REGULAR PERTURBATION THEORY

PERTURBATION OF ROOTS OF POLYNOMIALS

CONSIDER FINDING THE ROOTS OF $P_\varepsilon(X) = 0$ WHERE

$$P_\varepsilon(X) = p(X) + \varepsilon g(x)$$

WHERE $p(X)$ AND $g(X)$ ARE POLYNOMIALS WITH $\deg p \geq \deg g$.

THERE ARE TWO CASES:

(i) X_0 IS A SIMPLE ROOT OF $p(X) = 0$ SO THAT $p'(X_0) \neq 0$.

(ii) X_0 IS A ROOT OF MULTIPLICITY $m \geq 1$ WITH $p(X_0) = 0 = \dots = p^{(m-1)}(X_0) = 0$ WITH $p^{(m)}(X_0) \neq 0$.

CASE I X_0 IS A SIMPLE ROOT:

WE LET $X_\varepsilon = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots$ AND SET $P_\varepsilon(X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots) = 0$.

WE GET
$$p(X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots) + \varepsilon g(X_0 + \varepsilon X_1 + \dots) = 0$$

SO
$$p(X_0) + (\varepsilon X_1 + \varepsilon^2 X_2) p'(X_0) + \frac{\varepsilon^2}{2} X_1^2 p''(X_0) + \dots + \varepsilon g(X_0) + \varepsilon^2 g'(X_0) X_1 + \dots = 0$$

SETTING THE COEFFICIENTS OF ε^1 AND ε^2 TO ZERO, AND WITH $p(X_0) = 0$, WE GET

$$X_1 p'(X_0) + g(X_0) = 0 \longrightarrow X_1 = -g(X_0)/p'(X_0)$$

$$X_2 p'(X_0) = -\frac{1}{2} X_1^2 p''(X_0) - X_1 g'(X_0) \longrightarrow X_2 = -\frac{1}{2} \frac{[g(X_0)]^2 p''(X_0)}{(p'(X_0))^3} + \frac{g(X_0) g'(X_0)}{(p'(X_0))^2}$$

THIS YIELDS $X_\varepsilon = X_0 + \varepsilon X_1 + \dots$,

AND WE CAN GENERATE AN INFINITE SERIES AS $X_\varepsilon = \sum_{j=0}^{\infty} \varepsilon^j X_j$

EXAMPLE $X^3 + X^2 - X - 1 + \varepsilon X = 0$. FIND A 2-TERM EXPANSION FOR THE ROOT WITH $X_\varepsilon \to 1$ AS $\varepsilon \to 0$. WE WRITE $X_\varepsilon = 1 + \varepsilon X_1 + \varepsilon^2 X_2 + \varepsilon^3 X_3 + \dots$; WE SUBSTITUTE AND GET:

$$(1 + \varepsilon X_1 + \varepsilon^2 X_2 + \varepsilon^3 X_3)^3 + (1 + \varepsilon X_1 + \varepsilon^2 X_2 + \varepsilon^3 X_3)^2 - (1 + \varepsilon X_1 + \varepsilon^2 X_2 + \varepsilon^3 X_3) - 1$$
$$+ \varepsilon(1 + \varepsilon X_1 + \varepsilon^2 X_2) = 0$$

NOW USING $(1+h)^3 = 1 + 3h + 3h^2 + h^3$, AND COLLECT POWERS OF ε TO GET

$$\varepsilon^1: \quad 4X_1 + 1 = 0 \longrightarrow X_1 = -1/4$$

$$\varepsilon^2: \quad 4X_2 + 4X_1^2 + X_1 = 0 \longrightarrow X_2 = 0$$

$$\varepsilon^3: \quad 4X_3 + 8X_1X_2 + X_1^3 + X_2 = 0 \longrightarrow X_3 = -X_1^3/4 = 1/256$$

HENCE, $X_\varepsilon \sim 1 - \varepsilon/4 + \varepsilon^3/256 + \dots$

CASE (II) ROOT OF ORDER m

NOW SUPPOSE $p_\varepsilon(X) = p(X) + \varepsilon g(X) = 0$ WITH $p(X_0) = p'(X_0) = \dots = p^{(m-1)}(X_0) = 0$ AND $p^{(m)}(X_0) \neq 0$. WE LET $X_\varepsilon = X_0 + \varepsilon^\delta X_1 + \dots$

RECALL $p(X_0 + \varepsilon^\delta X_1) = p^{(m)}(X_0)\dfrac{(\varepsilon^\delta X_1)^m}{m!} + \dots$

THIS YIELDS THAT
$$p^{(m)}(X_0)\,\varepsilon^{\delta m}\frac{X_1^m}{m!} + \varepsilon g(X_0) + \dots = 0.$$

THIS YIELDS THAT $\delta m = 1 \longrightarrow m = 1/\delta$ OR $\delta = 1/m$.

IN ADDITION,
$$X_1^m = -\frac{m!\, g(X_0)}{p^{(m)}(X_0)}$$

(i) SUPPOSE m IS EVEN

- IF $g(X_0)\,p^{(m)}(X_0) < 0$ THEN $X_1^m > 0$ AND SO
$$X_1 = \pm\left(m!\left|\frac{g(X_0)}{p^{(m)}(X_0)}\right|\right)^{1/m}.$$
$m/2$ MULTIPLICITY FOR EACH OF $\pm$ SIGN.

- IF $g(X_0)\,p^{(m)}(X_0) > 0$ THEN $X_1^m < 0$ AND SO
$$X_1 = \left(m!\left|\frac{g(X_0)}{p^{(m)}(X_0)}\right|\right)^{1/m} e^{2\pi i j/m + i\pi/m} \quad j = 0, 1, \dots, m-1$$

THE ROOTS HAVE SPLIT UP INTO $m-1$ COMPLEX ROOTS.

(ii) SUPPOSE THAT m IS ODD

THEN $X_1^m > 0$ OR $X_1^m < 0$ DEPENDING ON SIGN OF $g(X_0)\, p^{(m)}(X_0)$.

THERE WILL BE ONE REAL ROOT AND $m-1$ COMPLEX CONJUGATE ROOTS.

EXAMPLE CONSIDER $(1-\varepsilon)X^2 - 2X + 1 = 0$. FIND THE CORRECTION TO THE ROOT $X = 1$. SO $p(X) = (X-1)^2$, $g(X) = -\varepsilon X^2$. HERE $m = 2$ EVEN.

WE LET $X = 1 + \varepsilon^{\delta} X_1 + \cdots$ SO $\varepsilon^{2\delta} X_1^2 - \varepsilon + \cdots = 0$

HENCE $X_1^2 = 1$ OR $X_1 = \pm 1$.

$$\text{THUS} \quad X_\varepsilon = 1 \pm \varepsilon^{1/2} + \cdots$$

THE MULTIPLE ROOT SPLITS INTO TWO REAL ROOTS.

EXAMPLE CONSIDER $X^3 + X^2 - X - 1 + \varepsilon X = 0$. FIND A 2-TERM EXPANSION FOR THE ROOT FOR WHICH $X \to -1$ AS $\varepsilon \to 0$. LET $p(X) = X^3 + X^2 - X - 1$ AND $p(-1) = p'(-1) = 0$, AND ALSO $p''(-1) = 2 + 6X\big|_{X=-1} = -4$. ALSO $g(X) = X$.

WE THEN PUT $X = -1 + \varepsilon^{1/2} X_1 + \cdots$, AND SO WE OBTAIN

$$\varepsilon\left(\frac{1}{2} p''(-1) X_1^2 + g(-1)\right) + \cdots = 0 \quad \longrightarrow \quad \frac{1}{2}(-4)X_1^2 - 1 = 0 \rightarrow X_1 = \pm i/\sqrt{2}$$

THIS YIELDS THAT $X = -1 \pm i\varepsilon^{1/2}\left(1/\sqrt{2}\right) + \cdots$

EXAMPLE (WILKINSON POLYNOMIAL)

CONSIDER $p(X) = (X-1)(X-2)\cdots(X-20)$ AND $g(X) = X^{19}$. WE WANT TO FIND THE ROOTS OF $p_\varepsilon(X) = p(X) + \varepsilon g(X) = 0$. WE COULD WRITE $p_\varepsilon(X)$ AS

$$p_\varepsilon(X) = X^{20} - (210 - \varepsilon)X^{19} + \cdots + 20!$$

THUS, THE PERTURBATION APPEARS ONLY IN THE COEFFICIENT OF X^{19}. SUPPOSE THAT ε IS THE MACHINE PRECISION AND $\varepsilon = 1.0 \times 10^{-8}$. NOTICE THAT $p(K) = 0$ WITH $K = 1, 2, \ldots, 20$ WITH $p(K) = 0$ AND $p'(K) \neq 0$. WE CALCULATE,

$X = K - \dfrac{\varepsilon g(K)}{p'(K)} + \cdots$. WE CALCULATE $p'(K) = (-1)^K (K-1)!\,(20-K)!$, SO THAT

$$X = K + (-1)^{K+1} \varepsilon K^{19} / (K-1)!\,(20-K)! + \cdots$$

REMARK (i) $p'(K) = (K-1)(K-2)\cdots(K-(K-1))(K-(K+1))\cdots(K-20) = (-1)^K (K-1)!\,(20-K)!$.

THE ROOT IS WRITTEN AS $X = K + (-1)^{K+1}\varepsilon a_K + \cdots \qquad a_K = \dfrac{K^{19}}{(K-1)!\,(20-K)!}$ (A3)

USING A COMPUTER WE CALCULATE THAT a_K IS LARGEST WHEN $K=16$.

IN PARTICULAR, FOR THE ROOTS $K=16$ AND $K=20$ WE CALCULATE THAT

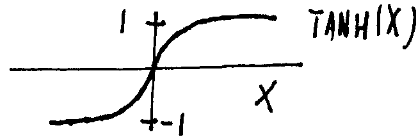
$$X = 16 - 1.5\times 10^{8}\varepsilon + \cdots \qquad X = 20 - 4.3\times 10^{7}\varepsilon + \cdots .$$

THUS FOR $K=16$ AND $\varepsilon = 1.0\times 10^{-8}$ THE PERTURBATION HAS A LARGE EFFECT ON THE ROOT.

SPECIFICALLY, IF ε IS A ROUNDOFF ERROR IN SINGLE PRECISION WE WOULD BE DOOMED IN COMPUTING THESE ROOTS.

TRANSCENDENTAL EQUATIONS

ITERATIVE METHODS CAN BE USED TO SOLVE TRANSCENDENTAL EQUATIONS APPROXIMATELY. WE RECALL THAT AN ITERATION PROCEDURE $X_{n+1} = F(X_n)$ CONVERGES TO A ROOT OF $X = F(X)$ WHEN $|F'(X_n)| < 1$ BY THE CONTRACTION MAPPING PRINCIPLE.



EXAMPLE 1 CONSIDER $X + \tanh X = \lambda$, $\lambda \to \infty$

NOW SINCE $\tanh X \sim 1 - 2e^{-2X} + \ldots$ AS $X \to \infty$ THEN THERE IS A ROOT NEAR $X = \lambda$

THIS IS THE ONLY ROOT SINCE $\frac{d}{dx}(X + \tanh X) = 1 + \operatorname{sech}^2 X > 0$.

THE ITERATION IS:

$$X_{n+1} = \lambda - \tanh X_n \quad \text{WITH} \quad X_0 = \lambda \quad \text{AND} \quad \lambda >> 1$$

THEN $$X_1 = \lambda - \tanh \lambda \sim \lambda - 1 + \ldots$$

$$X_2 = \lambda - \tanh X_1 = \lambda - \tanh(\lambda - 1) \sim \lambda - 1 + 2e^{-2(\lambda-1)} + \ldots$$

THUS $$X \sim \lambda - 1 + 2e^{-2(\lambda-1)}$$

WHEN $\lambda = 5$ THE APPROXIMATE ROOT IS $X = 4.0006709$

COMPARED TO THE EXACT ROOT $X = 4.0006698$.

SO λ DOES NOT HAVE TO BE VERY LARGE BEFORE THE APPROXIMATE FORMULA BECOMES RATHER ACCURATE.

EXAMPLE 2 FIND THE ROOT OF $X^2 - \log X = \lambda$ AS $\lambda \to \infty$.

WE WRITE THE ITERATION AS $X_{n+1} = (\lambda + \log X_n)^{1/2}$ WITH $X_0 = \lambda^{1/2}$.

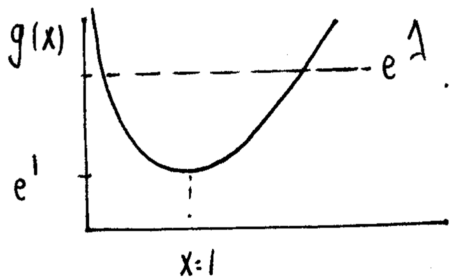
THEN $$X_1 = \left(\lambda + \frac{1}{2}\log\lambda\right)^{1/2} \sim \lambda^{1/2}\left(1 + \frac{1}{2\lambda}\log\lambda\right)^{1/2} \sim \lambda^{1/2}\left(1 + \frac{\log\lambda}{4\lambda} + \ldots\right)$$

THIS GIVES $$X \sim \lambda^{1/2} + \frac{\log\lambda}{4\lambda^{1/2}} + \ldots$$

EXAMPLE 3 FIND THE ROOTS OF $xe^{1/x} = e^{\lambda}$ AS $\lambda \to \infty$.

let $g(x) = xe^{1/x}$ $\quad g'(x) = e^{1/x} - \frac{1}{x}e^{1/x}$. so $g'(1) = 0$, $g'(x) \gtrless 0$ FOR $x \gtrless 1$.

THE PLOT IS AS FOLLOWS:



- SO FOR $\lambda \gg 1$ WE HAVE ONE ROOT WITH $x \approx e^{\lambda}$.
- THE OTHER ROOT IS WHERE $e^{1/x} \sim e^{\lambda}$ SO THAT $x \approx 1/\lambda$.

$(1+h)^p \approx 1 + ph + \cdots$

$\log(1+h) \approx h - h^2/2$

$e^h \sim 1 + h + h^2/2 + \cdots$

$h \to 0$

FOR THE ROOT WITH $x \sim e^{\lambda}$ WE LET

$$x_{n+1} = e^{\lambda} e^{-1/x_n} \quad \text{WITH} \quad x_0 = e^{\lambda}.$$

THEN $\quad x_1 = e^{\lambda} e^{-e^{-\lambda}} \approx e^{\lambda}\left(1 - e^{-\lambda} + \cdots\right) = e^{\lambda} - 1 + \cdots$

$$x_2 = e^{\lambda} e^{-1/x_1} = e^{\lambda} e^{-1/(e^{\lambda}-1)} \sim e^{\lambda}\left(1 - \frac{1}{e^{\lambda}-1} + \frac{1}{2(e^{\lambda}-1)^2} + \cdots\right)$$

HENCE $\quad x_2 \approx e^{\lambda}\left(1 - \frac{1}{e^{\lambda}(1-e^{-\lambda})} + \frac{1}{2}e^{-2\lambda} + \cdots\right) \approx e^{\lambda}\left(1 - e^{-\lambda}(1+e^{-\lambda}) + \frac{1}{2}e^{-2\lambda}\right)$

THIS YIELDS $\quad x_2 \approx e^{\lambda} - 1 + \frac{1}{2}e^{-\lambda} + \cdots \quad \lambda \to \infty.$

NOW FOR THE OTHER ROOT WE WRITE $e^{1/x} = x^{-1}e^{\lambda} \to \frac{1}{x} = -\log x + \lambda$

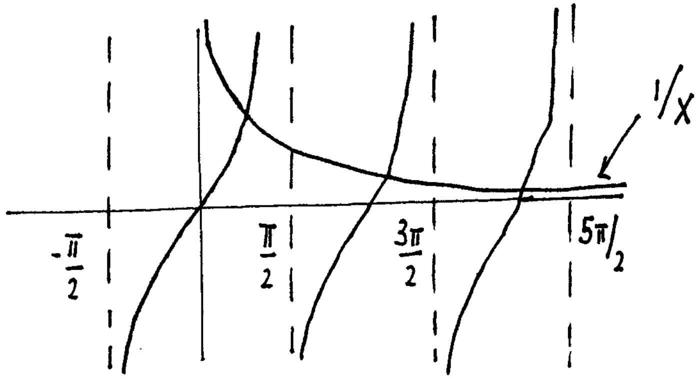
HENCE $\quad x_{n+1} = \dfrac{1}{-\log x_n + \lambda}. \quad x_0 = 1/\lambda.$

NOW $\quad x_1 = \dfrac{1}{\lambda + \log\lambda} = \dfrac{1}{\lambda(1 + \log\lambda/\lambda)} = \dfrac{1}{\lambda}\left(1 - \log\lambda/\lambda + \cdots\right) = \dfrac{1}{\lambda} - \dfrac{\log\lambda}{\lambda^2} + \cdots$

NOW $\quad x_2 = \dfrac{1}{\lambda - \log\left[\frac{1}{\lambda}(1-\log\lambda/\lambda)\right]} = \dfrac{1}{\lambda + \log\lambda - \log(1-\log\lambda/\lambda)} \approx \dfrac{1}{\lambda + \log\lambda + \log\lambda/\lambda}$

SO $\quad x_2 = \dfrac{1}{\lambda\left(1 + \frac{\log\lambda}{\lambda} + \frac{\log\lambda}{\lambda^2}\right)} = \dfrac{1}{\lambda}\left(1 - \dfrac{\log\lambda}{\lambda} - \dfrac{\log\lambda}{\lambda^2} + \left(\dfrac{\log\lambda}{\lambda}\right)^2 + \cdots\right)$

EXAMPLE FIND THE ROOTS OF $X \tan X = 1$ FOR X LARGE.



THE PICTURE SHOWS THAT THE ROOTS WILL BE NEAR $X = n\pi$ FOR n LARGE, WHERE n IS AN INTEGER.

WE WRITE $\tan X = \dfrac{1}{X}$ AND $X = n\pi + \tan^{-1}(1/X)$ WHERE $\tan^{-1} y$ IS THE PRINCIPLE VALUE OF THE ARCTANGENT. OUR ITERATION IS

$$X_{m+1} = n\pi + \tan^{-1}(1/X_m) \quad \text{WITH} \quad X_0 = n\pi.$$

WE RECALL $\tan^{-1} y = y - y^3/3 + \cdots$ AS $y \to 0$ (TAYLOR SERIES)

WE GET
$$X_1 = n\pi + \tan^{-1}\left(\frac{1}{X_0}\right) \approx n\pi + \frac{1}{n\pi}$$

$$X_2 = n\pi + \tan^{-1}(1/X_1) = n\pi + \frac{1}{n\pi\left(1+\frac{1}{n^2\pi^2}\right)} - \frac{1}{3\left(n\pi + 1/n\pi\right)^3}$$

THIS YIELDS
$$X_2 \cong n\pi + \frac{1}{n\pi} - \frac{1}{n^3\pi^3} - \frac{1}{3n^3\pi^3} + \cdots$$

HENCE
$$X \approx n\pi + \frac{1}{n\pi} - \frac{4}{3n^3\pi^3} + \cdots$$