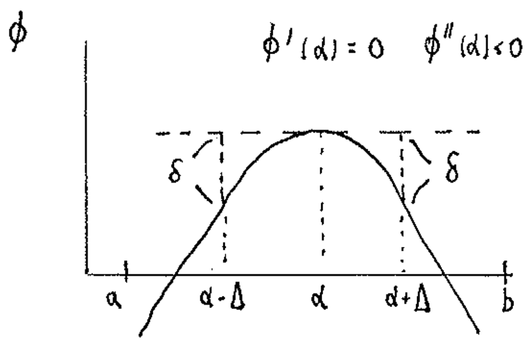
## LAPLACE'S METHOD

CONSIDER (*) $I(X) = \int_a^b f(t)\, e^{X\phi(t)}\, dt \qquad X \to \infty \qquad f,\ \phi, \phi'$ CONTINUOUS

THE DOMINANT CONTRIBUTION TO $I(X)$ ARISES FROM THOSE REGIONS WHERE $\phi(t)$ IS A MAXIMUM. NEAR THESE REGIONS A LOCAL EXPANSION IS USED FOR $f$ AND $\phi$ AND THE RESULTING INTEGRAL IS EVALUATED. IN A SMALL NEIGHBOURHOOD OF THE MAXIMUM WE CAN CONVERT (*) INTO A SUITABLE FORM FOR WATSONS LEMMA.

CASE 1

SUPPOSE THAT $a < \alpha < b$ AND $\phi(\alpha) > \phi(t)$ FOR $t \in (a,b)$ AND THAT $\phi$ IS SMOOTH NEAR $\alpha$. THEN WE HAVE AN INTERIOR MAXIMUM AS SHOWN. ASSUME ALSO THAT $f(\alpha) \neq 0$.




WE WRITE

$$(1) \qquad I(X) = \int_a^\alpha f(t)\, e^{X\phi(t)}\, dt + \int_\alpha^b f(t)\, e^{X\phi(t)}\, dt$$

LET $\quad I_1(X) = \int_a^\alpha f(t)\, e^{X\phi(t)}\, dt$

$\qquad I_2(X) = \int_\alpha^b f(t)\, e^{X\phi(t)}\, dt$

THEN

$$(2) \qquad I_1(X) = \int_a^{\alpha-\Delta} f(t)\, e^{X\phi(t)}\, dt + \int_{\alpha-\Delta}^{\alpha} f(t)\, e^{X\phi(t)}\, dt$$

CLEARLY $\int_a^{\alpha-\Delta} f(t)\, e^{X\phi(t)}\, dt = O\left(e^{X(\phi(\alpha)-\delta)}\right)$ AND WILL BE EXPONENTIALLY SMALLER THAN THE <u>SECOND INTEGRAL</u> FOR $I_1(X)$. NOW LET $\phi(t) - \phi(\alpha) = -\tau \qquad \tau > 0$ ON $t < \alpha$.

$$\int_{\alpha-\Delta}^{\alpha} f(t)\, e^{X\phi(t)}\, dt = \int_\delta^0 f[t(\tau)]\, e^{[\phi(\alpha)-\tau]X} \frac{dt}{d\tau}\, d\tau$$

THUS (3) $\int_{\alpha-\Delta}^{\alpha} f(t)\, e^{X\phi(t)}\, dt = -\, e^{X\phi(\alpha)} \int_0^\delta \underset{-}{g}(\tau)\, e^{-\tau X}\, d\tau \qquad \underset{-}{g}(\tau) \equiv f[t(\tau)] \frac{dt}{d\tau}$

NOTICE THAT WE WILL BE ABLE TO REPLACE $\delta$ BY $\infty$ BY MAKING AN ERROR OF $O\left(e^{X(\phi(\alpha)-\delta)}\right)$ WHICH WILL BE EXPONENTIALLY SMALLER THAN THE STUFF WE KEEP.

WE NOW CALCULATE $\underset{-}{g}(\tau)$ AS $\tau \to 0$. FOR $g_-(\tau)$ WE HAVE THAT $dt/d\tau < 0$

CALCULATION OF $dt/d\tau$ WHEN $\tau > 0$

$$\phi(t) - \phi(\alpha) = -\tau \qquad \phi(t) = \phi(\alpha) + \frac{(t-\alpha)^2}{2!}\phi^2(\alpha) + \frac{(t-\alpha)^3}{3!}\phi^3(\alpha) + \frac{(t-\alpha)^4}{4!}\phi^4(\alpha) + \ldots$$

HERE $\phi^2(\alpha) \equiv \phi''(\alpha)$ ETC. THUS WE MUST CALCULATE $dt/d\tau$ AS $\tau \to 0$ FROM

$$(4) \qquad \frac{(t-\alpha)^2}{2!}\phi^2(\alpha) + \frac{(t-\alpha)^3}{3!}\phi^3(\alpha) + \frac{(t-\alpha)^4}{4!}\phi^4(\alpha) + \ldots = -\tau$$

WE LOOK FOR A SOLUTION $t = t(\tau)$ IN THE FORM

$$t - \alpha = A\tau^{1/2} + B\tau + C\tau^{3/2} + \ldots \qquad \tau \to 0 \quad \text{FOR SOME } A, B, C \text{ TO BE FOUND}$$

SINCE FOR $I_1$, $t < \alpha$ WE MUST HAVE THAT $t < \alpha$. SUBSTITUTE INTO (4) AND COLLECT LIKE TERMS IN POWERS OF $\tau^{1/2}$. THIS GIVES

$$\left(\frac{A^2}{2}\phi^2\right)\tau + \left(\frac{1}{2}2\phi^2 AB + \frac{A^3}{6}\phi^3\right)\tau^{3/2} + \left(\frac{1}{2}\phi^2(B^2 + 2AC) + \frac{A^2 B}{2}\phi^3 + \frac{A^4}{24}\phi^4\right)\tau^2 + \ldots = -\tau$$

THUS $(3\frac{1}{2})$ $\quad A^2 = -2/\phi^2(\alpha) \qquad B = -\frac{1}{6}\frac{\phi^3(\alpha)}{\phi^2(\alpha)}A^2 \qquad C = \left(\frac{5}{72}\left(\frac{\phi^3(\alpha)}{\phi^2(\alpha)}\right)^2 - \frac{1}{24}\frac{\phi^4(\alpha)}{\phi^2(\alpha)}\right)A^3$

THE EXPANSION $t = \alpha + A\tau^{1/2} + B\tau + C\tau^{3/2} + \ldots$ IS THEN SPECIFIED. NOTICE THAT FOR $I_1$ WE HAVE $t < \alpha$ AND THUS NEED $A < 0$.

$$\frac{dt}{d\tau} = \frac{1}{2}A\tau^{-1/2} + B + \frac{3}{2}C\tau^{1/2} + \ldots \qquad \tau \to 0$$

NOW $\quad f(t) = f(\alpha) + (t-\alpha)f'(\alpha) + \frac{(t-\alpha)^2}{2}f''(\alpha) + \ldots$

SO $\quad f(t(\tau)) = f(\alpha) + A\tau^{1/2}f'(\alpha) + \tau\left[Bf'(\alpha) + \frac{A^2}{2}f''(\alpha)\right] + O(\tau^{3/2})$

FINALLY

$$f(t(\tau))\frac{dt}{d\tau} = \frac{1}{2}Af(\alpha)\tau^{-1/2} + \left(Bf(\alpha) + \frac{1}{2}A^2 f'(\alpha)\right) + \tau^{1/2}\left(\frac{3}{2}ABf'(\alpha) + \frac{1}{4}A^3 f''(\alpha) + \frac{3}{2}Cf(\alpha)\right) + \ldots$$

THUS (4)

$$g_-(\tau) = \frac{1}{2}Af(\alpha)\tau^{-1/2} + \left(Bf(\alpha) + \frac{1}{2}A^2 f'(\alpha)\right) + \tau^{1/2}\left(\frac{3}{2}ABf'(\alpha) + \frac{1}{4}A^3 f''(\alpha) + \frac{3}{2}Cf(\alpha)\right) + \ldots$$

where $A < 0$

NOW FROM (2) AND (3)

$$(5) \qquad I_1(x) \sim -e^{x\phi(\alpha)}\int_0^\infty g_-(\tau)e^{-\tau x}\,d\tau$$

NOW WE EVALUATE $I_2(X)$. WRITE $I_2(X)$ AS

$$I_2(X) = \int_\alpha^b f(t)e^{X\phi(t)}dt = \int_\alpha^{\alpha+\Delta} f(t)e^{X\phi(t)}dt + \int_{\alpha+\Delta}^b f(t)e^{X\phi(t)}dt$$

THUS $$I_2(X) = \int_\alpha^{\alpha+\Delta} f(t)e^{X\phi(t)}dt + O\left(e^{X(\phi(\alpha)-\delta)}\right)$$

NOW LET $\phi(t) - \phi(\alpha) = -\tau$ WITH $\tau > 0$. THEN

$$I_2(X) = \int_0^\delta f(t(\tau))\, e^{X[\phi(\alpha)-\tau]} \frac{dt}{d\tau}\, d\tau + O\left(e^{X(\phi(\alpha)-\delta)}\right)$$

SO THAT

$$(6) \qquad I_2(X) \sim e^{X\phi(\alpha)} \int_0^\infty g_+(\tau)e^{-X\tau}d\tau \qquad g_+(\tau) = f(t(\tau))\frac{dt}{d\tau}.$$

$g_+(\tau)$ IS SUCH THAT $dt/d\tau > 0$. THUS $g_+(\tau)$ HAS THE LOCAL FORM (4) WHERE $A$ IS REPLACED BY $-A$ AND $C$ REPLACED BY $-C$. NOTICE $B$ DEPENDS ON $A^2$ AND SO IS INVARIANT UNDER SIGN CHANGES IN $A$.

COMBINING (5) AND (6) WE OBTAIN

$$I(X) \sim e^{X\phi(\alpha)} \int_0^\infty [g_+(\tau) - g_-(\tau)]\, e^{-X\tau}d\tau$$

$$g_+(\tau) - g_-(\tau) = A f(\alpha)\tau^{-1/2} + \tau^{1/2}\left(3AB f'(\alpha) + \frac{1}{2}A^3 f''(\alpha) + 3C f(\alpha)\right) + \cdots$$

WHERE $A$ IS THE <u>POSITIVE</u> SQUARE ROOT OF $A^2 = -2/\phi^2(\alpha)$ AND $B$, $C$ ARE DEFINED IN $(3\tfrac{1}{2})$. NOW USING WATSON'S LEMMA

$$I(X) \sim e^{X\phi(\alpha)}\left[\frac{A f(\alpha)\,\Gamma(1/2)}{X^{1/2}} + \frac{\left(3AB f'(\alpha) + \frac{1}{2}A^3 f''(\alpha) + 3C f(\alpha)\right)\Gamma(3/2)}{X^{3/2}} + \cdots\right]$$

FINALLY WE USE $A = \left(-2/\phi^2(\alpha)\right)^{1/2}$ AND THE FORMULAS FOR $B$ AND $C$.

$\Gamma(1/2) = \sqrt{\pi}$ $\qquad \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$.

THIS GIVES THE FINAL RESULT ( PUTTING BACK $\phi^2(\alpha) \to \phi''(\alpha)$ ETC.

AS $X \to \infty$

$$I(X) \sim \left(\frac{2\pi}{-X\phi''(\alpha)}\right)^{1/2} e^{X\phi(\alpha)}\left[f(\alpha) + \frac{1}{X}\left(\frac{1}{2}\frac{\phi'''(\alpha)f'(\alpha)}{(\phi''(\alpha))^2} - \frac{1}{2}\frac{f''(\alpha)}{\phi''(\alpha)} - \frac{5}{24}\frac{f(\alpha)(\phi'''(\alpha))^2}{(\phi''(\alpha))^3} + \frac{f(\alpha)\phi''''(\alpha)}{8(\phi''(\alpha))^2}\right)\right] \quad (**)$$

<u>REMARKS</u> (i) THE LEADING TERM

$$I(X) \sim \left(\frac{2\pi}{-X\phi''(\alpha)}\right)^{1/2} e^{X\phi(\alpha)}$$ IS KNOWN AS LAPLACE'S METHOD

(ii) IF THIS PROCESS WAS CONTINUED (HEAVEN FORBID) WE WOULD OBTAIN A SERIES OF THE FORM

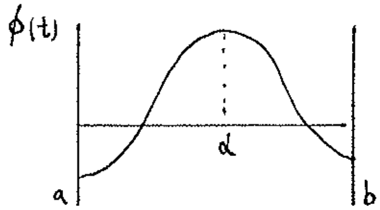
$$I(X) \sim \left(\frac{2\pi}{-X\phi''(\alpha)}\right)^{1/2} e^{X\phi(\alpha)}\left[f(\alpha) + \sum_{n=1}^{\infty} B_n / X^n\right]$$

WHERE $B_n$ ARE SOME COEFFICIENTS.

THERE IS A VERY SIMPLE, BUT UNRIGOROUS, WAY OF DERIVING THE LEADING TERM IN $I(X)$ FOR THE CASE OF AN INTERIOR MAXIMUM.

SUPPOSE WE HAVE

$$I(X) = \int_a^b f(t)\, e^{X\phi(t)}\, dt$$




THEN

$$I(X) \sim \int_{\alpha-\Delta}^{\alpha+\Delta} f(t)\, e^{X\phi(t)}\, dt$$ WHERE $\Delta > 0$ IS SMALL. NOW REPLACE

$f(t)$ AND $\phi(t)$ BY THEIR LOCAL EXPANSIONS.

$$f(t) \sim f(\alpha) \qquad \phi(t) \sim \phi(\alpha) + \frac{\phi''(\alpha)}{2}(t-\alpha)^2 \qquad \phi''(\alpha) < 0.$$

THEN SUBSTITUTE THESE RELATIONS AND LET $\Delta \to \infty$. THUS

$$I(X) \sim f(\alpha)\int_{-\infty}^{\infty} e^{X\phi(\alpha) + \frac{X}{2}\phi''(\alpha)(t-\alpha)^2}\, dt = f(\alpha)e^{X\phi(\alpha)}\int_{-\infty}^{\infty} e^{\frac{X}{2}\phi''(\alpha)(t-\alpha)^2}\, dt$$

NOW LET $v = t - \alpha$. THEN

$$I(X) \sim f(\alpha)e^{X\phi(\alpha)}\int_{-\infty}^{\infty} e^{-\frac{X}{2}|\phi''(\alpha)|v^2}\, dv$$

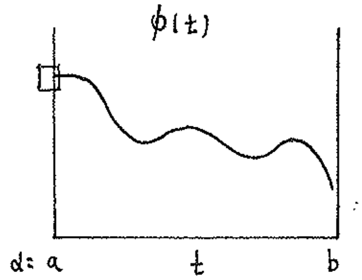
RECALL $\int_{-\infty}^{\infty} e^{-v^2/2\sigma^2}\, dv = \sqrt{2\pi}\,\sigma$ $\qquad \sigma = \frac{1}{X|\phi''(\alpha)|}$

THUS $$I(X) \sim \left(\frac{2\pi}{X|\phi''(\alpha)|}\right)^{1/2} f(\alpha)\, e^{X\phi(\alpha)} \qquad X \to \infty$$

WHICH AGREES WITH THE RESULT ABOVE

WE DERIVED THIS RESULT FOR AN INTERIOR MAXIMUM. WHAT HAPPENS IF $\alpha = a$ OR $\alpha = b$ WHERE $\phi'(\alpha) = 0$?

CASE 2 SUPPOSE $\alpha = a$ AND $\phi'(\alpha) = 0$, $\phi''(\alpha) < 0$ WITH $f(a) \neq 0$.


LET $I(x) = \int_a^b f(t) e^{x\phi(t)}\,dt$

WRITE $I(x) = \int_a^{a+\Delta} f(t) e^{x\phi(t)}\,dt + \int_{a+\Delta}^{\infty} f(t) e^{x\phi(t)}\,dt$

NOW $\int_{a+\Delta}^{\infty} f(t)\, e^{x\phi(t)}\,dt = O\left(e^{x(\phi(a)-\delta)}\right)$

THUS $I(x) \sim \int_a^{a+\Delta} f(t)\, e^{x\phi(t)}\,dt$ NOW SET $\phi(t) - \phi(a) = -\tau$ $\tau > 0$ locally

THUS $I(x) \sim \int_0^{\delta} f(t(\tau)) \frac{dt}{d\tau} e^{x\phi(a) - x\tau}\,d\tau = e^{x\phi(a)} \int_0^{\delta} f(t(\tau)) \frac{dt}{d\tau} e^{-x\tau}\,d\tau$

NOW NEAR $t = a$ WE HAVE $\phi(t) = \phi(a) + \frac{\phi^2(a)}{2!}(t-a)^2 + \frac{\phi^3(a)}{3!}(t-a)^3 + \ldots$

THUS REPEATING A CALCULATION DONE EARLIER

$$t - a = A\tau^{1/2} + B\tau + \ldots \qquad A = \left(-2/\phi^2(a)\right)^{1/2} \qquad B = -\frac{1}{6}\frac{\phi^3(a)}{\phi^2(a)} A^2$$

THUS $t = t(\tau)$ IS GIVEN LOCALLY BY

$$t = a + A\tau^{1/2} + B\tau + \ldots$$

NOW USING $f(t) = f(a) + (t-a) f'(a) + \frac{(t-a)^2}{2!} f''(a) + \ldots$

GIVES

$$g(\tau) = f[t(\tau)]\frac{dt}{d\tau} = \frac{1}{2} A f(a) \tau^{-1/2} + \left(B f(a) + \frac{1}{2} A^2 f'(a)\right) + \ldots \quad \tau \to 0$$

THUS $$I(x) \sim e^{x\phi(a)} \left( \frac{1}{2} A f(a) \frac{\Gamma(1/2)}{x^{1/2}} + \frac{\left(B f(a) + \frac{1}{2} A^2 f'(a)\right)}{x} + \ldots \right) \qquad x \to \infty$$

THIS CAN BE RE-WRITTEN AS

$$(*) \qquad I(x) \sim \frac{1}{2} e^{x\phi(a)} \left(\frac{-2\pi}{x\phi''(a)}\right)^{1/2} \left[ f(a) + \frac{1}{(\pi x)^{1/2}} \left( -\frac{1}{3}\frac{\phi'''(a)}{\phi''(a)} f(a) + f'(a) \right) \left(\frac{-2}{\phi''(a)}\right)^{1/2} \right]$$

REMARKS (i) THE LEADING TERM IS $I(x) \sim \frac{1}{2} e^{x\phi(a)} \left(\frac{-2\pi}{x\phi''(a)}\right)^{1/2}$ WHICH IS ONE-HALF OF THE LEADING TERM FOR AN INTERIOR MAXIMUM

(ii) THE CORRECTION TERM FOR (*) GIVES AN $O(X^{-3/2} e^{X\phi(a)})$ WHEREAS FOR (**) WE HAVE $O(X^{-1} e^{X\phi(a)})$.

(iii) A SIMILAR FORMULA CAN BE DERIVED IF $\alpha = b$ WHERE $\phi'(b) = 0$ $\phi''(b) < 0$.

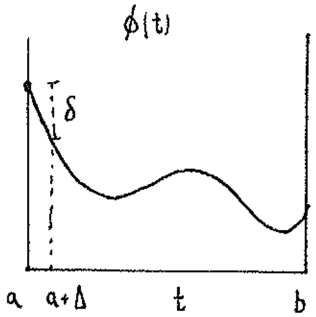
<u>CASE 3</u> WE NOW SUPPOSE THAT $\alpha = a$ BUT $\phi'(a) < 0$. IF $\phi'(t) < 0$ ON $[a,b]$ AND $f(a) \neq 0$ THEN INTEGRATION BY PARTS GAVE

$$(***) \qquad I(X) \sim -\frac{f(a)}{X\phi'(a)} e^{X\phi(a)} \qquad X \to \infty$$

WE NOW SHOW THAT $\phi'(t) < 0$ ON $[a,b]$ IS <u>NOT NEEDED</u> IN ORDER FOR (***) TO BE CORRECT. CONSIDER

$$I(X) = \int_a^b f(t) e^{X\phi(t)} dt$$

WHERE $\phi$ IS MAXIMIZED AT $t = a$ WITH $\phi'(a) < 0$.




WRITE
$$I(X) = \int_a^{a+\Delta} f(t) e^{X\phi(t)} dt + \int_{a+\Delta}^b f(t) e^{X\phi(t)} dt$$

THEN
$$I(X) = \int_a^{a+\Delta} f(t) e^{X\phi(t)} dt + O\left(e^{X(\phi(a) - \delta)}\right)$$

LET $\phi(t) - \phi(a) = -\tau$. THEN

$$I(X) \sim \int_0^{\delta} f(t(\tau)) \frac{dt}{d\tau} e^{X\phi(a) - X\tau} d\tau = e^{X\phi(a)} \int_0^{\delta} f(t(\tau)) \frac{dt}{d\tau} e^{-X\tau} d\tau$$

NOW NEAR $t = a$ WE HAVE $\phi'(a)(t-a) + \frac{\phi''(a)}{2}(t-a)^2 + \ldots = -\tau$. SOLVING FOR $t = t(\tau)$ WE GET

$$t - a = c_0 \tau + c_1 \tau^2 + \ldots \qquad c_0 = -1/\phi'(a) \qquad c_1 = -\frac{1}{2}\frac{\phi''(a)}{(\phi'(a))^3}$$

THUS
$$\frac{dt}{d\tau} = c_0 + 2c_1\tau + \ldots = -\frac{1}{\phi'(a)} - \frac{\phi''(a)}{(\phi'(a))^3}\tau + \ldots \qquad f[t(\tau)] = f(a) - \frac{\tau}{\phi'(a)} f'(a) + \ldots$$

NOW
$$g(\tau) = f[t(\tau)]\frac{dt}{d\tau} = -\frac{f(a)}{\phi'(a)} + \left[-\frac{f(a)\phi''(a)}{(\phi'(a))^3} + \frac{f'(a)}{(\phi'(a))^2}\right]\tau + \ldots$$

THUS
$$I(X) \sim e^{X\phi(a)} \int_0^{\infty} \left[-\frac{f(a)}{\phi'(a)} + \left(-\frac{f(a)\phi''(a)}{(\phi'(a))^3} + \frac{f'(a)}{(\phi'(a))^2}\right)\tau + \ldots\right] e^{-X\tau} d\tau$$

SO THAT

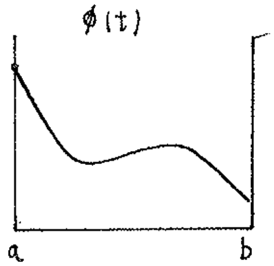
$$(+) \qquad I(X) \sim e^{X\phi(a)}\left[-\frac{f(a)}{X\phi'(a)} + \frac{1}{X^2}\left(-\frac{f(a)\phi''(a)}{(\phi'(a))^3} + \frac{f'(a)}{(\phi'(a))^2}\right) + \ldots\right]$$

**REMARK** (i) THE FORM (+) CAN BE OBTAINED BY REPEATED INTEGRATION BY PARTS PROVIDED THAT $\phi'(t) < 0$ ON $[a,b]$. THE ANALYSIS ABOVE ONLY REQUIRES THAT $\phi'(a) < 0$. NO ASSUMPTION ON $\phi(t)$ (except that it is maximized at $t = a$) IS MADE FOR $a < t < b$.

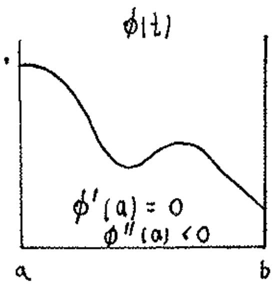
(ii) A SIMILAR FORMULA CAN BE DERIVED WHEN $\alpha = b$.

**SUMMARY** BELOW WE GIVE A LEADING ORDER FORMULA FOR $I(X)$ AS $X \to \infty$.

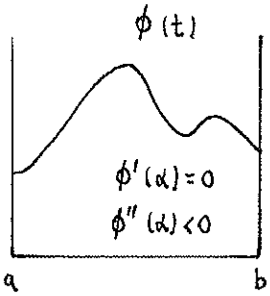
$$I(X) = \int_a^b f(t)\, e^{X\phi(t)}\, dt$$




$$I(X) \sim -\frac{f(a)\, e^{X\phi(a)}}{X\phi'(a)} \qquad X \to \infty.$$




$$I(X) \sim \frac{f(a)}{2} e^{X\phi(a)} \left( \frac{-2\pi}{X\phi''(a)} \right)^{1/2} \qquad X \to \infty$$




$$I(X) \sim \left( \frac{-2\pi}{X\phi''(\alpha)} \right)^{1/2} e^{X\phi(a)} f(\alpha) \qquad X \to \infty$$

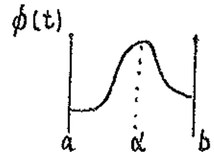
**REMARKS** (i) HIGHER ORDER TERMS TO $I(X)$ AS $X \to \infty$, CAN BE DERIVED BY CHANGING VARIABLES, RE-VERTING THE SERIES AND USING WATSONS LEMMA.

(ii) A MORE REFINED CALCULATION CAN BE DONE FOR THE SPECIAL CASE OF A HIGHER ORDER MAXIMUM POINT

$$\phi'(\alpha) = \phi''(\alpha) = \dots = \phi^{(2p-1)}(\alpha) = 0 \qquad \phi^{(2p)}(\alpha) < 0$$

OR WHEN $f(\alpha)$ VANISHES SO THAT

$$f(t) = (t-\alpha)^q h(t) \quad \text{WHERE } h(\alpha) \neq 0.$$




**REFINEMENTS OF THE BASIC THEORY** $\quad I(x) = \int_a^b f(t)\, e^{x\phi(t)}\, dt$

1) SUPPOSE THAT $f(t)$ VANISHES AT $t = \alpha \in (a,b)$ WHERE $\phi(t)$ HAS ITS MAXIMUM. THEN WE HAVE $f(t) = f'(\alpha)(t-\alpha) + f''(\alpha)(t-\alpha)^2/2! + \ldots \quad t \to \alpha$. THE RESULT (XX) HOLDS (PAGE 39) WHEN $f(\alpha) = 0$ AND

$$(1) \qquad I(x) \sim \left( \frac{2\pi}{x\,|\phi''(\alpha)|} \right)^{1/2} e^{x\phi(\alpha)} \left[ \frac{1}{2x} \frac{\phi'''(\alpha)\, f'(\alpha)}{(\phi''(\alpha))^2} - \frac{1}{2x} \frac{f''(\alpha)}{\phi''(\alpha)} \right] \qquad x \to \infty$$

THIS RESULT IS DIFFERENT THAN THAT OBTAINED BY THE "HEURISTIC" METHOD WHERE WE WRITE

$$I(x) \sim \int_{-\infty}^{\infty} \left[ f'(t)(t-\alpha) + \frac{f''(\alpha)(t-\alpha)^2}{2!} \right] e^{x\phi(\alpha) + x\phi''(\alpha)(t-\alpha)^2/2}\, dt$$

BY SYMMETRY THIS RESULT BECOMES $I(x) \sim \dfrac{f''(\alpha)}{2!} \displaystyle\int_{-\infty}^{\infty} (t-\alpha)^2 e^{x\phi(\alpha) + x\phi''(\alpha)(t-\alpha)^2/2}\, dt$

OR

$$(2) \qquad I(x) \sim \left( \frac{2\pi}{x\,|\phi''(\alpha)|} \right)^{1/2} e^{x\phi(\alpha)} \left[ \frac{f''(\alpha)\, e^{x\phi(\alpha)}}{2x\, |\phi''(\alpha)|} \right] \qquad x \to \infty$$

COMPARING (1) AND (2) WE CONCLUDE THAT THE HEURISTIC METHOD MISSES A TERM AND IS THEREFORE <u>INCORRECT</u>!! HOWEVER IF $f(\alpha) = f'(\alpha) = 0$ AND $f''(\alpha) \neq 0$, THE HEURISTIC METHOD DOES PRODUCE THE CORRECT LEADING ORDER BEHAVIOR FOR $I(x)$ AS $x \to \infty$.

2) NOW SUPPOSE THAT $\phi$ IS MAXIMIZED AT $t = a$ WHERE $\phi'(a) = 0$ AND $\phi''(a) \neq 0$. ASSUME ALSO THAT $f(a) = 0$ AND $f'(a) \neq 0$. THEN THE RESULT (X) FROM CASE 2 ON P. 40 APPLIES AND

$$(3) \qquad I(x) \sim \frac{1}{2}\, e^{x\phi(a)} \left( -\frac{2\pi}{x\,\phi''(a)} \right)^{1/2} \left[ \frac{1}{(\pi x)^{1/2}}\, f'(a) \left( \frac{-2}{\phi''(a)} \right)^{1/2} \right] \qquad x \to \infty$$

DOES THE 'HEURISTIC' METHOD GIVE THE CORRECT LEADING ORDER RESULT GIVEN IN (3)? THE HEURISTIC RESULT IS

$$I(x) \sim f'(a) \int_a^{\infty} (t-a)\, e^{x\phi(a) + x\phi''(a)(t-a)^2/2}\, dt$$

AND THUS

$$I(x) \sim \frac{f'(a)\, e^{x\phi(a)}}{x\,|\phi''(a)|} \qquad \text{WHICH AGREES WITH (3).}$$

THUS THE HEURISTIC METHOD IS CORRECT IN THIS CASE.

3) FINALLY CONSIDER THE CASE WHERE $\phi$ HAS A MAXIMUM AT SOME INTERIOR POINT $\alpha$ WHERE $f(\alpha) = 0$. HOWEVER, ASSUME THAT $\phi(t) = \phi(\alpha) + \phi^{(2p)}(\alpha)(t-\alpha)^{2p}/(2p)! + \cdots$ FOR $t$ NEAR $\alpha$ WHERE $p$ IS A POSITIVE INTEGER. THEN USING WATSON'S LEMMA YOU WILL SHOW THAT

$$I(X) \sim \frac{2\Gamma(1/2p)\left((2p)!\right)^{1/2p}}{2p\left[-X\phi^{2p}(\alpha)\right]^{1/2p}} f(\alpha)\, e^{X\phi(\alpha)} \qquad X \to \infty$$

NOTICE THE LARGER $p$ IS, THE 'FLATTER' $\phi$ IS AND THE INTEGRAL $I(X)$ IS ASYMPTOTICALLY LARGER FOR LARGER $p$.

FINAL COMMENTS

(i) FIRST LOCATE THE POINT OR POINTS WHERE $\phi$ IS MAXIMIZED.

(ii) IF A LEADING ORDER RESULT IS DESIRED IN GENERAL THE HEURISTIC METHOD CAN BE USED (exception on p. 43)

(iii) IF FURTHER TERMS ARE NEEDED WE MUST USE THE SYSTEMATIC METHOD BASED ON WATSON'S LEMMA DONE ON P. 36-41. THE FORMULAS ARE NOT TO BE MEMORIZED BUT THE PROCEDURE FOR DERIVING THEM SHOULD BE USED.

MOVEABLE MAXIMA

SOMETIMES AN INTEGRAL IS NOT IN A FORM WHERE WE CAN USE LAPLACE'S METHOD DIRECTLY. ONE CASE IS WHERE $\phi(t)$ IS MAXIMIZED AT $t = \alpha$ BUT $f(t)$ IS EXPONENTIALLY SMALL AT $t = \alpha$. FOR INSTANCE CONSIDER

$$I(X) = \int_0^\infty e^{-Xt - 1/t}\, dt \qquad \phi(t) = -t \text{ IS MAXIMIZED AT } t = 0 \text{ WHERE}$$

$f(t) = e^{-1/t}$ IS EXP. SMALL. TRY FINDING THE MAXIMUM OF THE EXPONENTIAL. SET $\frac{d}{dt}(-Xt - 1/t) = -X + 1/t^2 = 0 \Rightarrow t = 1/\sqrt{X}$. THUS THE LOCATION OF THE MAXIMUM DEPENDS ON X (MOVEABLE MAXIMUM). LET $t = \frac{1}{\sqrt{X}} s$. THEN $I(X) = \frac{1}{\sqrt{X}} \int_0^\infty e^{-\sqrt{X}(s + 1/s)}\, ds$ NOW $f(s) = 1$, $\phi(s) = -(s + 1/s)$ WHICH IS MAXIMIZED AT $s = 1$ AND WE CAN USE LAPLACES METHOD.

NOW CONSIDER $\Gamma(X) = \int_0^\infty e^{-t} t^{X-1}\, dt$ $\quad X \to \infty$. REWRITING $I(X) = \int_0^\infty \frac{1}{t} e^{X\log t - t}\, dt$.

$\frac{d}{dt}(X\log t - t) = 0 \Rightarrow X = t$ SO $X\log t - t$ IS MAXIMIZED AT $t = X$. NOTICE $\log t$ IS MAXIMIZED AT $t = \infty$ WHERE $e^{-t}$ IS EXP. SMALL. SET $t = Xs$ SO THAT $I(X) = X^X \int_0^\infty \frac{e^{-X(s - \log s)}}{s}$ WHICH IS AMENABLE TO LAPLACE'S METHOD. WE GIVE EXAMPLES BELOW.

## EXAMPLES LAPLACE'S METHOD

**EXAMPLE** CONSIDER $I(X) = \int_0^\infty e^{-Xt^2}\,dt$. RECALL THAT $I(X) = \frac{1}{2}(\pi/X)^{1/2}$ EXACTLY.

NOW LOOK AT
$$I(X) = \int_0^\infty e^{-X\sinh^2 t}\,dt = \int_0^\Delta e^{-X\sinh^2 t}\,dt + O(e^{-X\sinh^2\Delta})$$

THUS REPLACING $\sinh^2 t \sim t^2 + \dots$ WE OBTAIN
$$I(X) \sim \int_0^\Delta e^{-Xt^2}\,dt \sim \int_0^\infty e^{-Xt^2}\,dt = \frac{1}{2}\left(\frac{\pi}{X}\right)^{1/2}$$

THUS $I(X) \sim \frac{1}{2}(\pi/X)^{1/2}$

**EXAMPLE** RECALL ON PAGE 29 WE CONSIDERED $I(X) = \int_0^\infty \log[1+t]\,e^{-X\sinh^2 t}\,dt$

INTEGRATING BY PARTS WE OBTAIN
$$I(X) = \frac{1}{X}\left(\lim_{t\to 0} f(t)\right)e^{-X\sinh^2(0)} + \frac{1}{X}\int_0^\infty f'(t)\,e^{-X\sinh^2 t}\,dt$$

WHERE $f(t) = \dfrac{\log(1+t)}{\sinh(2t)} \sim \dfrac{t - t^2/2 + \dots}{2t + (2t)^3/3!} \sim \dfrac{t}{2t}(1 - t/2) \sim \dfrac{1}{2} - \dfrac{1}{4}t + \dots$ AS $t \to 0$

$$f'(t) \sim -\frac{1}{4} \quad \text{AS } t \to 0.$$

THUS
$$I(X) \sim \frac{1}{2X} + \frac{1}{X}\int_0^\infty \left(-\frac{1}{4}\right)e^{-Xt^2}\,dt$$

$$I(X) \sim \frac{1}{2X} - \frac{\pi}{8X^{3/2}} + \dots \qquad X \to \infty$$

WE NOW DERIVE THIS RESULT DIRECTLY. SINCE $\sinh^2 t$ IS MAXIMIZED AT $t=0$ WE HAVE
$$(1)\quad I(X) \sim \int_0^\Delta \log[1+t]\,e^{-X\sinh^2 t}\,dt = \int_0^\Delta f(t(\tau))\frac{dt}{d\tau}\,e^{-X\tau}\,d\tau \qquad \tau = \sinh^2 t,\quad f(t) = \log(1+t)$$

NOW $\sinh t \sim t + t^3/6 + \dots$ AS $t \to 0 \Rightarrow \tau = t^2 + t^4/3 + \dots$. THUS $t = \tau^{1/2} - \frac{1}{6}\tau^{3/2} + \dots$ AS $\tau \to 0$. NOW $\log(1+t) \sim t - t^2/2 + \dots$

THUS $f[t(\tau)] \sim \tau^{1/2} - \frac{\tau}{2} + \dots$ AS $\tau \to 0$

THIS GIVES
$$f[t(\tau)]\frac{dt}{d\tau} \sim \left(\frac{1}{2}\tau^{-1/2} - \frac{1}{4}\tau^{1/2} + \dots\right)\left(\tau^{1/2} - \frac{\tau}{2} + \dots\right) \sim \frac{1}{2} - \frac{1}{4}\tau^{1/2} + \dots$$

SUBSTITUTING IN (1) GIVES, UPON LETTING $\Delta \to \infty$,
$$I(X) \sim \int_0^\infty \left(\frac{1}{2} - \frac{1}{4}\tau^{1/2} + \dots\right)e^{-X\tau}\,d\tau \qquad I(X) \sim \frac{1}{2X} - \frac{\Gamma(3/2)}{4X^{3/2}} + \dots \quad X \to \infty$$

WHERE $\Gamma(3/2) = 1/2\,\Gamma(1/2) = \sqrt{\pi}/2$. THIS AGREES WITH THE RESULT ABOVE.

<u>EXAMPLE</u> FIND THE LEADING TERM AS $X \to \infty$ FOR

(i) $I(X) = \int_0^\infty e^{X(t-e^t)}\,dt$ (ii) $I(X) = \int_{-\infty}^\infty e^{-Xt^2} \log(1+t^2)\,dt$

FOR (i) $\phi(t) = t - e^t$ $\phi'(t) = 1 - e^t$ SO $\phi(t)$ IS MAXIMIZED AT $t=0$ WHERE $\phi(0) = -1$, $\phi'(0) = 0$

THUS LET $\tau = \phi(0) - \phi(t) = -1 - t + e^t$. THEN $I(X) \sim \int_0^\infty e^{-X} e^{-X\tau} \frac{dt}{d\tau}\,d\tau$

SO $I(X) \sim e^{-X} \int_0^\infty e^{-X\tau} \frac{dt}{d\tau}\,d\tau$ DETERMINE $t = t(\tau)$ FOR $\tau$ SMALL.

$$\tau = -1 - t + (1 + t + t^2/2! + \cdots) \qquad \tau \sim t^2/2 \qquad t \sim (2\tau)^{1/2} \qquad \frac{dt}{d\tau} \sim \frac{1}{\sqrt{2}}\tau^{-1/2} + \ldots \quad \tau \to 0$$

THUS $I(X) \sim \frac{e^{-X}}{\sqrt{2}} \int_0^\infty e^{-X\tau} \tau^{-1/2}\,d\tau$ $\quad I(X) \sim \left(\frac{\pi}{2X}\right)^{1/2} e^{-X}$ $\quad X \to \infty$

FOR (ii) THE MAXIMUM OCCURS AT $t=0$. WE CAN CHANGE VARIABLES OR DO IT THE HEURISTIC WAY. NEAR $t=0$ WE HAVE $\log(1+t^2) \sim t^2$. THUS

$$I(X) \sim + \int_{-\Delta}^{\Delta} e^{-Xt^2} \log(1+t^2)\,dt \sim \int_{-\Delta}^{\Delta} e^{-Xt^2} t^2\,dt$$ SINCE $\Delta$ IS SMALL.

NOW LET $y = Xt^2$ SO THAT $I(X) \sim 2\int_0^\Delta t^2 e^{-Xt^2}\,dt \sim \int_0^\infty \tau^{1/2} e^{-X\tau}\,d\tau$.

THUS $I(X) \sim \frac{1}{2}\sqrt{\pi}/X^{3/2}$ $\quad X \to \infty$.

<u>EXAMPLE</u> UNIFORM BEHAVIOR OF AN INTEGRAL

CONSIDER $I(X) = \int_0^{\pi/2} e^{X\cos t} \log(\lambda + \sin t)\,dt$ $\quad X \to \infty$ $\quad \lambda > 0$ REAL CONSTANT

SUPPOSE THAT $\lambda \neq 1$. THE MAXIMUM OF $\phi(t) = \cos t$ OCCURS AT $t=0$ WHERE $f(t) = \log(\lambda + \sin t)$ SATISFYING $f(0) = \log\lambda \neq 0$. TO FIND THE LEADING TERM,

$$I(X) \sim \int_0^\Delta e^{X\cos t} \log(\lambda + \sin t)\,dt \sim \log\lambda \int_0^\Delta e^{X - Xt^2/2}\,dt = e^X \log\lambda \int_0^\Delta e^{-Xt^2/2}\,dt$$

SO THAT

$$I(X) \sim e^X \log\lambda \int_0^\infty e^{-Xt^2/2}\,dt, \Rightarrow^{①} I(X) \sim \left(\frac{\pi}{2X}\right)^{1/2} e^X \log\lambda \quad X \to \infty.$$

NOW SUPPOSE $\lambda = 1$. THEN $f(0) = 0$ AND $f(t) = \log[1 + \sin t] \sim t$ $\quad t \to 0$. THUS WE HAVE

$$I(X) \sim \int_0^\infty t\, e^{X - Xt^2/2}\,dt \Rightarrow^{②} I(X) \sim e^X/X \quad X \to \infty$$

COMPARING ① AND ② WE OBSERVE THAT THE LEADING TERMS IN $I(X)$ AS $X \to \infty$ EXHIBIT A DISCONTINUOUS BEHAVIOR IN $\lambda$ AS $\lambda$ CROSSES THROUGH $\lambda = 1$. TO ELIMINATE THIS BEHAVIOR WE NOW COMPUTE AN EXTRA TERM FOR $I(X)$ AS $X \to$

LET $\cos t = 1 - \tau$ FOR $\tau \ll 1$. THEN

$$\text{①} \quad I(x) \sim e^{x} \int_0^{\Delta} e^{-x\tau} \log[\lambda + \sin t(\tau)] \frac{dt}{d\tau}\, d\tau$$

NOW $1 - \frac{t^2}{2} + \frac{t^4}{4!} + \ldots = 1 - \tau$ SO $\frac{t^2}{2} - \frac{t^4}{4!} + \ldots = \tau$

THUS $t = A\tau^{1/2} + B\tau^{3/2} + \ldots \qquad A = \sqrt{2} \qquad B = \sqrt{2}/12 \qquad dt/d\tau = \frac{1}{2} A \tau^{-1/2} + O(\tau^{1/2})$

NOW $\log[\lambda + \sin t(\tau)] \sim \log(\lambda + t) \sim \log \lambda + A\tau^{1/2}/\lambda + \ldots$

THUS $$\log[\lambda + \sin t(\tau)] \frac{dt}{d\tau} \sim \left(\log \lambda + \frac{\sqrt{2}\,\tau^{1/2}}{\lambda} + \ldots\right)\left(\frac{1}{2}\sqrt{2}\,\tau^{-1/2} + \ldots\right)$$

SO $$\text{②} \quad \log[\lambda + \sin t(\tau)] \frac{dt}{d\tau} \sim \frac{1}{\sqrt{2}} \log \lambda\, \tau^{-1/2} + \frac{1}{\lambda} + \ldots$$

USING ② IN ① GIVES WITH $\Delta = \infty$

$$I(x) \sim e^{x}\left[\sqrt{\frac{\pi}{2x}} \log \lambda + \frac{1}{\lambda x} + \ldots\right] \qquad x \to \infty$$

NOW NOTICE WHEN $\lambda = 1$ WE RECOVER THE RESULT ON THE PRECEEDING PAGE. THIS FORMULA IS <u>UNIFORMLY</u> VALID AS $\lambda$ CROSSES THROUGH $\lambda = 1$.

<u>EXAMPLE</u> STIRLINGS FORMULA: EXAMPLE OF MOVEABLE MAXIMA

CONSIDER $$\Gamma(x+1) = \int_0^{\infty} e^{-t} t^{x}\, dt = \int_0^{\infty} e^{-t} e^{x \log t}\, dt$$

NOTICE $\log t$ IS MAXIMIZED AT $t = \infty$ BUT $e^{-t}$ IS EXP. SMALL THERE. WE FIND MAXIMUM OF $x \log t - t$. THIS GIVES $t = x$. THUS SET $t = xs$. THEN

$$\Gamma(x+1) = x^{x+1} \int_0^{\infty} e^{x\phi(s)}\, ds \qquad \phi(s) = \log s - s$$

THE MAXIMUM OF $\phi(s)$ OCCURS AT $s = 1$ WHERE $\phi''(1) = -1$, $\phi'''(1) = 2$, $\phi^{(iv)}(1) =$. NOTICE $f \equiv 1$. THEREFORE USING WATSON'S LEMMA (AFTER TRANSFORMING $\int e^{x\phi(s)} ds$ AS ON P.36-38) WE CAN DERIVE MANY TERMS IN THE EXPANSION OF $\Gamma(x+1)$ AS $x \to \infty$

WE FIND USING (**) ON P. 39 THAT

$$\int_0^{\infty} e^{x\phi(s)}\, ds \sim \left(\frac{2\pi}{x}\right)^{1/2} e^{-x}\left[1 + \frac{1}{x}\left(-\frac{6}{8} + \frac{20}{24}\right)\right] \qquad x \to \infty$$

THUS $$\Gamma(x+1) \sim (2\pi)^{1/2} x^{x+1/2} e^{-x}\left[1 + 1/12x\right] \qquad x \to \infty$$

A FURTHER CYCLE OF THE METHOD GIVES

$$\Gamma(x+1) \sim (2\pi)^{1/2} e^{-x} x^{x+1/2}\left(1+\frac{1}{12x}+\frac{1}{288x^2}\right)$$

THUS $$n! \sim (2\pi)^{1/2} e^{-n} n^{n+1/2}\left(1+\frac{1}{12n}+\frac{1}{288n^2}\right) \qquad n\to\infty \quad \text{STIRLINGS FORMULA}$$

N THE HOMEWORK YOU WILL DETERMINE THE ACCURACY OF THE ASYMPTOTIC FORMULA IN PREDICTING $n!$

EXAMPLE FIND $I(x)=\int_{-\infty}^{\infty} e^{-x\cosh t}\,dt = 2\int_0^{\infty} e^{-x\cosh t}\,dt,$ AS $x\to\infty$.

NOW LET $\cosh t - 1 = \tau$. THEN $I(x) = 2e^{-x}\int_0^{\infty} e^{-x\tau}\frac{dt}{d\tau}\,d\tau.$

$$\sinh t\,\frac{dt}{d\tau} = 1 \qquad \frac{dt}{d\tau} = \frac{1}{\sinh t} = \frac{1}{((1+\tau)^2-1)^{1/2}} = \frac{1}{(\tau^2+2\tau)^{1/2}} = \frac{1}{(2\tau)^{1/2}(1+\tau/2)^{1/2}}$$

THIS GIVES $$I(x) = 2e^{-x}\int_0^{\infty} e^{-x\tau}\left(\frac{1}{2\tau}\right)^{1/2}[1+\tau/2]^{-1/2}\,d\tau$$

NOW USE WATSONS LEMMA

$$I(x)\sim 2e^{-x}\int_0^{\infty} e^{-x\tau}\left(\frac{1}{2\tau}\right)^{1/2}[1-\tau/4+\dots]\,d\tau$$

SO THAT

$$I(x)\sim (2\pi)^{1/2} e^{-x}\left(\frac{1}{x^{1/2}} - \frac{1}{8x^{3/2}}\right) \qquad x\to\infty$$

EXAMPLE LET $I(x) = \int_0^{\infty} e^{-xt-1/t}\,dt$. FIND $I(x)$ AS $x\to\infty$.

IOTICE THAT $\phi(t) = -t$ HAS A MAXIMUM AT $t=0$ WHERE $f(t) = e^{-1/t}$ IS EXPONENTIALLY SMALL. THE PREVIOUS RESULTS DO NOT APPLY. INSTEAD FIND THE MAXIMUM OF THE EXPONENT $-xt-1/t$. THUS $-x+1/t^2=0$ OR $t = x^{-1/2}$. THUS SET $t = x^{-1/2}s$. THEN

$$I(x) = x^{-1/2}\int_0^{\infty} e^{-x^{1/2}(s+1/s)}\,ds \qquad \phi(s) = -(s+1/s) \quad \phi'(1)=0 \quad \phi''(1) = -2.$$

THIS IS IN THE FORM FOR LAPLACE'S METHOD. THE LEADING TERM IS

$$I(x)\sim x^{-1/2} e^{-2x^{1/2}}\left(\frac{2\pi}{2x^{1/2}}\right)^{1/2} \qquad I(x)\sim\sqrt{\pi}\,x^{-3/4}e^{-2x^{1/2}} \quad x\to\infty$$

FURTHER TERMS CAN ALSO BE OBTAINED.

# GLOBAL CONTRIBUTIONS TO INTEGRALS

EULER'S CONSTANT $\gamma$ IS DEFINED BY

$$\gamma = \lim_{n \to \infty} \left[ 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n \right]$$

NOW WRITE

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = \int_0^\infty \left( e^{-t} + e^{-2t} + \dots + e^{-nt} \right) dt = \int_0^\infty \left( \frac{e^{-t} - e^{-(n+1)t}}{1 - e^{-t}} \right) dt$$

$$\log n = \int_1^n \frac{1}{x}\, dx = \int_1^n \left( \int_0^\infty e^{-xt}\, dt \right) dx = \int_0^\infty \frac{\left(e^{-t} - e^{-nt}\right)}{t}\, dt$$

THUS

$$1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n = \int_0^\infty \left[ \frac{e^{-t} - e^{-(n+1)t}}{1 - e^{-t}} - \left( \frac{e^{-t} - e^{-nt}}{t} \right) \right] dt$$

NOW LET $n \to \infty$ THEN

$$\gamma = \int_0^\infty \left[ \frac{e^{-t}}{1 - e^{-t}} - \frac{e^{-t}}{t} \right] dt$$

SO

$$\gamma = \int_0^\infty e^{-t} \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) dt$$

NOW OBTAIN AN ALTERNATIVE REPRESENTATION OF $\gamma$.

NOTICE $1+\frac{1}{2}+\dots+\frac{1}{n} = \int_0^1 (1+t+\dots+t^{n-1})\,dt = \int_0^1 \frac{1-t^n}{1-t}\,dt = \int_0^1 \left[\frac{1-(1-u)^n}{u}\right] du$

LET $u = t/n$. THEN $1+\frac{1}{2}+\dots+\frac{1}{n} = \int_0^n \frac{1-(1-t/n)^n}{t}\,dt = \int_0^1 \frac{1-(1-t/n)^n}{t}\,dt + \int_1^n \frac{1}{t}\,dt - \int_1^n \frac{(1-t/n)^n}{t}\,dt$

THUS WE HAVE $1+\frac{1}{2}+\dots+\frac{1}{n} - \log n = \int_0^1 \left[\frac{1-(1-t/n)^n}{t}\right] dt - \int_1^n \frac{[1-t/n]^n}{t}\,dt$

FINALLY LET $n \to \infty$, $\gamma = \int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_1^\infty \frac{e^{-t}}{t}\,dt$

THE LAST DERIVATION OF $\gamma$ IS TO SHOW THAT $\gamma = -\int_0^\infty (\log x)\, e^{-x}\,dx$.

WE FIRST NOTE THAT

$$\log x = \int_0^\infty \left(\frac{\exp(-\alpha) - \exp(-\alpha x)}{\alpha}\right) d\alpha$$

TO SHOW THIS WE USE $\int_0^\delta \frac{\exp(-\alpha) - \exp(-\alpha x)}{\alpha}\,dx \longrightarrow \int_0^\delta (x-1)\,d\alpha \to 0$ AS $\delta \to 0$.

NOW $\int_\delta^\infty \frac{\exp(-\alpha) - \exp(-\alpha x)}{\alpha}\,d\alpha = \int_\delta^\infty \frac{e^{-y}}{y}\,dy - \int_{x\delta}^\infty \frac{e^{-y}}{y}\,dy = \int_\delta^{x\delta} \frac{e^{-y}}{y}\,dy \longrightarrow \log x$ AS $\delta \to 0$.

THUS

$$\log x = \int_0^\infty \frac{1}{\alpha}[\exp(-\alpha) - \exp(-\alpha x)]\,d\alpha$$

RECALL $\Gamma(n) = \int_0^\infty t^{n-1}\exp(-t)\,dt \qquad \Gamma'(n) = \int_0^\infty (\log t)\, t^{n-1}\exp(-t)\,dt$

$$\Gamma'(n) = \int_0^\infty (\log t)\, t^{n-1} e^{-t}\,dt = \int_0^\infty \int_0^\infty t^{n-1} e^{-t} \left(\frac{e^{-\alpha} - e^{-\alpha t}}{\alpha}\right) d\alpha\, dt$$

THUS $\frac{\Gamma'(n)}{\Gamma(n)} = \int_0^\infty \left(\frac{e^{-\alpha}}{\alpha} - \frac{1}{\alpha(\alpha+1)^n}\right) d\alpha \qquad \frac{\Gamma'(1)}{\Gamma(1)} = \int_0^\infty \left(\frac{e^{-\alpha}}{\alpha} - \frac{1}{\alpha(\alpha+1)}\right) d\alpha$

NOW $\frac{\Gamma'(n)}{\Gamma(n)} - \frac{\Gamma'(1)}{\Gamma(1)} = \int_0^\infty \frac{1}{\alpha}\left(\frac{1}{\alpha+1} - \frac{1}{(\alpha+1)^n}\right) d\alpha = \int_0^1 \frac{1-u^{n-1}}{1-u}\,du = \int_0^1 \sum_{j=0}^{n-2} u^j\,du$

THUS $\frac{\Gamma'(n)}{\Gamma(n)} - \frac{\Gamma'(1)}{\Gamma(1)} = 1 + \frac{1}{2} + \dots + \frac{1}{n-1} \qquad \Gamma(n) \sim \sqrt{2\pi n}\, n^n e^{-n} \quad n \to \infty$

NOW $\Gamma(1) = 1 \qquad \Gamma'(1) = \int_0^\infty (\log t)\, e^{-t}\,dt \qquad \frac{\Gamma'(n)}{\Gamma(n)} \sim \log n \quad n \to \infty$

$-\int_0^\infty (\log t) e^{-t}\,dt = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \frac{\Gamma'(n)}{\Gamma(n)}$ THUS LETTING $n \to \infty$ WITH $\frac{\Gamma'(n)}{\Gamma(n)} \sim \log n$

WE OBTAIN $\gamma = -\int_0^\infty (\log t) e^{-t}\,dt$.

**EXAMPLE** CONSIDER THE INTEGRAL

$$E_1(X) = \int_X^\infty \frac{e^{-t}}{t}\,dt$$ WE LOOKED AT THE LARGE X BEHAVIOR PREVIOUSLY.

NOW EXAMINE $E_1(X)$ AS $X \to 0$. CLEARLY $E_1(X)$ DIVERGES AS $X \to 0$

$$E_1(X) = \int_x^1 \frac{e^{-t}}{t}\,dt + \int_1^\infty \frac{e^{-t}}{t}\,dt = \int_x^1 \frac{(e^{-t}-1)}{t}\,dt + \int_x^1 \frac{1}{t}\,dt + \int_1^\infty \frac{e^{-t}}{t}\,dt$$

$$E_1(X) = -\log x + \int_x^1 (e^{-t}-1)/_t\,dt + \int_1^\infty \frac{e^{-t}}{t}\,dt = -\log x + \int_0^1 \frac{(e^{-t}-1)}{t}\,dt - \int_0^X \frac{(e^{-t}-1)}{t}\,dt + \int_1^\infty \frac{e^{-t}}{t}\,d$$

OW RECALL

$$\gamma = \int_0^1 \left(\frac{1-e^{-t}}{t}\right)dt - \int_1^\infty \frac{e^{-t}}{t}\,dt.$$

HUS

$$(*) \qquad E_1(X) = -\log X - \gamma + \int_0^X (1-e^{-t})/_t\,dt$$

HIS REFORMULATION OF $E_1(X)$ IS EXACT! NOW LET X BE SMALL AND USE

$$\frac{1-e^{-t}}{t} = \frac{1}{t}\left(t - \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$

THUS $$E_1(X) \sim -\log X - \gamma + \sum_{n=1}^\infty \frac{(-1)^{n-1} X^n}{n \cdot n!} \qquad X \to 0$$

**REMARK** 1) NOTICE THAT THE SECOND TERM IN $E_1(X)$ IS A GLOBAL CONTRIBUTION. THE HIGHER TERMS ARE AGAIN LOCAL CONTRIBUTIONS.

2) THE REFORMULATED EXACT VERSION (*) IS WELL-SUITED FOR NUMERICAL COMPUTATIONS FOR X SMALL.

**XAMPLE** CONSIDER THE INTEGRAL FOR THE MODIFIED BESSEL FUNCTION $K_0(Z)$

$$K_0(z) = \int_1^\infty e^{-zs}(s^2-1)^{-1/2}\,ds = \int_0^\infty e^{-z\cosh t}\,dt$$

A VERY SIMILAR CALCULATION AS IN THE PRECEEDING EXAMPLE SHOWS THAT

$$K_0(z) = -\log(z/2) - \gamma + o(1) \qquad \text{AS } z \to 0$$

IN THE HOMEWORK, YOU WILL SHOW THAT

$$K_0(z) \sim -\log(z/2) - \gamma \qquad \text{AS } z \to 0.$$

**EXAMPLE** CONSIDER THE INTEGRAL $I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\, dt$

DETERMINE $I(x)$ FOR $x$ LARGE. THE INTEGRAL CAN BE RE-WRITTEN AS

$$I(x) = \frac{1}{x}\int_0^\infty \frac{e^{-u/x}}{1+u}\, du = \frac{1}{x} e^{1/x} \int_1^\infty \frac{e^{-v/x}}{v}\, dv = \frac{1}{x} e^{1/x} \int_{1/x}^\infty \frac{e^{-t}}{t}\, dt$$

THUS

$$I(x) = \frac{1}{x} e^{1/x} E_1(1/x) \qquad \text{WHERE} \qquad E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\, dt$$

HOWEVER WE JUST WORKED OUT THAT

$$E_1(z) \sim -\log z - \gamma + \sum_{n=1}^\infty \frac{(-1)^{n-1} z^n}{n \cdot n!}$$

SO

$$I(x) \sim \frac{1}{x} e^{1/x} \left[ \log x - \gamma + \sum_{n=1}^\infty \frac{(-1)^{n-1} x^{-n}}{n \cdot n!} \right] \qquad x \to \infty .$$

**EXAMPLE** ANOTHER DERIVATION OF $\gamma$.

CONSIDER $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt$

INTEGRATE BY PARTS

$$E_1(x) = -e^{-x}\log x + \int_x^\infty \log t\, e^{-t}\, dt = -e^{-x}\log x + \int_0^\infty \log t\, e^{-t}\, dt - \int_0^x \log t\, e^{-t}\, dt$$

IN THE LAST INTEGRAL LET $t = xu$. THEN

$$E_1(x) = -e^{-x}\log x + \int_0^\infty \log t\, e^{-t}\, dt - x\log x \int_0^1 e^{-ux}\, du - x\int_0^1 \log(u)\, e^{-ux}\, du$$

THUS

$$E_1(x) = -\log x + \int_0^\infty \log t\, e^{-t}\, dt - x\int_0^1 \log(u)\, e^{-ux}\, du$$

NOW $\int_0^1 \log(u)\, e^{-ux}\, du = \int_0^1 \log(u) \sum_{n=0}^\infty (-1)^n \frac{x^n u^n}{n!}\, du = \sum_{n=0}^\infty \frac{(-1)^n x^n}{n!} \int_0^1 (\log u)\, u^n\, du$

THUS

$$E_1(x) = -\log x + \int_0^\infty \log t\, e^{-t}\, dt - \sum_{n=0}^\infty \frac{(-1)^n x^{n+1}}{n!} \int_0^1 (\log u)\, u^n\, du$$

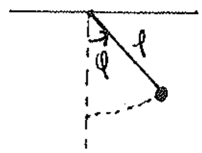
NOW $\int_0^1 (\log u)\, u^n\, du = -\int_0^\infty t\, e^{-(n+1)t}\, dt = -\frac{1}{(n+1)^2}$

SO,

$$E_1(x) = -\log x + \int_0^\infty (\log t)\, e^{-t}\, dt + \sum_{n=0}^\infty \frac{(-1)^n x^{n+1}}{(n+1)(n+1)!} \qquad x \to 0$$

THIS IS EQUIVALENT TO THE FORM ON P. 32 AND $\gamma = -\int_0^\infty (\log t)\, e^{-t}\, dt$.

EXAMPLE NONLINEAR PENDULUM




CONSIDER THE NONLINEAR PENDULUM EQUATION

$$\frac{d^2\varphi}{dt^2} + \frac{g}{\ell}\sin\varphi = 0 \qquad \varphi(0) = \varphi_0, \quad d\varphi/dt\,(0) = 0$$

HERE $\ell$ IS THE LENGTH AND $g$ IS GRAVITATIONAL ACCELERATION.

MULTIPLY BY $\frac{d\varphi}{dt}$ AND INTEGRATE $\quad \frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2 + \frac{g}{\ell}\left[\cos\varphi_0 - \cos\varphi\right] = 0$

THUS $\quad \frac{d\varphi}{dt} = -\sqrt{\frac{2g}{\ell}}\left[\cos\varphi - \cos\varphi_0\right]^{1/2}$ NOW $\varphi(T/2) = -\varphi_0$ $\quad T$ IS PERIOD OF OSCILLATION.

THEREFORE,

$$T = 2\sqrt{2}\,(\ell/g)^{1/2}\int_0^{\varphi_0}\left[\cos(u) - \cos(\varphi_0)\right]^{-1/2}du$$

REPLACING $\cos(u) = 1 - 2\sin^2(u/2)$, $\cos(\varphi_0) = 1 - 2\sin^2(\varphi_0/2)$ AND THEN LETTING $\sin\left(\frac{u}{2}\right) = \sin\frac{\varphi_0}{2}\sin\phi$ WE OBTAIN

$$T = 4\sqrt{\ell/g}\;I(m) \qquad I(m) = \int_0^{\pi/2}\left[1 - m^2\sin^2\phi\right]^{-1/2}d\phi \qquad m = \sin(\varphi_0/2)$$

$I(m)$ IS THE COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND. IN THE HOMEWORK YOU WILL DERIVE A TWO TERM EXPANSION FOR $I(m)$ AS $m \to 0$ (WEAKLY NONLINEAR CASE). ALSO YOU WILL DETERMINE $I(m)$ AS $\varphi_0 \to \pi^-$ (I.E. $m \to 1^-$). CLEARLY THE PERIOD MUST BE INFINITE WHEN $m = 1$.

EXAMPLE $\quad u'' + 2(u - u^3) = 0 \qquad -\infty < X < \infty \qquad u' = du/dX$

$$u(0) = 0 \qquad u(X) \longrightarrow 1 \quad \text{AS} \quad X \to \infty.$$

THE EXACT SOLUTION IS $u = \tanh(X)$ THUS $u \sim 1 - 2e^{-2X}$ $\quad X \to \infty$. WITHOUT USING THE EXACT SOLUTION DETERMINE THE $a$ SUCH THAT $u \sim 1 - ae^{-2X}$, AS $X \to \infty$.

MULTIPLYING THE EQUATION BY $u'$ WE GET

$$u_X^2 + (-u^4 + 2u^2 - 1) = 0 \qquad u \to 1,\ u_X \to 0 \text{ AS } X \to \infty \text{ HAS BEEN USED.}$$

THUS $\quad u_X = 1 - u^2 \quad$ SO $\quad \int_0^u \frac{1}{1-\lambda^2}\,d\lambda = X \quad$ CLEARLY AS $u \to 1$ $\quad X \to \infty$.

LET $\quad f(\lambda) = 1 - \lambda^2 \qquad f(\lambda) \sim -2(\lambda - 1) + \ldots$ AS $\lambda \to 1$. THUS

$$\int_0^u \left[\frac{1}{1-\lambda^2} + \frac{1}{2(\lambda-1)}\right]d\lambda - \frac{1}{2}\int_0^u \frac{1}{\lambda-1}\,d\lambda = X \quad \text{SO} \quad \int_0^u \left[\frac{1}{1-\lambda^2} + \frac{1}{2(\lambda-1)}\right]d\lambda - \frac{1}{2}\log(1-\lambda)\Big|_0^u = X$$

NOW LET $u \sim 1 - ae^{-2X}$. THEN $2\int^1\left[\frac{1}{1-\lambda^2} + \frac{1}{2(\lambda-1)}\right]d\lambda = \log(a)$. EVALUATING THE INTEGRAL GIVES $a = 2$

EXAMPLE CONSIDER $I(\varepsilon) = \int_0^1 (\varepsilon + x)^{-1/2}\,dx$ $\quad I(\varepsilon) = 2[(1+\varepsilon)^{1/2} - \varepsilon^{1/2}] \sim 2 - 2\varepsilon^{1/2} + \ldots \quad \varepsilon \to 0$

AS $\varepsilon \to 0$ THEN $I \sim \int_0^1 x^{-1/2}\,dx = 2$

NOW LETS CALCULATE ANOTHER TERM

$$I(\varepsilon) = \int_0^1 x^{-1/2}\,dx + \int_0^1 [(\varepsilon + x)^{-1/2} - x^{-1/2}]\,dx$$

NOW $I(\varepsilon) = 2 + \int_0^1 [(\varepsilon + x)^{-1/2} - x^{-1/2}]\,dx$

NOW LET $x = O(1)$ THEN $f(x,\varepsilon) \equiv (\varepsilon + x)^{-1/2} - x^{-1/2} = O(\varepsilon)$ $\quad dx = O(1) \implies \int f(x,\varepsilon)\,dx = O(\varepsilon)$

NOW LET $x = O(\varepsilon)$ THEN $f(x,\varepsilon) = O(\varepsilon^{-1/2})$ $\quad dx = O(\varepsilon)$ SO $\int f(x,\varepsilon)\,dx = O(\varepsilon^{1/2})$.

THUS THE INTEGRAL $\int_0^1 [(\varepsilon + x)^{-1/2} - x^{-1/2}]\,dx$ is determined by the region near $x = O(\varepsilon)$.

THUS LET $y = \varepsilon^{-1}x$ THEN

$$\int_0^1 [(\varepsilon + x)^{-1/2} - x^{-1/2}]\,dx = \varepsilon^{1/2}\int_0^{1/\varepsilon} [(1+y)^{-1/2} - y^{-1/2}]\,dy \sim \varepsilon^{1/2}\int_0^{\infty} [(1+y)^{-1/2} - y^{-1/2}]\,dy$$

SO $I(\varepsilon) \sim 2 + \varepsilon^{1/2}\int_0^\infty [(1+y)^{-1/2} - y^{-1/2}]\,dy$

EVALUATING

$$\int_0^\infty [(1+y)^{-1/2} - y^{-1/2}]\,dy = 2\left[\lim_{R\to\infty}\left((1+R)^{1/2} - R^{1/2}\right) - 1\right] = 2\left[R^{1/2}\left(1 + \frac{1}{2R} + \ldots\right) - R^{1/2} - 1\right]$$

SO AS $R \to \infty$ $\quad \int_0^\infty [(1+y)^{-1/2} - y^{-1/2}]\,dy = -2$. THIS GIVES $I(\varepsilon) \sim 2 - 2\varepsilon^{1/2} + \ldots \quad \varepsilon \to 0$.

EXAMPLE CONSIDER $I(\varepsilon) = \int_0^\infty \frac{dx}{(\varepsilon + x)^\alpha (1+x)}$ $\quad \varepsilon \to 0^+$ FOR DIFFERENT VALUES OF $\alpha > 0$.

NOTICE FOR $\alpha > 0$ THAT $I(\varepsilon)$ CONVERGES FOR FIXED $\varepsilon$.

(i) SUPPOSE $0 < \alpha < 1$ THEN $\left.\begin{array}{ll} x = O(\varepsilon) \to dx = O(\varepsilon) & I = O(\varepsilon^{1-\alpha}) \\ x = O(1) \to dx = O(1) & I = O(1) \end{array}\right\} \Rightarrow I(\varepsilon) \sim \int_0^\infty \frac{dx}{x^\alpha (1+x)}$ GLOBAL CONTRIBUTION

(ii) SUPPOSE $\alpha > 1$. THEN $\left.\begin{array}{ll} x = O(\varepsilon) \quad dx = O(\varepsilon) \to & I = O(\varepsilon^{1-\alpha}) \\ x = O(1) \quad dx = O(1) \to & I = O(1) \end{array}\right\} \Rightarrow I(\varepsilon) \sim \int_0^\infty \frac{dx}{(\varepsilon + x)^\alpha} = \frac{\varepsilon^{1-\alpha}}{\alpha - 1}$

LOCAL CONTRIBUTION

IN CASE (i) THE CONTRIBUTION FROM $x = O(1)$ DOMINATES THE LOCAL CONTRIBUTION FROM $x = O(\varepsilon)$. THIS SITUATION IS REVERSED IN CASE (ii).