITE

$$(1) \quad I(X) = \int_a^\alpha f(t)\, e^{X\phi(t)}\, dt + \int_\alpha^b f(t)\, e^{X\phi(t)}\, dt$$

LET $I_1(X) = \int_a^\alpha f(t)\, e^{X\phi(t)}\, dt$

$I_2(X) = \int_\alpha^b f(t)\, e^{X\phi(t)}\, dt$

THEN

$$(2) \quad I_1(X) = \int_a^{\alpha-\Delta} f(t)\, e^{X\phi(t)}\, dt + \int_{\alpha-\Delta}^{\alpha} f(t)\, e^{X\phi(t)}\, dt$$

CLEARLY $\int_a^{\alpha-\Delta} f(t)\, e^{X\phi(t)}\, dt = O\left(e^{X(\phi(\alpha)-\delta)}\right)$ AND WILL BE EXPONENTIALLY SMALLER THAN THE <u>SECOND INTEGRAL</u> FOR $I_1(X)$. NOW LET $\phi(t) - \phi(\alpha) = -\tau \quad \tau > 0$ ON $t < \alpha$.

$$\int_{\alpha-\Delta}^{\alpha} f(t)\, e^{X\phi(t)}\, dt = \int_\delta^0 f[t(\tau)]\, e^{[\phi(\alpha)-\tau]X} \frac{dt}{d\tau}\, d\tau$$

THUS (3) $\displaystyle\int_{\alpha-\Delta}^{\alpha} f(t)\, e^{X\phi(t)}\, dt = -e^{X\phi(\alpha)} \int_0^\delta g_-(\tau)\, e^{-\tau X}\, d\tau \qquad g_-(\tau) \equiv f[t(\tau)] \frac{dt}{d\tau}$

NOTICE THAT WE WILL BE ABLE TO REPLACE δ BY ∞ BY MAKING AN ERROR OF $O\left(e^{X(\phi(\alpha)-\delta)}\right)$ WHICH WILL BE EXPONENTIALLY SMALLER THAN THE STUFF WE KEEP.

WE NOW CALCULATE $g_-(\tau)$ AS $\tau \to 0$. FOR $g_-(\tau)$ WE HAVE THAT $dt/d\tau < 0$

CALCULATION OF $dt/d\tau$ WHEN $\tau > 0$

$$\phi(t) - \phi(\alpha) = -\tau \qquad \phi(t) = \phi(\alpha) + \frac{(t-\alpha)^2}{2!}\phi^2(\alpha) + \frac{(t-\alpha)^3}{3!}\phi^3(\alpha) + \frac{(t-\alpha)^4}{4!}\phi^4(\alpha) + \dots$$

HERE $\phi^2(\alpha) \equiv \phi''(\alpha)$ ETC. THUS WE MUST CALCULATE $dt/d\tau$ AS $\tau \to 0$ FROM

$$(4) \qquad \frac{(t-\alpha)^2}{2!}\phi^2(\alpha) + \frac{(t-\alpha)^3}{3!}\phi^3(\alpha) + \frac{(t-\alpha)^4}{4!}\phi^4(\alpha) + \dots = -\tau$$

WE LOOK FOR A SOLUTION $t = t(\tau)$ IN THE FORM

$$t - \alpha = A\tau^{1/2} + B\tau + C\tau^{3/2} + \dots \qquad \tau \to 0 \quad \text{FOR SOME } A, B, C \text{ TO BE FOUND}$$

SINCE FOR I_1, $t < \alpha$ WE MUST HAVE THAT $t < \alpha$. SUBSTITUTE INTO (4) AND COLLECT LIKE TERMS IN POWERS OF $\tau^{1/2}$. THIS GIVES

$$\left(\frac{A^2}{2}\phi^2\right)\tau + \left(\frac{1}{2}2\phi^2 AB + \frac{A^3}{6}\phi^3\right)\tau^{3/2} + \left(\frac{1}{2}\phi^2(B^2 + 2AC) + \frac{A^2 B}{2}\phi^3 + \frac{A^4}{24}\phi^4\right)\tau^2 + \dots = -\tau$$

THUS $(3\tfrac{1}{2})$ $\quad A^2 = -2/\phi^2(\alpha) \qquad B = -\frac{1}{6}\frac{\phi^3(\alpha)}{\phi^2(\alpha)}A^2 \qquad C = \left(\frac{5}{72}\left(\frac{\phi^3(\alpha)}{\phi^2(\alpha)}\right)^2 - \frac{1}{24}\frac{\phi^4(\alpha)}{\phi^2(\alpha)}\right)A^3$

THE EXPANSION $t = \alpha + A\tau^{1/2} + B\tau + C\tau^{3/2} + \dots$ IS THEN SPECIFIED. NOTICE THAT FOR I_1 WE HAVE $t < \alpha$ AND THUS NEED $A < 0$.

$$\frac{dt}{d\tau} = \frac{1}{2}A\tau^{-1/2} + B + \frac{3}{2}C\tau^{1/2} + \dots \qquad \tau \to 0$$

NOW $\quad f(t) = f(\alpha) + (t-\alpha)f'(\alpha) + \frac{(t-\alpha)^2}{2}f''(\alpha) + \dots$

SO $\quad f(t(\tau)) = f(\alpha) + A\tau^{1/2}f'(\alpha) + \tau\left[Bf'(\alpha) + \frac{A^2}{2}f''(\alpha)\right] + O(\tau^{3/2})$

FINALLY

$$f(t(\tau))\frac{dt}{d\tau} = \frac{1}{2}Af(\alpha)\tau^{-1/2} + \left(Bf(\alpha) + \frac{1}{2}A^2 f'(\alpha)\right) + \tau^{1/2}\left(\frac{3}{2}ABf'(\alpha) + \frac{1}{4}A^3 f''(\alpha) + \frac{3}{2}Cf(\alpha)\right) + \dots$$

THUS (4)

$$g_-(\tau) = \frac{1}{2}Af(\alpha)\tau^{-1/2} + \left(Bf(\alpha) + \frac{1}{2}A^2 f'(\alpha)\right) + \tau^{1/2}\left(\frac{3}{2}ABf'(\alpha) + \frac{1}{4}A^3 f''(\alpha) + \frac{3}{2}Cf(\alpha)\right) + \dots$$

where $A < 0$

NOW FROM (2) AND (3)

$$(5) \qquad I_1(x) \sim -e^{x\phi(\alpha)}\int_0^\infty g_-(\tau)e^{-\tau x}\,d\tau$$

NOW WE EVALUATE $I_2(X)$. WRITE $I_2(X)$ AS

$$I_2(X) = \int_\alpha^b f(t)\, e^{X\phi(t)}\, dt = \int_\alpha^{\alpha+\Delta} f(t)\, e^{X\phi(t)}\, dt + \int_{\alpha+\Delta}^b f(t)\, e^{X\phi(t)}\, dt$$

THUS

$$I_2(X) = \int_\alpha^{\alpha+\Delta} f(t)\, e^{X\phi(t)}\, dt + O\left(e^{X(\phi(\alpha)-\delta)}\right)$$

NOW LET $\phi(t) - \phi(\alpha) = -\tau$ WITH $\tau > 0$. THEN

$$I_2(X) = \int_0^\delta f(t(\tau))\, e^{X[\phi(\alpha)-\tau]}\, \frac{dt}{d\tau}\, d\tau + O\left(e^{X(\phi(\alpha)-\delta)}\right)$$

SO THAT

$$(6) \qquad I_2(X) \sim e^{X\phi(\alpha)} \int_0^\infty g_+(\tau)\, e^{-X\tau}\, d\tau \qquad g_+(\tau) = f(t(\tau))\,\frac{dt}{d\tau}.$$

$g_+(\tau)$ IS SUCH THAT $dt/d\tau > 0$. THUS $g_+(\tau)$ HAS THE LOCAL FORM (4) WHERE A IS REPLACED BY $-A$ AND C REPLACED BY $-C$. NOTICE B DEPENDS ON A^2 AND SO IS INVARIANT UNDER SIGN CHANGES IN A.

COMBINING (5) AND (6) WE OBTAIN

$$I(X) \sim e^{X\phi(\alpha)} \int_0^\infty [g_+(\tau) - g_-(\tau)]\, e^{-X\tau}\, d\tau$$

$$g_+(\tau) - g_-(\tau) = A f(\alpha)\,\tau^{-1/2} + \tau^{1/2}\left(3AB f'(\alpha) + \tfrac{1}{2}A^3 f''(\alpha) + 3C f(\alpha)\right) + \cdots$$

WHERE A IS THE <u>POSITIVE</u> SQUARE ROOT OF $A^2 = -2/\phi^2(\alpha)$ AND B, C ARE DEFINED IN ($3\frac{1}{2}$). NOW USING WATSON'S LEMMA

$$I(X) \sim e^{X\phi(\alpha)}\left[\frac{A f(\alpha)\,\Gamma(1/2)}{X^{1/2}} + \frac{\left(3AB f'(\alpha) + \frac{1}{2}A^3 f''(\alpha) + 3C f(\alpha)\right)\Gamma(3/2)}{X^{3/2}} + \cdots\right]$$

FINALLY WE USE $A = \left(-2/\phi^2(\alpha)\right)^{1/2}$ AND THE FORMULAS FOR B AND C.

$\Gamma(1/2) = \sqrt{\pi}$ $\qquad \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$.

THIS GIVES THE FINAL RESULT (PUTTING BACK $\phi^2(\alpha) \to \phi''(\alpha)$ ETC.

AS $X \to \infty$

$$I(X) \sim \left(\frac{2\pi}{-X\phi''(\alpha)}\right)^{1/2} e^{X\phi(\alpha)}\left[f(\alpha) + \frac{1}{X}\left(\frac{1}{2}\frac{\phi'''(\alpha)f'(\alpha)}{(\phi''(\alpha))^2} - \frac{1}{2}\frac{f''(\alpha)}{\phi''(\alpha)} - \frac{5}{24}\frac{f(\alpha)(\phi'''(\alpha))^2}{(\phi''(\alpha))^3} + \frac{f(\alpha)\phi''''(\alpha)}{8(\phi''(\alpha))^2}\right)\right] \quad (**)$$

<u>REMARKS</u> (i) THE LEADING TERM

$$I(X) \sim \left(\frac{2\pi}{-X\phi''(\alpha)}\right)^{1/2} e^{X\phi(\alpha)}$$

IS KNOWN AS LAPLACE'S METHOD

(ii) IF THIS PROCESS WAS CONTINUED (HEAVEN FORBID) WE WOULD OBTAIN A SERIES OF THE FORM

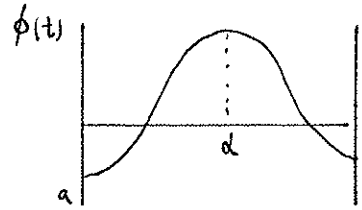
$$I(x) \sim \left(\frac{2\pi}{-X\phi''(\alpha)}\right)^{1/2} e^{X\phi(\alpha)}\left[f(\alpha) + \sum_{n=1}^{\infty} B_n / X^n\right]$$

WHERE B_n ARE SOME COEFFICIENTS.

THERE IS A VERY SIMPLE, BUT UNRIGOROUS, WAY OF DERIVING THE LEADING TERM IN $I(x)$ FOR THE CASE OF AN INTERIOR MAXIMUM.

SUPPOSE WE HAVE

$$I(X) = \int_a^b f(t)\, e^{X\phi(t)}\, dt$$



THEN

$$I(x) \sim \int_{\alpha-\Delta}^{\alpha+\Delta} f(t)\, e^{X\phi(t)}\, dt$$

WHERE $\Delta > 0$ IS SMALL. NOW REPLACE $f(t)$ AND $\phi(t)$ BY THEIR LOCAL EXPANSIONS.

$$f(t) \sim f(\alpha) \qquad \phi(t) \sim \phi(\alpha) + \frac{\phi''(\alpha)}{2}(t-\alpha)^2 \qquad \phi''(\alpha) < 0.$$

THEN SUBSTITUTE THESE RELATIONS AND LET $\Delta \to \infty$. THUS

$$I(X) \sim f(\alpha)\int_{-\infty}^{\infty} e^{X\phi(\alpha) + \frac{X}{2}\phi''(\alpha)(t-\alpha)^2}\, dt = f(\alpha)e^{X\phi(\alpha)}\int_{-\infty}^{\infty} e^{\frac{X}{2}\phi''(\alpha)(t-\alpha)^2}\, dt$$

NOW LET $V = t - \alpha$. THEN

$$I(X) \sim f(\alpha)e^{X\phi(\alpha)}\int_{-\infty}^{\infty} e^{-\frac{X}{2}|\phi''(\alpha)|V^2}\, dV$$

RECALL $\int_{-\infty}^{\infty} e^{-V^2/2\sigma^2}\, dV = \sqrt{2\pi}\,\sigma$

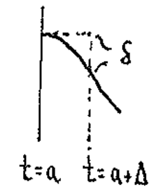
$$\sigma = \frac{1}{X|\phi''(\alpha)|}$$

THUS

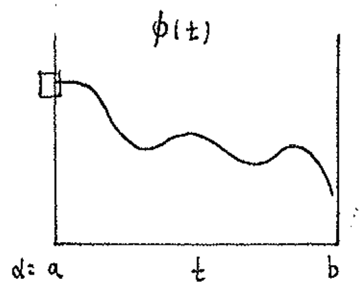
$$I(X) \sim \left(\frac{2\pi}{X|\phi''(\alpha)|}\right)^{1/2} f(\alpha)\, e^{X\phi(\alpha)} \qquad X \to \infty$$

WHICH AGREES WITH THE RESULT ABOVE

WE DERIVED THIS RESULT FOR AN INTERIOR MAXIMUM. WHAT HAPPENS IF $\alpha = a$ OR $\alpha = b$ WHERE $\phi'(\alpha) = 0$?



CASE 2 SUPPOSE $\alpha = a$ AND $\phi'(\alpha) = 0$, $\phi''(\alpha) < 0$ WITH $f(a) \neq 0$.



LET $I(X) = \int_a^b f(t) e^{X\phi(t)}\, dt$

WRITE $I(X) = \int_a^{a+\Delta} f(t) e^{X\phi(t)}\, dt + \int_{a+\Delta}^{\infty} f(t) e^{X\phi(t)}\, dt$

NOW $\int_{a+\Delta}^{\infty} f(t)\, e^{X\phi(t)}\, dt = O\left(e^{X(\phi(a) - \delta)} \right)$

THUS

$$I(X) \sim \int_a^{a+\Delta} f(t)\, e^{X\phi(t)}\, dt$$

NOW SET $\phi(t) - \phi(a) = -\tau$ $\tau > 0$ locally

THUS

$$I(X) \sim \int_0^{\delta} f(t(\tau)) \frac{dt}{d\tau}\, e^{X\phi(a) - X\tau}\, d\tau = e^{X\phi(a)} \int_0^{\delta} f(t(\tau)) \frac{dt}{d\tau}\, e^{-X\tau}\, d\tau$$

NOW NEAR $t = a$ WE HAVE $\phi(t) = \phi(a) + \frac{\phi^2(a)}{2!}(t-a)^2 + \frac{\phi^3(a)}{3!}(t-a)^3 + \dots$

THUS REPEATING A CALCULATION DONE EARLIER

$$t - a = A\tau^{1/2} + B\tau + \dots \qquad A = \left(-2/\phi^2(a)\right)^{1/2} \qquad B = -\frac{1}{6}\frac{\phi^3(a)}{\phi^2(a)} A^2$$

THUS $t = t(\tau)$ IS GIVEN LOCALLY BY

$$t = a + A\tau^{1/2} + B\tau + \dots$$

NOW USING $f(t) = f(a) + (t-a) f'(a) + \frac{(t-a)^2}{2!} f''(a) + \dots$

GIVES

$$g(\tau) = f[t(\tau)] \frac{dt}{d\tau} = \frac{1}{2} A f(a) \tau^{-1/2} + \left(B f(a) + \frac{1}{2} A^2 f'(a) \right) + \dots \qquad \tau \to 0$$

THUS

$$I(X) \sim e^{X\phi(a)} \left(\frac{1}{2} A f(a) \frac{\Gamma(1/2)}{X^{1/2}} + \frac{\left(B f(a) + \frac{1}{2} A^2 f'(a) \right)}{X} + \dots \right) \qquad X \to \infty$$

THIS CAN BE RE-WRITTEN AS

$$(*) \qquad I(X) \sim \frac{1}{2} e^{X\phi(a)} \left(\frac{-2\pi}{X\phi''(a)} \right)^{1/2} \left[f(a) + \frac{1}{(\pi X)^{1/2}} \left(-\frac{1}{3} \frac{\phi'''(a)}{\phi''(a)} f(a) + f'(a) \right) \left(\frac{-2}{\phi''(a)} \right)^{1/2} \right]$$

REMARKS (i) THE LEADING TERM IS $I(X) \sim \frac{1}{2} e^{X\phi(a)} \left(\frac{-2\pi}{X\phi''(a)} \right)^{1/2}$ WHICH IS ONE-HALF OF THE LEADING TERM FOR AN INTERIOR MAXIMUM

(ii) THE CORRECTION TERM FOR (*) GIVES AN $O(X^{-3/2}e^{X\phi(a)})$ WHEREAS FOR (**) WE HAVE $O(X^{-1}e^{X\phi(a)})$.

(iii) A SIMILAR FORMULA CAN BE DERIVED IF $\alpha = b$ WHERE $\phi'(b)=0$ $\phi''(b) < 0$.

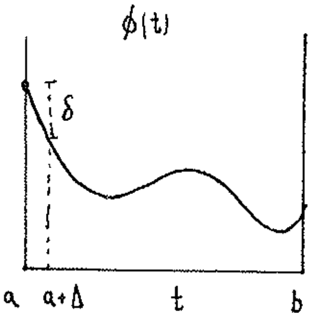
<u>CASE 3</u> WE NOW SUPPOSE THAT $\alpha = a$ BUT $\phi'(a) < 0$. IF $\phi'(t) < 0$ ON $[a,b]$ AND $f(a) \neq 0$ THEN INTEGRATION BY PARTS GAVE

$$(***) \qquad I(X) \sim -\frac{f(a)}{X\phi'(a)} e^{X\phi(a)} \qquad X \to \infty$$

WE NOW SHOW THAT $\phi'(t) < 0$ ON $[a,b]$ IS <u>NOT NEEDED</u> IN ORDER FOR (***) TO BE CORRECT. CONSIDER

$$I(X) = \int_a^b f(t)e^{X\phi(t)}\,dt$$

WHERE ϕ IS MAXIMIZED AT $t=a$ WITH $\phi'(a) < 0$.



WRITE
$$I(X) = \int_a^{a+\Delta} f(t)e^{X\phi(t)}\,dt + \int_{a+\Delta}^b f(t)e^{X\phi(t)}\,dt$$

THEN
$$I(X) = \int_a^{a+\Delta} f(t)e^{X\phi(t)}\,dt + O\left(e^{X(\phi(a)-\delta)}\right)$$

LET $\phi(t) - \phi(a) = -\tau$. THEN

$$I(X) \sim \int_0^\delta f(t(\tau))\frac{dt}{d\tau}\, e^{X\phi(a) - X\tau}\,d\tau = e^{X\phi(a)}\int_0^\delta f(t(\tau))\frac{dt}{d\tau}\, e^{-X\tau}\,d\tau$$

NOW NEAR $t=a$ WE HAVE $\phi'(a)(t-a) + \frac{\phi''(a)}{2}(t-a)^2 + \dots = -\tau$. SOLVING FOR $t = t(\tau)$ WE GET

$$t - a = c_0\tau + c_1\tau^2 + \dots \qquad c_0 = -1/\phi'(a) \qquad c_1 = -\frac{1}{2}\frac{\phi''(a)}{(\phi'(a))^3}$$

THUS
$$\frac{dt}{d\tau} = c_0 + 2c_1\tau + \dots = -\frac{1}{\phi'(a)} - \frac{\phi''(a)}{(\phi'(a))^3}\tau + \dots \qquad f[t(\tau)] = f(a) - \frac{\tau}{\phi'(a)}f'(a) + \dots$$

NOW
$$g(\tau) = f[t(\tau)]\frac{dt}{d\tau} = -\frac{f(a)}{\phi'(a)} + \left[-f(a)\frac{\phi''(a)}{(\phi'(a))^3} + \frac{f'(a)}{(\phi'(a))^2}\right]\tau + \dots$$

THUS
$$I(X) \sim e^{X\phi(a)}\int_0^\infty \left[\frac{-f(a)}{\phi'(a)} + \left(\frac{-f(a)\phi''(a)}{(\phi'(a))^3} + \frac{f'(a)}{(\phi'(a))^2}\right)\tau + \dots\right]e^{-X\tau}\,d\tau$$

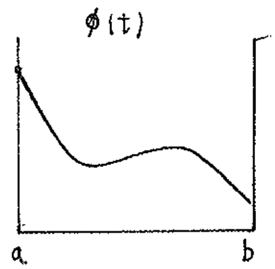
SO THAT
$$(+) \qquad I(X) \sim e^{X\phi(a)}\left[\frac{-f(a)}{X\phi'(a)} + \frac{1}{X^2}\left(\frac{-f(a)\phi''(a)}{(\phi'(a))^3} + \frac{f'(a)}{(\phi'(a))^2}\right) + \dots\right]$$

REMARK (i) THE FORM (+) CAN BE OBTAINED BY REPEATED INTEGRATION BY PARTS PROVIDED THAT $\phi'(t) < 0$ ON $[a,b]$. THE ANALYSIS ABOVE ONLY REQUIRES THAT $\phi'(a) < 0$. NO ASSUMPTION ON $\phi(t)$ (except that it is maximized at $t = a$) IS MADE FOR $a < t < b$.

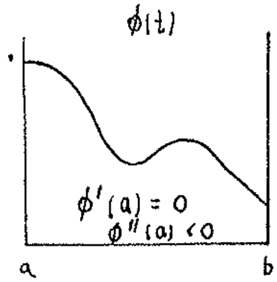
(ii) A SIMILAR FORMULA CAN BE DERIVED WHEN $\alpha = b$.

SUMMARY BELOW WE GIVE A LEADING ORDER FORMULA FOR $I(X)$ AS $X \to \infty$.

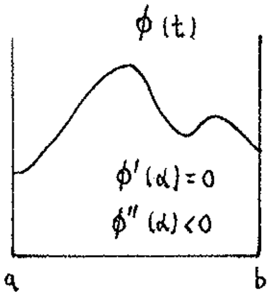
$$I(X) = \int_a^b f(t)\, e^{X\phi(t)}\, dt$$



$$I(X) \sim -\frac{f(a)\, e^{X\phi(a)}}{X\phi'(a)} \qquad X \to \infty.$$



$$I(X) \sim \frac{f(a)}{2} e^{X\phi(a)} \left(\frac{-2\pi}{X\phi''(a)} \right)^{1/2} \qquad X \to \infty$$



$$I(X) \sim \left(\frac{-2\pi}{X\phi''(\alpha)} \right)^{1/2} e^{X\phi(a)}\, f(\alpha) \qquad X \to \infty$$

REMARKS (i) HIGHER ORDER TERMS TO $I(X)$ AS $X \to \infty$, CAN BE DERIVED BY CHANGING VARIABLES, RE-VERTING THE SERIES AND USING WATSONS LEMMA.

(ii) A MORE REFINED CALCULATION CAN BE DONE FOR THE SPECIAL CASE OF A HIGHER ORDER MAXIMUM POINT

$$\phi'(\alpha) = \phi''(\alpha) = \dots = \phi^{(2p-1)}(\alpha) = 0 \qquad \phi^{(2p)}(\alpha) < 0$$

OR WHEN $f(\alpha)$ VANISHES SO THAT

$$f(t) = (t-\alpha)^{\beta}\, h(t) \qquad \text{WHERE } h(\alpha) \neq 0.$$