MATRIX EIGENVALUE PERTURBATION THEORY

CONSIDER THE MATRIX EIGENVALUE PROBLEM

$$A\underline{x} + \varepsilon B\underline{x} = \lambda \underline{x}$$

A, B $N \times N$ (REAL MATRICES) A NOT NECESSARILY SYMMETRIC.

ASSUME THAT $\lambda = \lambda_0$ IS A <u>SIMPLE EIGENVALUE</u> OF A WITH EIGENVECTOR $\underline{x}_0$

WE THEN WRITE:

$$\underline{x} = \underline{x}_0 + \varepsilon \underline{x}_1 + \varepsilon^2 \underline{x}_2 + \dots$$

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \dots$$

THIS YIELDS: (1) $(A - \lambda_0 I)\underline{x}_1 = -B\underline{x}_0 + \lambda_1 \underline{x}_0$

(2) $(A - \lambda_0 I)\underline{x}_2 = -B\underline{x}_1 + \lambda_2 \underline{x}_0 + \lambda_1 \underline{x}_1$.

NOW LET $\underline{x}_0^*$ BE THE EIGENVECTOR OF A^T. SINCE A, A^T HAVE SAME EIGENVALUES $\Longrightarrow$ $A^T \underline{x}_0^* = \lambda_0 \underline{x}_0^*$.

WE TAKE THE DOT PRODUCT:

$$(\underline{x}_0^*)^T (A - \lambda_0 I)\underline{x}_1 = [\underbrace{(A^T - \lambda_0 I)\underline{x}_0^*}_{=0}]^T \underline{x}_1 = -(\underline{x}_0^*)^T B\underline{x}_0 + \lambda_1 \underline{x}_0^{*T}\underline{x}_0$$

HENCE $$\lambda_1 = \frac{\underline{x}_0^{*T} B \underline{x}_0}{\underline{x}_0^{*T}\underline{x}_0}$$

THEN WE OBTAIN THAT $$(A - \lambda_0 I)\underline{x}_1 = \underline{r}_1 \equiv -B\underline{x}_0 + \frac{\underline{x}_0^{*T} B\underline{x}_0}{\underline{x}_0^{*T}\underline{x}_0}\underline{x}_0$$

WITH $\underline{x}_0^{*T}\underline{r}_1 = 0$. HENCE, $\exists$ A SOLUTION FOR $\underline{x}_1$ AND TO MAKE IT UNIQUE WE CAN CONVIENTLY IMPOSE THAT

$$\underline{x}_0^{*T}\underline{x}_1 = 0.$$

SIMILARLY, FOR THE X_2 EQUATION IN (2) WE OBTAIN

$$\underline{X}_0^{*T} (A-\lambda_0 I)\underline{X}_2 = [(A^T-\lambda_0 I)\underline{X}_0^*]^T \underline{X}_2 = 0 = -X_0^{*T} B \underline{X}_1 + \lambda_2 \underline{X}_0^{*T} \underline{X}_0 + \lambda_1 X_0^{*T} X_1$$

SINCE $\underline{X}_0^{*T} \underline{X}_1 = 0$ WE OBTAIN

$$\lambda_2 = \frac{\underline{X}_0^{*T} B \underline{X}_1}{\underline{X}_0^{*T} \underline{X}_0} .$$

REMARK (i) WE DID NOT NEED TO IMPOSE $X_0^{*T} X_1 = 0$. THE EXPRESSION FOR λ_2 IS INDEPENDENT OF THIS CHOICE. IF WE SET $\underline{X}_1 = \underline{X}_{1p} + \alpha \underline{X}_0$ where $X_0^{*T} X_{1p} = 0$ AND α IS ANY SCALAR, THEN WE WOULD OBTAIN

$$-\underline{X}_0^{*T} B \underline{X}_{1p} - \alpha \underline{X}_0^{*T} B \underline{X}_0 + \lambda_2 \underline{X}_0^{*T} \underline{X}_0 + \lambda_1 \underline{X}_0^{*T} X_{1p} + \alpha \lambda_1 X_0^{*T} X_0 = 0$$

NOTICE THAT FROM FORMULA FOR λ_1 THE $\sim$ TERMS CANCEL, AND $\underline{X}_0^{*T} \underline{X}_{1p} = 0$. THIS GIVES OUR FORMULA FOR λ_2.

THEREFORE WE OBTAIN

$$\lambda \sim \lambda_0 + \varepsilon \frac{\underline{X}_0^{*T} B \underline{X}_0}{\underline{X}_0^{*T} \underline{X}_0} + \varepsilon^2 \frac{\underline{X}_0^{*T} B \underline{X}_1}{\underline{X}_0^{*T} \underline{X}_0} + \ldots$$

MULTIPLE-ROOTS SUPPOSE THAT $\lambda = \lambda_0$ IS AN EIGENVALUE THAT GENERATES n INDEPENDENT EIGENVECTORS $\underline{V}_1, \ldots, \underline{V}_n$ SO THAT

$$\underline{X}_0 = C_1 \underline{V}_1 + \ldots + C_n \underline{V}_n .$$

WE THEN WRITE

$$\underline{X} = c_1 \underline{V}_1 + \dots + c_n \underline{V}_n + \varepsilon \underline{X}_1 + \dots$$

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \dots$$

THIS YIELDS THAT $\quad A \underline{X}_1 = + \lambda_0 \underline{X}_1 + \lambda_1 \sum_{j=1}^{n} c_j \underline{V}_j - \sum_{j=1}^{n} c_j B \underline{V}_j .$

NOW LET $\underline{V}_1^*, \dots, \underline{V}_n^*$ BE EIGENVECTORS OF A^T WITH EIGENVALUE λ_0.

WE THEN OBTAIN FOR $i = 1, \dots, n$

$$(V_i^*)^T [A - \lambda_0 I] X_1 = [(A^T - \lambda_0 I) V_i^*] X_1 = 0$$

$$= \lambda_1 \sum_{j=1}^{n} c_j V_i^{*T} V_j - \sum_{j=1}^{n} c_j V_i^{*T} B V_j$$

THEREFORE, WE HAVE THAT $\underline{c}, \lambda_1$ SATISFY THE GENERALIZED MATRIX EIGENVALUE PROBLEM:

$$\mathcal{B} \underline{c} = \lambda_1 \mathcal{M} \underline{c} \qquad \mathcal{B}_{ij} \equiv (\underline{V}_i^*)^T B \underline{V}_j$$

$$\mathcal{M}_{ij} \equiv \underline{V}_i^{*T} \underline{V}_j$$

<u>REMARK</u> BY MAKING $\underline{V}_1, \dots, \underline{V}_n$ ORTHOGONAL WE CAN ENSURE $\mathcal{M} = I$ WHEN A IS A SYMMETRIC MATRIX.

(i) THIS IS A SUBPROBLEM OF SIZE $n \times n$ WITH $n \le N$ (ORIGINAL SIZE OF MATRIX). IT INDICATES HOW THE PERTURBATION WILL BREAK THE DEGENARACY IN n POSSIBLE WAYS TYPICALLY.

(ii) <u>EXAMPLE</u> $\quad \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \underline{X} = \lambda \underline{X}$

THIS GIVES $\quad (1-\lambda)^2 + \varepsilon^2 = 0 \quad$ OR $\quad \lambda = 1 \pm i\varepsilon$

<u>EXAMPLE</u> FOR $\varepsilon \ll 1$ CALCULATE THE SOLUTION TO

$$A\underline{x} + \varepsilon B \underline{x} = b$$

WHERE A $N \times N$ SYMMETRIC, B IS $N \times N$, AND $A\underline{\phi}_0 = 0$ WITH $\dim \mathcal{N}(A) = 1$

AND $\underline{b}^T \underline{\phi}_0 \neq 0$. NOTICE THAT $A\underline{x} = b$ HAS NO SOLUTION IN THIS CASE!

WE FIRST CONSIDER THE EIGENVALUE PROBLEM:

$$A\underline{\phi} + \varepsilon B \underline{\phi} = \lambda \underline{\phi}$$

WHERE WE TRY

$$\lambda = \varepsilon \lambda_0 + \varepsilon^2 \lambda_1 + \cdots$$

$$\phi = \phi_0 + \varepsilon \phi_1 + \cdots$$

THUS $A\underline{\phi}_0 = 0$ $\quad A\underline{\phi}_1 = -B\underline{\phi}_0 + \lambda_0 \underline{\phi}_0$

NOW $(\underline{\phi}_0^T A \underline{\phi}_1) = (A\underline{\phi}_0)^T \underline{\phi}_1 = -\underline{\phi}_0^T B \underline{\phi}_0 + \lambda_0 \underline{\phi}_0^T \underline{\phi}_0 = 0.$

THIS YIELDS THAT $\lambda_0 = \underline{\phi}_0^T B \underline{\phi}_0 / \underline{\phi}_0^T \underline{\phi}_0$

AND INDICATES THAT $\lambda = O(\varepsilon)$.

THIS SUGGESTS THAT WE LOOK FOR

$$\underline{X} = \frac{\underline{X}_0}{\varepsilon} + \underline{X}_1 + \cdots$$

THEN $A\underline{X}_0 = 0$ SO THAT $\underline{X}_0 = C\underline{\phi}_0$ WITH $\underline{\phi}_0^T \underline{\phi}_0 = 1$

$$A\underline{X}_1 = \underline{b} - B\underline{X}_0 = \underline{b} - CB\underline{\phi}_0$$

NOW $\underline{\phi}_0^T A \underline{X}_1 = (A\underline{\phi}_0)^T \underline{X}_1 = 0 = \underline{\phi}_0^T \underline{b} - C\underline{\phi}_0^T B \underline{\phi}_0 = 0$

OR $C = \underline{\phi}_0^T \underline{b} / \underline{\phi}_0^T B \underline{\phi}_0 \Rightarrow \underline{X} \sim C\underline{\phi}_0 / \varepsilon$

CONSIDER EIGENVALUE PROBLEMS OF THE FORM

$$(1)\quad \begin{cases} u'' + (\lambda + \varepsilon a(x))\, u = 0 & 0 < x < L \\ u(0) = u(L) = 0 \end{cases}$$

OR MORE GENERALLY,

$$\Delta u + [\lambda + \varepsilon a(\underline{x})]\, u = 0 \quad \text{IN } \Omega$$

$$u = 0 \quad \text{ON } \partial\Omega$$

HOW DO WE CALCULATE THE CORRECTION TO THE EIGENVALUES?

WE BEGIN BY RECALLING LAGRANGE'S IDENTITY:

$$\int_0^L (u \mathcal{L} v - v \mathcal{L} u)\, dx = (u v' - v u')\Big|_0^L \qquad \text{WHERE} \quad \mathcal{L}u = u'' + \lambda u.$$

THIS IS PROVED BY INTEGRATION BY PARTS.

THEREFORE, IF $u(0) = u(L) = 0$ AND $v(0) = v(L) = 0$ THEN IF $\mathcal{L}v = 0$ AND $\mathcal{L}u = f$ WE CONCLUDE THAT

$$\int_0^L [u \,(0) - v f]\, dx = 0 \qquad \text{OR} \qquad \int_0^L v f \, d\underline{x} = 0.$$

TO APPLY THIS TO (1), WE SUPPOSE THAT WHEN $\varepsilon = 0$ THEN

$$u_0 = \sin\left(\frac{n\pi x}{L}\right) \quad \text{AND} \quad \lambda_0 = \frac{n^2\pi^2}{L^2}.$$

WE WANT TO DETERMINE THE CORRECTIONS TO THE EIGENVALUE. WE EXPAND

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots \qquad u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

WE THEN DERIVE THAT

$$\mathcal{L}(u_1) \equiv u_1'' + \lambda_0 u_1 = -\lambda_1 u_0 - a(x) u_0$$
$$u_1(0) = u_1(L) = 0$$

THE SOLVABILITY CONDITION DETERMINES λ_1:

WE WRITE: $\displaystyle\int_0^L (u_0 \mathcal{L} u_1 - u_1 \mathcal{L} u_0)\, dx = (u_0 u_1' - u_1 u_0')\Big|_0^L$

HENCE $\qquad (u_0, \mathcal{L} u_1) - (u_1, \mathcal{L} u_0) = 0 \qquad$ WHERE $\displaystyle (f,g) = \int_0^L f g \, dx.$

THIS YIELDS THAT $\mathcal{L} u_0 = 0$, $\mathcal{L} u_1 = -\lambda_1 u_0 - a(x) u_0$

$$-\lambda_1 (u_0, u_0) - (u_0, a u_0) = 0,$$

OR EQUIVALENTLY

$$\lambda_1 = -\frac{(u_0, a u_0)}{(u_0, u_0)} = -\frac{\int_0^L a(x) \sin^2(n\pi x/L)\, dx}{\int_0^L \sin^2(n\pi x/L)\, dx}.$$

REMARK (i) IN THE CASE WHERE $u(0) = u(L)$, $u'(0) = u'(L)$ WE HAVE $\lambda_0 = 4n^2\pi^2/L^2$ AND $u_0 = c_1 \cos(2n\pi x) + c_2 \sin(2n\pi x)$ THE PERTURBATION $\varepsilon a(x)$ BREAKS THE EIGENVALUE DEGENERACY.

EXAMPLE 1 CALCULATE FOR $\varepsilon \ll 1$ THE FIRST TWO TERMS FOR THE SOLUTION TO

$$u'' - \varepsilon u = f(x), \quad 0<x<1, \quad \varepsilon \ll 1 \qquad \text{WITH} \int_0^1 f(x)\, dx \neq 0$$
$$u'(0) = u'(1) = 0$$

REMARK NOTICE THAT IF $\varepsilon = 0$ THEN $u'' = f(x)$, $0 < x < 1$
$$u'(0) = u'(1) = 0.$$

THIS HAS NO SOLUTION SINCE $0 = \int_0^1 u''\,dx = u'(1) - u'(0) = \int_0^1 f(x)\,dx$,

BUT $\int_0^1 f(x)\,dx \neq 0$ BY ASSUMPTION.

WE INSTEAD EXPAND $u = \dfrac{u_0}{\varepsilon} + u_1 + \varepsilon u_2 + \cdots$

WE SUBSTITUTE AND COLLECT POWERS OF ε TO OBTAIN

$$\varepsilon^0: \quad u_0'' = 0, \quad 0 < x < 1, \qquad u_0'(0) = u_0'(1) = 0$$

$$\varepsilon^1: \quad u_1'' = f(x) - u_0, \quad 0 < x < 1, \qquad u_1'(0) = u_1'(1) = 0$$

$$\varepsilon^2: \quad u_2'' = -u_1, \quad 0 < x < 1 \qquad u_2'(0) = u_2'(1) = 0$$

THE SOLUTION TO u_0 IS $u_0 = C$, WITH C ARBITRARY CONSTANT.

AT NEXT ORDER, $u_1'' = f(x) - C$

WE INTEGRATE $\int_0^1 \longrightarrow \int_0^1 u_1''\,dx = u_1'(1) - u_1'(0) = \int_0^1 (f(x) - C)\,dx = 0.$

THIS YIELDS THAT $C = \int_0^1 f(x)\,dx.$

THEN $u_1'' = f(x) - \int_0^1 f(x)\,dx$

THE SOLUTION TO ε^2 PROBLEM YIELDS $\int_0^1 u_2''\,dx = u_2'(1) - u_2'(0) = \int_0^1 u_1\,dx = 0.$

THEREFORE, u_1 MUST BE THE UNIQUE SOLUTION OF

$$u_1'' = f(x) - \int_0^1 f(x)\,dx \qquad u_1'(0) = u_1'(1) = 0$$

WITH $\int_0^1 u_1\,dx = 0$

THEN $u(x) = \left(\int_0^1 f(x)\,dx\right)/\varepsilon + u_1(x) + \cdots \qquad \varepsilon \ll 1.$

REMARK HOW WOULD ONE HAVE ANTICIPATED $u \sim u_0/\varepsilon + u_1 + \dots$?

TO SEE THIS CONSIDER EIGENVALUE PROBLEM

$$\mathcal{L}_\varepsilon \phi \equiv \phi'' - \varepsilon \phi = \lambda \phi \quad , \quad 0 < x < 1.$$

$$\phi'(0) = \phi'(1) = 0$$

THIS IS A STURM-LIOUVILLE PROBLEM WITH FOLLOWING PROPERTIES

(i) countably infinite # eigenvalues $\mathcal{L}_\varepsilon \phi_j = \lambda_j \phi_j$ $\lambda_j \geq \lambda_{j+1}$ $\forall j$.
the eigenvalues are real $\lambda_j \to -\infty$ as $j \to \infty$

(ii) ORTHOGONALITY $(\phi_j, \phi_K) = \int_0^1 \phi_j \phi_K \, dx = 0$ FOR $j \neq K$.

(iii) ANY FUNCTION $g(x) \in L^2$ CAN BE WRITTEN AS

$$g(x) = \sum_{j=1}^{\infty} c_j \phi_j(x) \qquad \text{WITH} \qquad \frac{(\phi_j, g(x))}{(g(x), g(x))} = c_j \quad \text{BY ORTHOGONALITY}$$

THEREFORE OUR PROBLEM IS

$$\mathcal{L}_\varepsilon u = f(x) \qquad 0 < x < 1$$

$$u'(0) = u'(1) = 0$$

HENCE $\quad u = \sum_{j=1}^{\infty} c_j \phi_j \longrightarrow \mathcal{L}_\varepsilon u = \sum_{j=1}^{\infty} c_j \mathcal{L}_\varepsilon \phi_j = \sum_{j=1}^{\infty} c_j \lambda_j \phi_j = f(x)$

BY ORTHOGONALITY, $\quad \sum_{j=1}^{\infty} c_j \lambda_j (\phi_j, \phi_K) = (f, \phi_K)$

$$\longrightarrow c_K \lambda_K (\phi_K, \phi_K) = (f, \phi_K)$$

THEREFORE,

$$u(x) = \sum_{j=1}^{\infty} \frac{(f, \phi_j)}{\lambda_j (\phi_j, \phi_j)} \phi_j(x)$$

WE CONCLUDE THAT IF $\lambda_1 = O(\varepsilon)$, THEN $u = O(1/\varepsilon)$. NOTICE $\lambda_1 = \varepsilon$, $\phi_1 = 1$ IS AN EIGENFUNCTION PAIR!!

EXAMPLE 2 FIND A 2-TERM EXPANSION FOR

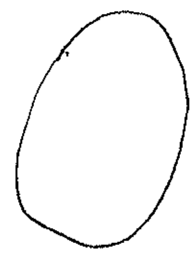
$$\Delta u = F(\underline{x}) \quad \text{IN} \quad \Omega \qquad \text{(see homework).}$$

$$\partial_n u - \varepsilon(u - g) = 0 \quad \text{ON} \quad \partial\Omega \qquad \varepsilon \ll 1.$$

($\partial_n u$ = outward normal derivative). Assume $\int_\Omega F(\underline{x})\, d\underline{x} \neq 0$.

EIGENVALUES IN TWO-SPACE DIMENSIONS

CONSIDER THE PERTURBED EIGENVALUE PROBLEM



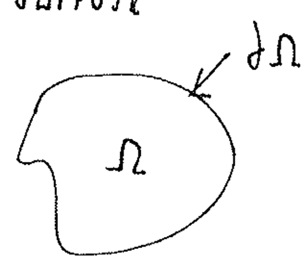
$$\Delta \phi + (\lambda + \varepsilon a(\underline{x}))\, \phi = 0 \quad \text{IN} \quad \Omega$$

$$\phi = 0 \quad \text{ON} \quad \partial\Omega$$

HOW DOES ONE CALCULATE THE CORRECTIONS TO THE EIGENVALUES?

RECALL FIRST SOME BASIC RESULTS:

LEMMA LET $\mathcal{L} u \equiv \Delta u + q(\underline{x}) u$ WITH $\Omega \in \mathbb{R}^N$. SUPPOSE WE WANT TO SOLVE



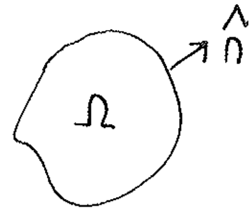
$$(*) \quad \begin{cases} \mathcal{L} u = f(\underline{x}) & \text{IN } \Omega \\ u = 0 & \text{ON } \partial\Omega \end{cases}$$

THIS PROBLEM HAS A SOLUTION IF AND ONLY IF

$$\int_\Omega f(\underline{x})\, \phi_0(\underline{x})\, d\underline{x} = 0 \quad \text{whenever} \quad \mathcal{L}\phi_0 = 0 \text{ IN } \Omega, \quad \phi_0 = 0 \text{ ON } \partial\Omega.$$

PROOF DEFINE $(u, v) \equiv \int_\Omega u v \, d\underline{x}$ FOR ANY TWO FUNCTIONS u, v.

WE THEN RECALL GREEN'S SECOND IDENTITY:



$$\int_\Omega (u\,\Delta v - v\,\Delta u)\,d\underline{x} = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$$

WHERE $\partial u/_{\partial n} = \nabla u \cdot \hat{n}$ AND $\hat{n}$ OUTWARD NORMAL (UNIT) TO $\partial\Omega$.

<u>PROOF</u> $\nabla\cdot[u\nabla v] = \nabla u\cdot\nabla v + u\Delta v$

$\nabla\cdot[v\nabla u] = \nabla v\cdot\nabla u + v\Delta u$

SUBTRACT AND USE DIVERGENCE THEOREM: $\int_\Omega (u\Delta v - v\Delta u)\,d\underline{x} = \int_\Omega \left(\nabla\cdot[u\nabla v] - \nabla\cdot[v\nabla u]\right) d\underline{x}$

$$= \int_{\partial\Omega} (u\,\nabla v\cdot\hat{n} - v\,\nabla u\cdot\hat{n})\,ds \;\checkmark$$

GREEN'S SECOND IDENTITY PLACES SAME ROLE AS LAGRANGE'S IDENTITY.

BY ADDING AND SUBTRACTING $q\,u$ WE GET

$$\int_\Omega (u\,\mathcal{L}v - v\,\mathcal{L}u)\,d\underline{x} = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds. \qquad \mathcal{L} \text{ IS SELF-ADJOINT}$$

- IT IS NECESSARY THAT $\int_\Omega f\,\phi_0\,d\underline{x} = 0$ WHENEVER $\mathcal{L}\phi_0 = 0$ IN Ω
  $\phi_0 = 0$ ON $\partial\Omega$.

<u>PROOF</u> $$(u, \mathcal{L}\phi_0) - (\phi_0, \mathcal{L}u) = \int_{\partial\Omega}\left(u\frac{\partial\phi_0}{\partial n} - \phi_0\frac{\partial u}{\partial n}\right) ds = 0.$$

BUT $\mathcal{L}u = f$ AND $\mathcal{L}\phi_0 = 0 \longrightarrow (\phi_0, f) = \int_\Omega \phi_0(\underline{x})\,f(\underline{x})\,d\underline{x} = 0.$

- THE CONDITION $(\phi_0, f) = 0$ IS SUFFICIENT FOR THERE TO BE A SOLUTION.

TO PROVE THIS WE CONSIDER THE EIGENVALUE PROBLEM

$$\mathcal{L}\phi \equiv \Delta\phi + q(\underline{x})\,\phi = \lambda\phi \qquad x\in\Omega$$

$$\phi = 0 \quad \text{ON } \partial\Omega$$

FROM COURANT AND HILBERT VOL. I CHAPTER 5 WE CONCLUDE THAT

(i) $\exists$ A COUNTABLE INFINITY OF EIGENVALUES λ_j. THE CORRESPONDING EIGENVECTORS ARE ORTHOGONAL IN THE SENSE THAT

$$\int_\Omega \phi_j \phi_k \, d\underline{x} = 0 \quad \text{FOR } j \neq k \quad \left(\text{WE CAN NORMALIZE} \int_\Omega \phi_j^2 d\underline{x} = 1 \right)$$

(ii) ANY L^2 FUNCTION $f(\underline{x})$ CAN BE EXPANDED AS

$$f(\underline{x}) = \sum_{j=0}^{\infty} c_j \phi_j(\underline{x}) \qquad c_j = \int_\Omega f(x)\phi_j \, d\underline{x} \Big/ \int_\Omega \phi_j^2 \, d\underline{x}.$$

(COMPLETENESS)

(iii) IF λ_j IS A REPEATED EIGENVALUE, THEN WE CAN ALWAYS FIND ENOUGH INDEPENDENT EIGENFUNCTIONS.

SO TO SOLVE (*) ON P. 11 WE EXPAND

$$u(\underline{x}) = \sum_{j=0}^{\infty} c_j \phi_j(x) \qquad c_j = (u, \phi_j) \quad \text{WHEN } (\phi_j, \phi_j) = 1.$$

THEN $$\int_\Omega (\phi_j \mathcal{L} u - u \mathcal{L} \phi_j) \, dx = \int_{\partial\Omega} \left(\phi_j \frac{\partial}{\partial n} u - u \frac{\partial \phi_j}{\partial n}\right) ds = 0$$

SO $$\int_\Omega f \phi_j \, d\underline{x} = \lambda_j \int u \phi_j \, d\underline{x} = \lambda_j c_j.$$

HENCE, $$c_j = \frac{1}{\lambda_j}(f, \phi_j) \longrightarrow \quad (+) \; u(\underline{x}) = \sum_{j=0}^{\infty} \frac{1}{\lambda_j}(f, \phi_j)\, \phi_j(x)$$

THUS IF $\exists$ ϕ_0 WITH $\mathcal{L}\phi_0 = 0$ AND $\lambda_0 = 0$ WE MUST HAVE $(f, \phi_0) = 0$. THE SOLUTION FOR u IN $(+)$ IS VALID PROVIDED THAT $(f, \phi_0) = 0$ WHENEVER $\lambda_0 = 0$. HENCE, IF THIS OCCURS SIMPLY DELETE THE $j = 0$ TERM IN THE SUM.

BASIC RESULT 2 THERE ARE SOME 2-D DOMAINS WHERE WE CAN CALCULATE THE EIGENFUNCTIONS AND EIGENVALUES OF $\Delta\phi + \lambda\phi = 0$ IN Ω WITH $\phi = 0$ ON $\partial\Omega$.

EXAMPLE (SQUARE OR RECTANGLE) $\Omega : [0,a] \times [0,b] \rightarrow \begin{matrix} 0<x<a \\ 0<y<b \end{matrix}$

$$\phi = A_{m,n} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

$$\lambda_{m,n} = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \qquad m,n = 1,2,\ldots$$

NOTICE THAT THE LOWEST EIGENVALUE $\lambda_{1,1} = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}$ IS SIMPLE, BUT THAT HIGHER EIGENVALUES MAY NOT BE.

e.g. $a = b$ $\quad \lambda_{1,2} = \lambda_{2,1} = \frac{5\pi^2}{a^2}$ $\qquad \phi = \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right)$

OR $\phi = \sin\left(\frac{2\pi y}{a}\right) \sin\left(\frac{\pi x}{a}\right)$.

EXAMPLE (THE UNIT DISK) $\Delta\phi = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\varphi\varphi} = \lambda\phi$

WITH $\phi = 0$ ON $r = 1$.

WE SEPARATE VARIABLES $\phi = R(r)\Phi(\varphi)$ TO OBTAIN

$$\Phi'' + \mu\Phi = 0 \quad \Phi \ 2\pi \text{ periodic} \longrightarrow \mu = n^2 \quad \Phi = A_n \cos n\varphi + B_n \sin n\varphi$$

$$r^2 R'' + r R' + (r^2\lambda - \mu)R = 0 \longrightarrow R = J_n(\sqrt{\lambda}\, r)$$

$R(1) = 0$, R BOUNDED AS $r \rightarrow 0$

SO $J_n(\sqrt{\lambda_{mn}}) = 0$

THEN $\phi_{m,n}(r,\varphi) = A_n \cos(n\varphi) J_n(\sqrt{\lambda_{mn}}\, r) + B_n \sin(n\varphi) J_n(\sqrt{\lambda_{mn}}\, r)$

NOTICE THAT UNLESS $n = 0$ EACH EIGENVALUE HAS TWO INDEPENDENT EIGENFUNCTI

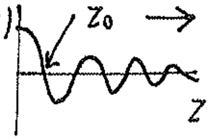
EXAMPLE 1 CONSIDER

$$\Delta\phi + (\lambda - \varepsilon F(r,\varphi))\phi = 0 \qquad 0 \le r \le 1,\ 0 \le \varphi \le 2\pi.$$

$$\phi = 0 \text{ ON } r = 1$$

LET λ_0 BE THE SMALLEST EIGENVALUE OF THE $\varepsilon = 0$ PROBLEM.

THEN $\phi_0 = J_0(\sqrt{\lambda_0}\, r)$ WITH $J_0(\sqrt{\lambda_0}) = 0$



$\rightarrow \lambda_0 = z_0^2$.

HOW DOES THIS EIGENVALUE CHANGE UNDER THE PERTURBATION?

WE LET $\phi = \phi_0 + \varepsilon\phi_1 + \cdots \qquad \lambda = \lambda_0 + \varepsilon\lambda_1 + \cdots$

WE OBTAIN $\mathcal{L}\phi_1 = \Delta\phi_1 + \lambda_0\phi_1 = -\lambda_1\phi_0 + F\phi_0$.

SINCE $\mathcal{L}\phi_0 = 0$ WITH $\phi_0 = 0$ ON $r = 1$, WE OBTAIN (DENOTING $(u,v) = \int_\Omega uv\, d\underline{x}$)

$$(\phi_0, \mathcal{L}\phi_1) - \underbrace{(\phi_1, \mathcal{L}\phi_0)}_{=0} = \int_{\partial\Omega} (\phi_0\, \partial_n\phi_1 - \phi_1\, \partial_n\phi_0)\, ds = 0$$

SO $(\phi_0, \mathcal{L}\phi_1) = 0 \rightarrow -\lambda_1(\phi_0,\phi_0) + (\phi_0, F\phi_0) = 0$

THIS CAN BE WRITTEN LONGHAND AS

$$\lambda_1 = \frac{\int_0^{2\pi}\int_0^1 r\, F(r,\varphi)\, J_0^2(\sqrt{\lambda_0}\, r)\, dr}{\int_0^{2\pi}\int_0^1 J_0^2(\sqrt{\lambda_0}\, r)\, r\, dr}$$

FOR EXAMPLE, IF $F(r,\varphi) = \sin^2\varphi$ (NO r-dependence) THEN $\lambda_1 = 1/2$

AND $\lambda = \lambda_0 + \varepsilon/2$.

EXAMPLE 2 CONSIDER $\Delta\phi + (\lambda - \varepsilon a(x,y))\phi = 0$ IN $0<x<1,\ 0<y<1$

$\phi = 0$ ON boundary OF unit square.

LET $\lambda_0 = 5\pi^2$. THEN FOR $\varepsilon = 0$, $\lambda_{m,n} = m^2\pi^2 + n^2\pi^2$

$$\phi_{m,n} = \sin(m\pi x)\sin(n\pi y)$$

HENCE, CHANGING NOTATION SLIGHTLY, WE SEE THAT

$$\lambda_0 = 5\pi^2, \quad \phi_0 = c_1 V_1 + c_2 V_2 \qquad V_1 = \sin(\pi x)\sin(2\pi y)$$
$$V_2 = \sin(2\pi x)\sin(\pi y)$$

THERE ARE TWO INDEPENDENT EIGENFUNCTIONS FOR THIS EIGENVALUE.

WE EXPAND

$$\phi = \phi_0 + \varepsilon\phi_1 + \cdots = c_1 V_1 + c_2 V_2 + \varepsilon\phi_1$$
$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \cdots$$

THIS YIELDS THAT

$$\mathcal{L}\phi_1 \equiv \Delta\phi_1 + \lambda_0\phi_1 = -\lambda_1\phi_0 + a\phi_0 \quad \text{IN } \Omega$$
$$\phi_1 = 0 \quad \text{ON } \partial\Omega$$

NOW WE NOTICE $\mathcal{L}V_1 = \mathcal{L}V_2 = 0$. THUS

$$-\lambda_1(V_1, \phi_0) + (V_1, a\phi_0) = 0$$
$$-\lambda_1(V_2, \phi_0) + (V_2, a\phi_0) = 0$$

NOW WE HAVE $(V_1, V_2) = \int_\Omega V_1 V_2 \, dx\, dy = 0$. HENCE, WE CALCULATE

$$\begin{pmatrix} (V_1, aV_1) & (V_1, aV_2) \\ (V_2, aV_1) & (V_2, aV_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} (V_1,V_1) & 0 \\ 0 & (V_2,V_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

SUPPOSE THAT WE NORMALIZE SO THAT $(V_1, V_1) = (V_2, V_2) = 1$.

THEN $\underset{\sim}{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ IS DETERMINED FROM

$$\mathcal{M}\underset{\sim}{c} = \lambda_1 \underset{\sim}{c} \qquad \mathcal{M} = \mathcal{M}^T = \begin{pmatrix} (V_1, aV_1) & (V_1, aV_2) \\ (V_2, aV_1) & (V_2, aV_2) \end{pmatrix}$$

DOMAIN PERTURBATIONS

NEXT, WE CONSIDER THE EFFECT OF CHANGING THE SHAPE OF THE DOMAIN. WE CONSIDER

$$\Delta\phi + \lambda\phi = 0 \quad \text{IN } \Omega_\varepsilon \qquad \partial\Omega_\varepsilon : \; r = 1 + \varepsilon f(\varphi)$$

$$\phi = 0 \quad \text{ON } \partial\Omega_\varepsilon$$

CALCULATE THE CORRECTION TO THE PRINCIPAL EIGENVALUE.

WE WRITE $\phi(1+\varepsilon f(\varphi), \varphi) = \phi(1,\varphi) + \varepsilon f(\varphi)\, \phi_{1r}(1,\varphi) + \dfrac{\varepsilon^2 f^2(\varphi)}{2} \phi_{11}$

WE THEN SUBSTITUTE $\phi = \phi_0 + \varepsilon\phi_1 + \dots, \quad \lambda = \lambda_0 + \varepsilon\lambda_1 + \dots$

IF WE SUBSTITUTE WE OBTAIN THAT

$$\mathcal{L}\phi_1 = \Delta\phi_1 + \lambda_0\phi_1 = -\lambda_1\phi_0$$

$$\phi_1 = -f(\varphi)\,\phi_{0r}(1,\varphi)$$

ϕ_1 nice at ORIGIN.

NOW WE CALCULATE $(\phi_0, \mathcal{L}\phi_1) - (\phi_1, \mathcal{L}\phi_0) = -\lambda_1(\phi_0, \phi_0)$

THEN $(\phi_0, \mathcal{L}\phi_1) - (\phi_1, \mathcal{L}\phi_0) = \displaystyle\int_{\partial D} (\phi_0\, \partial_n \phi_1 - \phi_1 \partial_n \phi_0)\, ds$

THIS YIELDS $-\lambda_1(\phi_0, \phi_0) = -\displaystyle\int_{\partial\Omega} \phi_1\, \partial_n \phi_0\, ds = +\int_{\partial\Omega} f(\varphi)\left(\phi_{0r}(1,\varphi)\right)^2 dr$

THIS YIELDS, $\lambda_1 = -\displaystyle\int_{\partial\Omega} f(\varphi)\left(\phi_{0r}(1,\varphi)\right)^2 dr \Big/ (\phi_0, \phi_0)$

BUT $\phi_0 = J_0(\sqrt{\lambda_0}\, r)$ WITH $\sqrt{\lambda_0}$ BEING FIRST ROOT OF $J_0(z) = 0$.

THEN
$$\lambda_1 = \frac{-\lambda_0 \int_0^{2\pi} \left(J_0'(\sqrt{\lambda_0})\right)^2 F(\varphi)\, d\varphi}{2\pi \int_0^1 r \left(J_0'(\sqrt{\lambda_0}\, r)\right)^2 dr}.$$

<u>REMARK</u>
$$\int_0^1 r \left(J_0(\sqrt{\lambda_0}\, r)\right)^2 dr = \frac{1}{2}\left(J_0'(\sqrt{\lambda_0})\right)^2$$

FROM ABRAMOWITZ + STEGUN p. 485.

THIS YIELDS THAT
$$\lambda_1 = -\frac{\left(J_0'(\sqrt{\lambda_0})\right)^2}{\frac{1}{2}\left(J_0'(\sqrt{\lambda_0})\right)^2} \frac{\lambda_0 \int_0^{2\pi} F(\varphi)\, d\varphi}{2\pi}.$$

THIS YIELDS
$$\lambda_1 = -2\lambda_0 \left(\frac{1}{2\pi} \int_0^{2\pi} F(\varphi)\, d\varphi \right).$$

HENCE
$$\lambda_1 = -2\lambda_0 \bar{F} \qquad \bar{F} = \text{average of } F.$$

THEN,
$$\lambda \sim \lambda_0 - 2\varepsilon \lambda_0 \bar{F} + \ldots$$

WHERE
$$\bar{F} \equiv \frac{1}{2\pi} \int_0^{2\pi} F(\varphi)\, d\varphi.$$

NOTICE THAT IF $\bar{F} \equiv 0$ WE MUST PROCEED TO NEXT ORDER TO OBTAIN λ_1.

EXAMPLE: (WELL-STIRRED APPROXIMATION)

THE WELL-STIRRED APPROXIMATION. CONSIDER THE REACTION DIFFUSION SYSTEM

$$(*)\quad \begin{cases} U_t = D\,U_{XX} + \Gamma\, f\left(U/U_0, V/V_0, a(X/L)\right) \\ V_t = D\,V_{XX} + \Gamma\, g\left(U/U_0, V/V_0, b(X/L)\right) \end{cases}$$

WHERE U_0, V_0 SCALES OF U, V, L = length OF domain, Γ = strength OF REACTION. WE WANT TO SOLVE $(*)$ WITH NO-FLUX BOUNDARY CONDITIONS

$$V_X = U_X = 0 \quad \text{ON} \quad X = 0, L.$$

SUPPOSE THAT THE SYSTEM IS WELL-MIXED (OR EQUIVALENTLY THE DIFFUSION IS STRONG). FIND AN APPROXIMATION FOR THE SOLUTION.

WE LET $u = U/U_0$ $\quad v = V/V_0$, $\quad x = X/L$, $\quad t = \omega\tau$

THEN

$$\frac{U_0\,\omega}{\Gamma}\, u_\tau = \frac{D\,U_0}{L^2\,\Gamma}\, u_{xx} + f(u, v, a(x)) \qquad u_x = 0, \ x = 0, 1$$

$$\frac{V_0\,\omega}{\Gamma}\, v_\tau = \frac{D\,V_0}{L^2\,\Gamma}\, v_{xx} + g(u, v, b(x)) \qquad v_x = 0, \ x = 0, 1.$$

NOW WE LET $\omega = \Gamma/U_0$. WE DEFINE $\varepsilon^{-1} = \dfrac{D\,U_0}{L^2\,\Gamma} \gg 1$,

AND $\gamma \equiv V_0/U_0$.

THEN $$U_\tau = \frac{1}{\varepsilon} U_{XX} + f(U, V, a(X)), \qquad U_X = 0 \quad \text{ON} \quad X = 0, 1$$

$$\gamma V_\tau = \frac{1}{\varepsilon} V_{XX} + g(U, V, b(X)), \qquad V_X = 0 \quad \text{ON} \quad X = 0, 1.$$

WE THEN EXPAND

$$U = U_0 + \varepsilon U_1 + \dots \qquad V = V_0 + \varepsilon V_1 + \dots$$

THEN $$U_{0XX} = 0, \quad U_{0X} = 0 \text{ ON } X = 0, 1 \longrightarrow U_0 = U_0(t)$$

$$V_{0XX} = 0, \quad V_{0X} = 0 \text{ ON } X = 0, 1 \longrightarrow V_0 = V_0(t).$$

THEREFORE, AT NEXT ORDER

$$\mathcal{L} U_1 \equiv U_{1XX} = U_0' - f(U_0, V_0, a(X)), \qquad U_{1X} = 0 \quad X = 0, 1$$

$$\mathcal{L} V_1 \equiv V_{1XX} = \gamma V_0' - g(U_0, V_0, b(X)), \qquad V_{1X} = 0, \quad X = 0, 1.$$

SINCE $\mathcal{L} 1 = 0$ THE SOLVABILITY CONDITION IS THAT THE RIGHT HAND-SIDES ARE ORTHOGONAL TO ONE.

HENCE $$\int_0^1 (1) \left(U_0' - f(U_0, V_0, a(X)) \right) dX = 0$$

$$\int_0^1 (1) \left(\gamma V_0' - g(U_0, V_0, b(X)) \right) dX = 0$$

THIS YIELDS THAT $$U_0' = \int_0^1 f(U_0, V_0, a(X))\, dX$$

$$\gamma V_0' = \int_0^1 g(U_0, V_0, b(X))\, dX$$

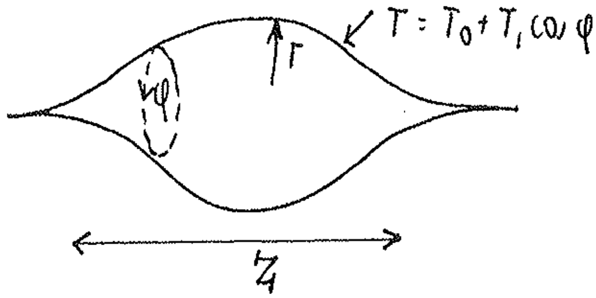
REMARK (i) IF f, g independent OF X WE OBTAIN THAT

$$U_0' = f(U_0, V_0) \qquad \gamma V_0' = g(U_0, V_0)$$

(ii) THE HOMEWORK SHOWS THAT $U_0(0) = \int_0^1 U(X, 0)\, dX$, $V_0(0) = \int_0^1 V(X, 0)\, dX$.

LONG THIN DOMAINS

PROBLEM 1 A SPINDLE OF REVOLUTION IS HEATED SO THAT ITS BOUNDARY IS AT THE TEMPERATURE $T = T_0 + T_1 \cos\varphi$. SUPPOSE THAT THE SPINDLE IS LONG AND SLENDER. CALCULATE APPROXIMATELY THE TEMPERATURE DISTRIBUTION



L = length of spindle
R_0 : typical radius of spindle
Neglect end effects at $z = 0, L$.

THE SYSTEM IS
$$T_{rr} + \frac{1}{r} T_r + \frac{1}{r^2} T_{\varphi\varphi} + T_{zz} = 0 \qquad 0 < r < R_0 F(z/L)$$
$$T = T_0 + T_1 \cos\varphi \qquad \text{ON} \qquad r = R_0 F(z/L)$$

WE LET $r = R_0 \rho$, $z = z/L$, $T = T_0 u$. THIS YIELDS THAT

$$u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\varphi\varphi} + \varepsilon^2 u_{zz} = 0 \qquad 0 < \rho < F(z)$$
$$u(F(z), z) = 1 + \frac{T_1}{T_0} \cos\varphi \qquad \varepsilon = R_0/L \ll 1$$
$$\text{(long domain)}$$
$$u \text{ nice at } \rho = 0$$

WE THEN EXPAND $u = u_0 + \varepsilon^2 u_1 + \ldots$ TO OBTAIN THAT

$$O(1) \left\{ \begin{array}{ll} L u_0 \equiv u_{0\rho\rho} + \frac{1}{\rho} u_{0\rho} + \frac{1}{\rho^2} u_{0\varphi\varphi} = 0 & 0 < \rho < F(z) \\ u_0 = 1 + \frac{T_1}{T_0} \cos\varphi \quad \text{ON} & \rho = F(z) \end{array} \right.$$

$$O(\varepsilon^2) \left\{ \begin{array}{ll} L u_1 = -u_{0zz}, & 0 < \rho < F(z) \\ u_1 = 0 \quad \text{ON} & \rho = F(z) \end{array} \right.$$

TO SOLVE THE $O(1)$ PROBLEM LET

$$u_0 = 1 + \phi_0(\rho)\cos\varphi$$

THEN $$\phi_0'' + \frac{1}{\rho}\phi_0' - \frac{1}{\rho^2}\phi_0 = 0 \longrightarrow \phi_0 = A\rho.$$

WE LET $$u_0 = 1 + \frac{T_1}{T_0}\cos\varphi \quad \text{ON } \rho = f(z) \longrightarrow A f(z) = T_1/T_0.$$

HENCE, $$u_0 = 1 + \frac{T_1}{T_0 f(z)}\rho\cos\varphi.$$

THEN $$L u_1 = -\frac{T_1}{T_0}\rho\cos\varphi\frac{d^2}{dz^2}[f(z)]^{-1}$$

WE LET $$u_1 = \phi_1\cos\varphi \quad \text{WITH } \phi_1 = \phi_1(\rho).$$

THIS YIELDS, $$\phi_1'' + \frac{1}{\rho}\phi_1' - \frac{1}{\rho^2}\phi_1 = -\frac{T_1}{T_0}\frac{d^2}{dz^2}[f(z)]^{-1}\rho.$$

WE SOLVE IN FORM $$\phi_1 = A\rho^m \longrightarrow A(m^2-1)\rho^{m-2} = -\frac{T_1}{T_0}\frac{d^2}{dz^2}[f(z)]^{-1}\rho.$$

HENCE, $m = 3$ AND $$A = -\frac{T_1}{8T_0}\frac{d^2}{dz^2}[f(z)]^{-1}.$$

IN SUMMARY, WE HAVE $$\phi_1 = C\rho - \frac{T_1}{8T_0}\frac{d^2}{dz^2}[f(z)]^{-1}\rho^3$$

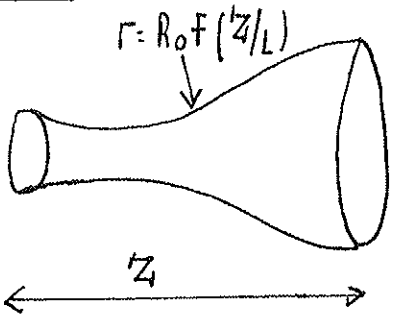
WITH C CHOSEN SO THAT $\phi_1 = 0$ ON $\rho = f(z)$.

BY INSPECTION THIS YIELDS $$\phi_1 = \frac{T_1}{8T_0}\rho\left[(f(z))^2 - \rho^2\right]\frac{d^2}{dz^2}[f(z)]^{-1}.$$

FINALLY $$u \sim 1 + \frac{T_1}{T_0 f(z)}\rho\cos\varphi + \varepsilon\phi_1\cos\varphi + \cdots = 1 + \frac{T_1}{T_0}\cos\varphi\left[\frac{\rho}{f(z)} + \frac{\varepsilon^2\rho}{8}(f^2(z)-\rho^2)\frac{d^2}{dz^2}(f\right.$$

REMARK (i) DEPENDING ON WHAT THE GEOMETRY IS FOR Z NEAR 0 AND L, WE MAY HAVE ADDITIONAL BOUNDARY CONDITIONS TO SATISFY. THIS APPROXIMATION IS VALID ONLY AWAY FROM $Z=0,L$.

PROBLEM 2 CONSIDER A SLENDER BODY OF REVOLUTION AS SHOWN



L : length of body

R_0 : typical radius of body

$$T_{rr} + \frac{1}{r} T_r + T_{ZZ} = 0 \qquad 0 < Z < L, \quad 0 < r < R_0 f(Z/L)$$

$$\nabla T \cdot \hat{n} = 0 \quad \text{ON} \quad r = R_0 f(Z/L)$$

$$T = 0 \quad \text{ON} \quad Z = 0; \quad T = T_0 \quad \text{ON} \quad Z = L.$$

T_0 CONSTANT

NOTICE $$\hat{n} = \frac{(1, -R_0/L \, f'(Z/L))}{\left[1 + \frac{R_0^2}{L^2}(f'(Z/L))^2\right]^{1/2}}$$

WE NON-DIMENSIONALIZE $T = T_0 u, \quad \rho = r/R_0, \quad z = Z/L.$

THIS YIELDS THAT

$$(*) \left\{ \begin{array}{ll} u_{\rho\rho} + \frac{1}{\rho} u_\rho + \varepsilon^2 u_{zz} = 0 & 0 < z < 1, \quad 0 < \rho < f(z) \\ u = 0 \text{ ON } z = 0, \quad u = 1 \text{ ON } z = 1 & \\ u_\rho = \varepsilon^2 f'(z) u_z \quad \text{ON} \quad \rho = f(z) & \end{array} \right. \qquad \varepsilon = \frac{R_0}{L} \ll 1.$$

REMARK (i) WHERE DOES $u_\rho = \varepsilon^2 f'(z) u_z$ ON $\rho = f(z)$ COME FROM?

WE LET $\nabla T \cdot \hat{n} = 0 \rightarrow T_r - \frac{R_0}{L} f'(Z/L) T_Z = 0.$

NOW $$\frac{1}{R_0} u_\rho - \frac{1}{L}\frac{R_0}{L} f'(Z/L) u_z = 0$$

THIS GIVES $$u_\rho = \frac{R_0^2}{L^2} f'(z) u_z \quad \text{ON} \quad \rho = f(z).$$

WE NOW EXPAND THE SOLUTION TO (*) AS

$$u = u_0 + \varepsilon^2 u_1 + \cdots$$

THIS YIELDS,

$$L u_0 = u_{0\rho\rho} + \frac{1}{\rho} u_{0\rho} = 0, \quad 0 < \rho < f(z); \quad u_{0\rho} = 0 \text{ ON } \rho = f(z)$$

$$L u_1 = -u_{0zz}, \quad 0 < \rho < f(z); \quad u_{1\rho} = f'(z)\, u_{0z} \text{ ON } \rho = f(z)$$

THE SOLUTION FOR u_0 IS $u_0 = A(z)$. NOW SINCE $L = 0$

AND L IS SELF-ADJOINT WE INTEGRATE

$$-\int_0^{f(z)} u_{0zz}\, \rho\, d\rho = \int_0^{f(z)} (1)\, L u_1\, \rho\, d\rho = \int_0^{f(z)} (\rho\, u_{1\rho})_\rho\, d\rho = \rho\, u_{1\rho}\Big|_0^{f(z)}$$

SO

$$-u_{0zz} \int_0^{f(z)} \rho\, d\rho = f(z)\, u_{1\rho}\Big|_{f(z)} = f(z)\, f'(z)\, u_{0z}.$$

THUS,

$$-u_{0zz} \frac{[f(z)]^2}{2} = f(z)\, f'(z)\, u_{0z}$$

THIS CAN BE WRITTEN AS

$$(1) \qquad \left[(f(z))^2\, u_{0z} \right]_z = 0$$

IT BEHAVES LIKE A STEADY-STATE HEAT EQUATION

WITH A VARIABLE DIFFUSIVITY THAT IS DEPENDENT

ON THE AREA $= \pi (f(z))^2$ OF THE CROSS-SECTION AT POINT z

WE MUST SOLVE (1) WITH $u_0 = 0$ ON $z = 0$, $u_0 = 1$ ON $z = 1$.

THIS GIVES $\quad u_{0z} = C[f(z)]^{-2} \qquad u_0 = C \int_0^z [f(\tilde{z})]^{-2} d\tilde{z}$.

HENCE,

$$u_0(z) = \int_0^z (f(\tilde{z}))^2\, d\tilde{z} \Big/ \int_0^1 (f(\tilde{z}))^2\, d\tilde{z}.$$

QUASI-STEADY STATE APPROXIMATION

WE BEGIN WITH A HEATED METAL SPHERE WITH TIME VARYING SURFACE TEMPERATURE. THE MODEL IS FOR $T(r,t)$:

$$T_t = D\left(T_{rr} + \frac{2}{r} T_r \right) \qquad \text{IN} \quad 0 < r < a$$

$$T(a,t) = T_0 \sin(\omega t) \qquad T(r,0) = T_0 \, f(r/a)$$

WE ASSUME THAT "DIFFUSION" IS STRONG. FIND AN APPROXIMATION TO THE SOLUTION IN THIS LIMIT.

WE LET $\tau = \omega t$, $r = a\rho$, $T = T_0 u$. THEN

$$\omega \, u_\tau = \frac{D}{a^2}\left(u_{\rho\rho} + \frac{2}{\rho} u_\rho \right) \qquad 0 < \rho < 1$$

$$u(1,\tau) = \sin(\tau) \qquad u(\rho,0) = f(\rho)$$

WE DEFINE $\qquad \varepsilon = \dfrac{\omega a^2}{D}$, $\qquad [D] = \text{length}^2/\text{time} \qquad [\omega] = 1/\text{time}$

AND ASSUME THAT $\varepsilon \ll 1$.

OUR PROBLEM IS

$$\begin{cases} \varepsilon \, u_\tau = u_{\rho\rho} + \dfrac{2}{\rho} u_\rho & 0 < \rho < 1 \\ u(1,\tau) = \sin\tau, \quad u(\rho,0) = f(\rho) \end{cases}$$

THERE ARE TWO TIME-SCALES. $\qquad \tau = O(1) \longrightarrow$ quasi-steady state

$\qquad \tau = O(\varepsilon) \longrightarrow$ transient.

WE ASSUME THE QUASI-STEADY STATE EXPANSION

$$u = u_0 + \varepsilon u_1 + \ldots$$

WE SUBSTITUTE TO OBTAIN

$$U_{0\rho\rho} + \frac{2}{\rho} U_{0\rho} = 0 \quad , \qquad U_0 = \sin\tau \text{ ON } \rho = 1 \quad \longrightarrow \quad U_0 \equiv \sin\tau$$

$$U_{1\rho\rho} + \frac{2}{\rho} U_{1\rho} = U_{0\tau} = \cos\tau \quad , \qquad U_1 = 0 \text{ ON } \rho = 1.$$

WE LET $U_1 = A\rho^2 \longrightarrow 6A = \cos\tau \qquad A = \frac{1}{6}\cos\tau.$

HENCE $U_1 = \frac{\cos\tau}{6}(\rho^2 - 1) = -\sin(\tau - \pi/2)\frac{(\rho^2-1)}{6}.$

THIS YIELDS THAT $(*)$ $U \sim \sin\tau - \varepsilon \sin(\tau - \pi/2)\frac{(\rho^2-1)}{6} + \dots$

NOTICE THAT THE PERTURBATION LAGS BEHIND THE SURFACE TEMPERATURE.

NOTICE ALSO THAT $(*)$ HAS NO FREEDOM TO IMPOSE $U(\rho, 0) = f(\rho)$.

HENCE WE MUST RESCALE TIME.

WE LET $\hat{\tau} = \tau/\varepsilon \quad , \quad \hat{V}(\hat{\tau}, \rho) = U(\varepsilon\hat{\tau}, \rho) \simeq \hat{V}_0 + \varepsilon \hat{V}_1 + \dots$

THIS YIELDS

$$\begin{cases} \hat{V}_{0\hat{\tau}} = \hat{V}_{0\rho\rho} + \frac{2}{\rho}\hat{V}_{0\rho} \\ \hat{V}_0(1, \hat{\tau}) = 0 \\ \hat{V}_0(\rho, 0) = f(\rho) \end{cases}$$

WE USE SEPARATION OF VARIABLES: $\hat{V}_0 = H(\rho)\, T(\hat{\tau})$.

WE OBTAIN $\frac{T'}{T} = \frac{H'' + \frac{2}{\rho}H'}{H} = -\lambda \qquad H'' + \frac{2}{\rho}H' + \lambda H = 0$

HENCE $H(\rho) = \frac{A\sin(\sqrt{\lambda}\rho)}{\rho}$ WITH $\sqrt{\lambda} = n\pi$ OR $\lambda = n^2\pi^2$.

THUS, $\hat{V}_0 = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\hat{\tau}} \sin(n\pi\rho)/\rho$

THE INITIAL CONDITION IMPLIES THAT

$$F(\rho) = \sum_{n=1}^{\infty} C_n \sin(n\pi\rho)/\rho$$

SINCE THE WEIGHT FUNCTION IS ρ^2 (i.e. $(\rho^2 H')' + \lambda\rho^2 H = 0$)

WE OBTAIN $$C_n = \int_0^1 F(\rho)\left(\frac{\sin(n\pi\rho)}{\rho}\right)\rho^2\,d\rho \Big/ \int_0^1 \left(\frac{\sin(n\pi\rho)}{\rho}\right)^2 \rho^2\,d\rho$$

NOTICE THAT $\hat{V}_0 \to 0$ AS $\hat{\tau} \to \infty$ $\hat{\tau} = \tau/\varepsilon$. HENCE AFTER A SHORT TIME THE SOLUTION TENDS TO ITS QUASI-STEADY STATE VALUE IN (*)

$$u \sim \sin(\varepsilon\hat{\tau}) - \varepsilon\sin(\varepsilon\hat{\tau} - \pi/2)\left(\frac{\rho^2-1}{6}\right) = O(\varepsilon)$$