SINGULAR PERTURBATIONS I

WE FIRST CONSIDER SINGULAR PERTURBATIONS FOR THE ROOTS OF POLYNOMIALS. FOR INSTANCE CONSIDER

$$\varepsilon X^4 - X^2 - X + 2 = 0.$$

FIND A 2-TERM EXPANSION FOR EACH OF THE ROOTS AS $\varepsilon \to 0$.

$$\varepsilon X^4 - (X^2 + X - 2) = \varepsilon X^4 - (X+2)(X-1) = 0.$$

WE WILL HAVE ONE ROOT NEAR $X=1$ AND ANOTHER NEAR $X=-2$.

<u>ROOT NEAR $X=1$</u> : $\quad X = 1 + \varepsilon X_1 + \cdots$ SO THAT

$$\varepsilon(1+\varepsilon X_1)^4 - (3+\varepsilon X_1)(\varepsilon X_1) = \varepsilon(1-3X_1) + O(\varepsilon^2) = 0$$

$$\text{SO} \quad X_1 = 1/3 \qquad X = 1 + \varepsilon/3 + \cdots$$

<u>ROOT NEAR $X=-2$</u> $\quad$ LET $\quad X = -2 + \varepsilon X_1 + \cdots$

$$\varepsilon(-2+\varepsilon X_1)^4 - \varepsilon X_1(-3+\varepsilon X_1) = \varepsilon(3X_1 - 16) + O(\varepsilon^2) = 0$$

$$\text{SO} \quad X_1 = 16/3 \qquad X = -2 + 16\varepsilon/3 + \cdots$$

FOR THE OTHER TWO ROOTS LET $\quad X = \varepsilon^{-p} y \qquad p > 0$.

THEN $\quad \varepsilon^{1-4p} y^4 - (\varepsilon^{-2p} y^2 + \varepsilon^{-p} y - 2) = 0.$

WE BALANCE $\quad \varepsilon^{1-4p} = \varepsilon^{-2p} \quad$ SO $\quad 1-4p = -2p \to p = \frac{1}{2}.$

THEN $\quad y^4 - (y^2 + \varepsilon^{1/2} y - 2\varepsilon) = 0.$

SO $\quad y^4 - y^2 = 0 \quad$ OR $\quad y^2 = 1 \rightarrow y = \pm 1.$

- WE LET $\quad y = 1 + \varepsilon^{1/2} y_1 + \dots$

$$(1 + \varepsilon^{1/2} y_1)^4 - (1 + \varepsilon^{1/2} y_1)^2 - \varepsilon^{1/2}(1 + \dots) + O(\varepsilon) = 0$$

$$\longrightarrow \quad 4\varepsilon^{1/2} y_1 - 2\varepsilon^{1/2} y_1 - \varepsilon^{1/2} = 0$$

$$2y_1 - 1 = 0 \qquad y_1 = 1/2.$$

SO $\quad y = 1 + \varepsilon^{1/2}/2 \longrightarrow \quad x = \varepsilon^{-1/2} + 1/2 + \dots$

- WE LET $\quad y = -1 + \varepsilon^{1/2} y_1 + \dots$

$$(-1 + \varepsilon^{1/2} y_1)^4 - (-1 + \varepsilon^{1/2} y_1)^2 - \varepsilon^{1/2}\left(-1 + \varepsilon^{1/2} y_1\right) + O(\varepsilon) = 0$$

$$-4\varepsilon^{1/2} y_1 - (-2\varepsilon^{1/2} y_1) + \varepsilon^{1/2} = 0$$

$$-2\varepsilon^{1/2} y_1 + \varepsilon^{1/2} = 0 \qquad y_1 = 1/2$$

SO $\quad y = -1 + \varepsilon^{1/2}/2 + \dots$

$$x = -\varepsilon^{-1/2} + 1/2 + \dots$$

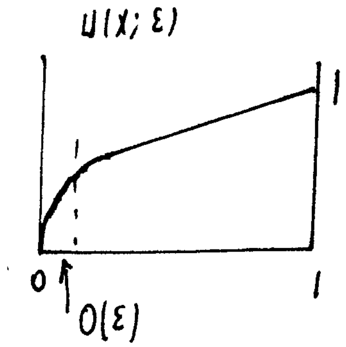
THE ROOTS ARE:

$$\left\{ \begin{array}{l} x = 1 + \varepsilon/3 + \dots \\ x = -2 + 16\varepsilon/3 + \dots \\ x_{\pm} = \pm \varepsilon^{-1/2} + 1/2 + \dots \end{array} \right.$$

EXAMPLE 1 FIND THE EXACT AND A BOUNDARY LAYER SOLUTION FOR

$$\varepsilon u'' + u' = \frac{1}{2}, \quad 0 \le x \le 1 \qquad \varepsilon \to 0^+.$$

$$u(0) = 0, \quad u(1) = 1$$



THE EXACT SOLUTION IS READILY FOUND TO BE

$$u(x;\varepsilon) = \frac{1 - e^{-x/\varepsilon}}{2(1 - e^{-1/\varepsilon})} + \frac{x}{2}$$

NOTICE THAT $\lim\limits_{\substack{\varepsilon \to 0 \\ x \neq 0}} u(x;\varepsilon) = \frac{1}{2} + \frac{x}{2}$ outer limit

$\lim\limits_{\substack{\varepsilon \to 0 \\ y = x/\varepsilon \text{ FIXED}}} u(x;\varepsilon) = \frac{1 - e^{-y}}{2}$ INNER LIMIT.

WE NOW DEVELOP A BOUNDARY LAYER THEORY TO CONSTRUCT THE SOLUTION.

AWAY FROM $x = 0$ WE EXPAND

$$u_{out} = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

COLLECTING POWERS OF ε WE GET

$$u_0' = \frac{1}{2} \longrightarrow u_0 = \frac{x + c_0}{2}$$

$$u_1' = -u_0'' \longrightarrow u_1 = c_1$$

WE CAN'T SATISFY BOTH BOUNDARY CONDITIONS AT $x = 0$, AND $x = 1$. ASSUME THAT THERE IS NO BL NEAR $x = 1$ AND SET $u_{out}(1) = 1$. THIS GIVES $u_0(1) = 1$ AND $u_1(1) = 0$. WE OBTAIN $c_1 = 0$ AND $c_0 = 1$

SO THAT $u_0 = \frac{(1+x)}{2}$, $u_1 = 0$.

THUS $u_{out} = (1+x)/2 + O(\varepsilon^2)$.

NOW SINCE $u_{out}(0) = 1/2 \neq 0$ WE MUST INSERT A BL NEAR $x=0$ WHERE u CHANGES RAPIDLY AND $\varepsilon u''$ IS NO LONGER SMALL.

WE RESCALE NEAR $x=0$ BY LETTING $y = x/\varepsilon^p$ AND $V(y) = u(\varepsilon^p y)$.

THEN $u_x = V_y \varepsilon^{-p}$ AND SO

$$\varepsilon^{1-2p} V'' + \varepsilon^{-p} V' = 1/2 \quad , \quad y \geq 0$$

$$V(0) = 0$$

THE ONLY CONSISTENT BALANCE IS TO TAKE $p = 1$ AND SO

$$V'' + V' = \varepsilon/2 \qquad \text{ON } 0 \leq y < \infty$$

$$V(0) = 0$$

WE EXPAND $V = V_0 + \varepsilon V_1 + \dots$ AND OBTAIN

$$V_0'' + V_0' = 0 \qquad y \geq 0$$

$$V_0(0) = 0$$

THE SOLUTION IS $V_0(y) = A_0(1 - e^{-y})$

THE INNER EXPANSION IS $V = A_0(1 - e^{-y}) + O(\varepsilon) + \dots$

NOW THE MATCHING ~~STATEMENT~~ VAN-DYKE (1964) MATCHING PRINCIPLE: IS THAT THE OUTER EXPANSION WHEN EXPRESSED IN INNER VARIABLES (AND LETTING $\varepsilon \to 0$) MUST AGREE ASYMPTOTICALLY WITH THE INNER SOLUTION WHEN EXPRESSED IN OUTER VARIABLES (AND LETTING $\varepsilon \to 0$).

IN THIS SIMPLE CASE THE MATCHING CONDITION IS

$$\frac{1}{2} = \lim_{x \to 0} u_{out} = \lim_{y \to \infty} V(y) = A_0 \longrightarrow A_0 = 1/2$$

$$u_{out}(x) = (1+x)/2 \quad , \quad u_{INN}(x) = \frac{1}{2}\left(1 - e^{-x/\varepsilon}\right) \qquad x/\varepsilon = O(1)$$

A COMPOSITE EXPANSION IS OBTAINED BY ADDING $U_{out}(x) + U_{INN}(x)$ AND SUBTRACTING THE COMMON PART $1/2$ USED IN MATCHING.

THIS YIELDS
$$U_{COMP} = \frac{(1+x)}{2} + \frac{1}{2}\left(1 - e^{-x/\varepsilon}\right) - \frac{1}{2} = \frac{(1+x)}{2} - \frac{1}{2} e^{-x/\varepsilon}.$$

REMARK ANOTHER WAY TO MATCH INNER AND OUTER SOLUTIONS IS FROM THE CONCEPT OF INTERMEDIATE LIMITS. THE INNER AND OUTER EXPANSIONS MUST AGREE ASYMPTOTICALLY ON ALL INTERMEDIATE SCALES.

IF WE DEFINE $x_\Lambda = x/\Lambda(\varepsilon)$ $\quad \varepsilon \ll \Lambda(\varepsilon) \ll 1$ AND x_Λ FIXED

THEN
$$y = x_\Lambda \Lambda(\varepsilon)/\varepsilon \to \infty \quad \text{AS} \quad \varepsilon \to 0 \quad \text{SINCE} \quad \Lambda(\varepsilon) \gg \varepsilon$$
$$x = x_\Lambda \Lambda(\varepsilon) \longrightarrow 0 \quad \text{AS} \quad \varepsilon \to 0$$

THE INTERMEDIATE MATCHING PRINCIPLE OF KAPLUN (1964) STATES THAT THE INNER AND OUTER SOLUTIONS SHOULD AGREE WHEN WRITTEN IN TERMS OF x_Λ.

EXAMPLE CONSTRUCT THE SOLUTION TO
$$\varepsilon u'' + (x^2 - 2) u = -1 \qquad u(-1) = u(1) = 0, \quad -1 \le x \le 1.$$

IN THE OUTER REGION WE WRITE
$$U_{out} = u_0 + \varepsilon u_1 + \cdots$$

THIS YIELDS
$$(2 - x^2) u_0 = 1$$
$$(2 - x^2) u_1 = -u_0''$$

THUS $u_0 = 1/(2 - x^2)$ AND SO $U_{out} = \dfrac{1}{2 - x^2} + O(\varepsilon)$

NOW AS $x \longrightarrow \pm 1$ THE $u_0 \to \pm 1$ WHICH DOES NOT SATISFY THE BOUNDARY CONDITIONS. WE NEED A BL NEAR $x = \pm 1$.

CONSIDER THE REGION NEAR $X=-1$.

LET $y=(X+1)/\varepsilon^{p}$ AND $V(y)=U(-1+\varepsilon^{p}y)$.

THEN $\varepsilon^{1-2p}V'' - (2-(1+\varepsilon^{p}y)^2)V=-1 \qquad 0\leq y<\infty$

TO BALANCE TERMS TAKE $p=1/2. \longrightarrow \quad V''-(2-(1+2\varepsilon^{1/2}y+\ldots))V=-1$

WE EXPAND $V=V_0+\varepsilon^{1/2}V_1+\ldots$

$$\left.\begin{array}{l} V_0''-V_0=-1 \\ V_0(0)=0 \end{array}\right\} \longrightarrow \quad V_0=+1+Ae^{-y}+Be^{y}.$$

NOW THE MATCHING CONDITION IS THAT

$$\underset{X\to 0}{U_{out}} \sim \underset{y\to\infty}{V}$$

$$U_{out}=\frac{1}{2-X^2}=\frac{1}{2-(1+\varepsilon^{1/2}y)}\sim\frac{1}{1-\varepsilon^{1/2}y}\sim 1+\varepsilon^{1/2}y+\ldots$$

$V_0 = 1+Ae^{-y}+Be^{y} \longrightarrow 1$ AS $y\to\infty$ WHEN $B=0$

TO ELIMINATE EXPONENTIAL GROWTH.

THUS V_0 AND U_0 MATCH.

FINALLY MAKE $V_0(0)=0 \longrightarrow A=-1$.

$$U_{INNER}=1-e^{-y}+O(\varepsilon^{1/2}), \qquad y=\varepsilon^{-1/2}(1+X)$$

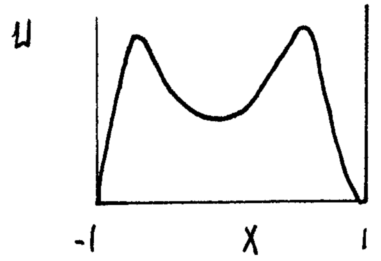
$$U_{out}=\frac{1}{2-X^2}+O(\varepsilon), \qquad -1+O(\varepsilon^{1/2})<X<1-O(\varepsilon^{1/2})$$

$$U_{INNER}=1-e^{-\hat{y}}+O(\varepsilon^{1/2}), \qquad \hat{y}=\varepsilon^{1/2}(1-X)$$

NOTICE THE SYMMETRY UNDER X TO $-X$.

THE COMPOSITE EXPANSION IS

$$u_{comp} \sim \frac{1}{2-x^2} - e^{-(1+x)\varepsilon^{-1/2}} - e^{-(1-x)\varepsilon^{-1/2}} + \ldots$$



EXAMPLE FIND THE LEADING ORDER INNER AND OUTER SOLUTION FOR

$$\varepsilon u'' - (1-x^2)u = -1 \qquad u(\pm 1) = 0, \quad -1 \leq x \leq 1.$$

OUTER REGION: LET $u = u_0 + \varepsilon u_1 + \cdots$

THEN $(1-x^2)u_0 = 1 \longrightarrow u_0 = 1/_{1-x^2}$.

$$+(1-x^2)u_1 = u_0''$$

HENCE $u_{out} \sim u_0 = \frac{1}{1-x^2} + O(\varepsilon)$.

IN BOUNDARY LAYER REGION NEAR $x = 1$ WE LET $y = \varepsilon^{-p}(x+1)$, $p > 0$. IN TERMS OF y NOTICE THAT

$$u_{out} \sim \frac{1}{1-x^2} = \frac{1}{1-(-1+\varepsilon^p y)^2} = \frac{1}{2\varepsilon^p y + \ldots}$$

THIS SUGGESTS THAT THE INNER SOLUTION SHOULD HAVE THE FORM

$$y = \varepsilon^{-p}(x+1), \qquad u(x) = \varepsilon^{-p} V\left(\varepsilon^{-p}(x+1)\right).$$

THEN $\varepsilon u'' - (1-x^2)u = -1$ TRANSFORMS TO

$$\varepsilon^{1-3p}\, V'' - \left(1 - (-1 + \varepsilon^{p} y)^2\right) \varepsilon^{-p}\, V = -1 .$$

$$\varepsilon^{1-3p}\, V'' - (2\varepsilon^{p} y + \dots)\,\varepsilon^{-p} V = -1$$

TO BALANCE TERMS TAKE $p = 1/3$ AND EXPAND $V = V_0 + \dots$ THIS YIELDS

$$(*) \quad V_0'' - 2y\, V_0 = -1 \qquad 0 \le y < \infty$$

$$V_0(0) = 0, \qquad V_0(y) \sim 1/2y \quad \text{AS } y \to \infty \quad \text{MATCHING CONDITION}.$$

THE SOLUTION CAN BE WRITTEN AS

$$V_0(y) = C_1 A_i(y\, 2^{1/3}) + C_2 B_i(y\, 2^{1/3}) + V_p(y)$$

WHERE $A_i(z)$, $B_i(z)$ ARE THE TWO FUNDAMENTAL SOLUTIONS OF AIRY'S EQUATION $\quad W''(z) - z\,W(z) = 0$

RECALL THAT $A_i(z)$ DECAYS EXPONENTIALLY AS $z \to +\infty$

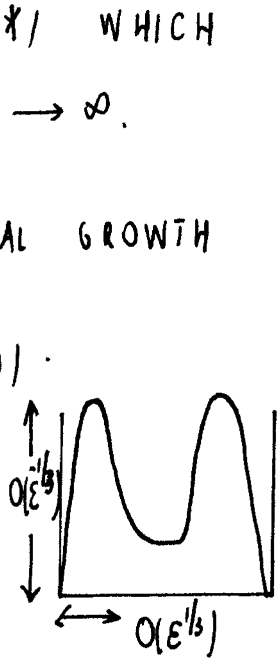
$B_i(z)$ GROWS EXPONENTIALLY AS $z \to \infty$

ALSO $V_p(y)$ IS THE PARTICULAR SOLUTION OF $(*)$ WHICH CLEARLY SATISFIES $\quad V_p(y) \sim 1/2y + d/y^4 + \dots \quad$ AS $y \to \infty$.

HENCE WE SET $C_2 = 0$ TO KILL EXPONENTIAL GROWTH

AND $\quad C_1 A_i(0) = -V_p(0) \longrightarrow C_1 = -V_p(0)/A_i(0)$.

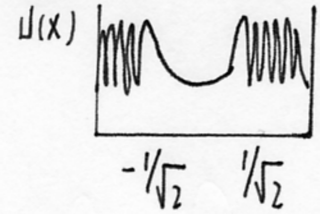
$$V_0(y) = V_p(y) - V_p(0)\, A_i(y\, 2^{1/3}) / A_i(0).$$



REMARK (i) IF $\varepsilon u'' - \left(\frac{1}{2} - x^2\right) u = -1 \quad -1 \le x \le 1, \quad u(\pm 1) = 0$

THEN HIGHLY OSCILLATORY ON $-1 < x < -\sqrt{\frac{1}{2}}$ AND $\frac{1}{\sqrt{2}} < x < 1$.

B.L. THEORY IS NOT APPROPRIATE. NEED WKB TURNING POINT THEORY



GENERAL CASE: LINEAR TWO-POINT BVP

$$\varepsilon u'' + a(x) u' + b(x) u = 0 \qquad 0 \le x \le 1 \qquad \varepsilon \to 0^+$$

$$u(0) = A, \quad u(1) = B$$

WE WILL ASSUME THAT $a(x) > 0$ ON $0 \le x \le 1$ OR $a(x) < 0$ ON $0 \le x \le 1$. WE WILL EXCLUDE THE "TURNING POINT" CASE WHERE $a(x_0) = 0 \quad x_0 \in (0,1)$ FOR THE MOMENT.

WE WILL SHOW

(i) IF $a(x) > 0$ ON $[0,1] \longrightarrow$ B.L. NEAR $x = 0$

(ii) IF $a(x) < 0$ ON $[0,1] \longrightarrow$ B.L. NEAR $x = 1$.

IN THE OUTER REGION WE EXPAND $u = u_0 + \varepsilon u_1 + \cdots$ TO GET

$$a(x) u_0' + b(x) u_0 = 0 \quad , \qquad a(x) u_i' + b(x) u_i = -u_{i-1}'' \ , \ i \ge 1$$

IN CASE (i) WE SET $u_0(1) = B$ AND IN CASE (ii) WE TAKE $u_0(0) = A$. THEN UPON INTEGRATING

(i) $\quad u_0(x) = B \exp\left(\int_x^1 \frac{b(s)}{a(s)} \, ds \right)$

(ii) $\quad u_0(x) = A \exp\left(- \int_0^x \frac{b(s)}{a(s)} \, ds \right)$

CASE (i) $a(x) > 0$. WE PUT A BL NEAR $x=0$.

WE LET $y = x/\varepsilon^p$ AND $V(y) = u(\varepsilon^p x)$, WITH $p > 0$

THEN $$\varepsilon^{1-2p} V'' + \varepsilon^{-p} a(\varepsilon^p y) V' + b(\varepsilon^p y) V = 0.$$

$a(\varepsilon^p y) \sim a(0) + \dots$ $\quad b(\varepsilon^p y) \sim b(0) + \dots$ WE THEN BALANCE $\varepsilon^{1-2p} = \varepsilon^{-p}$

TO OBTAIN $p = 1$. WE THEN EXPAND $V = V_0 + \varepsilon V_1 + \dots$ TO OBTAIN

$$V_0'' + a(0) V_0' = 0, \qquad V_0(0) = A$$

$$V_0(\infty) = u_0(0) \quad \text{MATCHING CONDITION}$$

SINCE $a(0) > 0 \longrightarrow V_0'$ DECAYS EXPONENTIALLY.

THE SOLUTION IS $$V_0(y) = (A - C_0) + C_0 e^{-a(0)y}$$

THE MATCHING CONDITION GIVES $A - C_0 = u_0(0)$, SO THAT

$$C_0 = A - u_0(0) = A - B\exp\left(\int_0^1 \frac{b(s)}{a(s)}\,ds\right)$$

IN SUMMARY: LEADING ORDER INNER, OUTER, AND COMPOSITE

$$u_{INN} = B\exp\left(\int_0^1 \frac{b(s)}{a(s)}\,ds\right) + \left(A - B\exp\int_0^1 \frac{b(s)}{a(s)}\,ds\right)e^{-a(0)y}$$

$$u_{out} = B\exp\left(\int_x^1 \frac{b(s)}{a(s)}\,ds\right)$$

$$u_{COMP} = B\exp\left(\int_x^1 \frac{b(s)}{a(s)}\,ds\right) + \left(A - B\exp\left(\int_0^1 \frac{b(s)}{a(s)}\,ds\right)\right)e^{-a(0)x/\varepsilon}$$

REMARK (i) IF WE INCORRECTLY ASSUMED A BOUNDARY LAYER AT $x=1$ INSTEAD OF AT $x=0$ WE WOULD HAVE

$$V_0'' - a(1) V_0' = 0, \quad 0 < y < \infty \qquad y = (1-x)/\varepsilon$$

THIS GROWS EXPONENTIALLY AS $y \to +\infty$!!!

CASE 2 NOW ASSUME THAT $a(x) < 0$ ON $[0,1]$.

WE INSERT A BL NEAR $x=1$.

LET $y = \varepsilon^{-1}(1-x)$, $W(y) = u[1-\varepsilon y]$. THIS GIVES

$$\frac{1}{\varepsilon} W'' - \frac{1}{\varepsilon} a(1-\varepsilon y) W' + b(1-\varepsilon y) W = 0 \qquad 0 < y < \infty.$$

WE EXPAND $W = W_0 + \varepsilon W_1 + \cdots$

$$W_0'' - a(1) W_0' = 0, \qquad 0 < y < \infty$$

$$W_0(0) = B, \qquad W_0(\infty) = u_0(1) \quad \text{MATCHING CONDITION.}$$

SINCE $a(1) < 0$ WE HAVE EXPONENTIAL DECAY AS $y \to +\infty$.

THE SOLUTION IS $W_0 = (B - C_1) + C_1 e^{a(1) y}$.

MATCHING CONDITION IS $W_0(\infty) = u_0(1) = A \exp\left(-\int_0^1 \frac{b(s)}{a(s)} ds\right) = B - C_1$

THUS $$u_{INN} = W_0(y) = A \exp\left(-\int_0^1 \frac{b(s)}{a(s)} ds\right) + \left[B - A \exp\left(-\int_0^1 \frac{b(s)}{a(s)} ds\right)\right] e^{a(1)(1-x)/\varepsilon}$$

$$u_{out} = A \exp\left(-\int_0^x \frac{b(s)}{a(s)} ds\right)$$

$$u_{COMP} = A \exp\left(-\int_0^x \frac{b(s)}{a(s)} ds\right) + \left(B - A \exp\left(-\int_0^1 \frac{b(s)}{a(s)} ds\right)\right) e^{a(1)(1-x)/\varepsilon}$$

EXAMPLE $$\varepsilon u'' + (1+2x) u' + u = x$$

$$u(0) = 0, \quad u(1) = 1$$

SINCE $a(x) = 1 + 2x > 0$ ON $[0,1]$ WE WILL HAVE A BL AT $x = 0$. IN THE OUTER REGION $0 < x \le 1$ WE EXPAND

$$u = u_0 + \varepsilon u_1 + \cdots$$

$$(1+2x)U_0' + U_0 = x \,, \qquad U_0(1) = 1$$

$$(1+2x)U_1' + U_1 = -U_0'' \,, \qquad U_1(1) = 0$$

SOLVING FOR U_0 WE GET

$$U_0 = \frac{C_0}{(1+2x)^{1/3}} + \frac{1}{3}(x-1) \qquad U_0(1) = 1 \longrightarrow C_0 = \sqrt{3}$$

THEN, $\quad (1+2x)U_1' + U_1 = -3\,C_0\,(1+2x)^{-5/2} \,, \quad U_1(1) = 0$ GIVES

$$U_1 = \frac{1}{3}\,\frac{(2-x-x^2)\,C_0}{(1+2x)^{5/2}} \qquad C_0 = \sqrt{3}\,.$$

THUS TO TWO TERMS WE HAVE

$$U_{out}(x) = \frac{\sqrt{3}}{(1+2x)^{1/2}} + \frac{1}{3}(x-1) + \frac{\varepsilon}{\sqrt{3}}\,\frac{(2-x-x^2)}{(1+2x)^{5/2}} + \cdots$$

NOW PUT A BL NEAR $x = 0$. LET $y = x/\varepsilon$, $V(y) = U(\varepsilon y)$.

THEN $\quad V'' + (1+2\varepsilon y)V' + \varepsilon V = \varepsilon^2 y$

$$V(0) = 0.$$

WE EXPAND $\quad V = V_0 + \varepsilon V_1 + \cdots$

(i) $\quad V_0'' + V_0' = 0 \,, \qquad V_0(0) = 0$

(ii) $\quad V_1'' + V_1' = -2y\,V_0' - V_0 \,, \qquad V_1(0) = 0.$

NOW WE WRITE $U_{out}(x)$ IN TERMS OF $y = x/\varepsilon$ OR $x = \varepsilon y$ AND LET $\varepsilon \to 0$.

$$U_{out} \sim \frac{\sqrt{3}}{(1+2\varepsilon y)^{1/2}} + \frac{1}{3}(-1+\varepsilon y) + \frac{\varepsilon}{\sqrt{3}}\,\frac{(2-\varepsilon y-\varepsilon^2 y^2)}{(1+2\varepsilon y)^{5/2}} + \cdots$$

THIS YIELDS THAT

$$U_{out} \sim \sqrt{3}\,(1-\varepsilon y + \dots) - \frac{1}{3} + \varepsilon\left(\frac{y}{3} + \frac{2}{\sqrt{3}}\right) + O(\varepsilon^2)$$

$$U_{out} \sim \sqrt{3} - \frac{1}{3} + \varepsilon\left(\frac{2}{\sqrt{3}} + y\left(\frac{1}{3} - \sqrt{3}\right)\right) + O(\varepsilon^2)$$

COMPARING WITH $U_{INN} = V_0 + \varepsilon V_1 + \dots$

WE REQUIRE THAT

(ia) $V_0 \longrightarrow \sqrt{3} - 1/3 \quad , \quad y \longrightarrow +\infty$

(iia) $V_1 \longrightarrow \dfrac{2}{\sqrt{3}} + y\left(\dfrac{1}{3} - \sqrt{3}\right) \quad , \quad y \longrightarrow +\infty$

WE NOW SOLVE (i) AND (ii) WITH (ia) AND (iia).

WE OBTAIN $V_0(y) = \left(\sqrt{3} - \dfrac{1}{3}\right)\left(1 - e^{-y}\right)$

$$(e^y V_1')' = -2y\,e^y V_0' - V_0 e^y = -2y\left(\sqrt{3} - \frac{1}{3}\right) - \left(\sqrt{3} - \frac{1}{3}\right)(e^y - 1)$$

SOLVING THIS WITH $V_1(0) = 0$ AND $V_1 \longrightarrow \dfrac{2}{\sqrt{3}} + y\left(\dfrac{1}{3} - \sqrt{3}\right)$

WE OBTAIN

$$V_1 = \left(\sqrt{3} - \frac{1}{3}\right)e^{-y}(y^2 + y + 1) - \left(\sqrt{3} - \frac{1}{3}\right)(y+1) + \left(\sqrt{3} - \frac{1}{3} + \frac{2}{\sqrt{3}}\right)\left(1 - e^{-y}\right)$$

EXAMPLE FIND A TWO-TERM EXPANSION FOR THE SOLUTION TO

$$\varepsilon u'' + u' - x u = 0, \quad 0 \le x \le 1; \quad u(0) = 0, \quad u(1) = e^{1/2}.$$

SINCE $a(x) = 1 > 0$ ON $[0,1]$ WE GET A BL AT $x = 0$.

IN THE OUTER REGION $0 < x \le 1$ WE EXPAND $u = u_0 + \varepsilon u_1 + \cdots$

THIS YIELDS:

$$u_0' - x u_0 = 0; \qquad u_0(1) = e^{1/2}$$

$$u_1' - x u_1 = -u_0'', \qquad u_1(1) = 0$$

WE OBTAIN $u_0 = e^{x^2/2}$ AND $u_1 = e^{x^2/2}\left(\frac{4}{3} - x - x^3/3\right)$.

THUS $$u_{out} = e^{x^2/2}\left(1 + \varepsilon\left(4/3 - x - x^3/3\right) + \cdots\right)$$

IN THE INNER REGION WE LET $y = x/\varepsilon$, $V(y) = u(\varepsilon y)$.

THEN $$\frac{1}{\varepsilon} V'' + \frac{1}{\varepsilon} V' - \varepsilon y V = 0.$$

HENCE $$V'' + V' - \varepsilon^2 y V = 0, \qquad 0 \le y < \infty$$

$$V(0) = 0.$$

NOW TAKE THE OUTER SOLUTION AND LET $x = \varepsilon y$ AND

EXPAND AS $\varepsilon \to 0$: $$u_{out} \sim e^{\varepsilon^2 y^2/2}\left(1 + \frac{4\varepsilon}{3} + O(\varepsilon^2)\right) \sim 1 + \frac{4\varepsilon}{3} + O(\varepsilon^2)$$

MATCHING WITH V WE CONCLUDE

$$V = V_0 + \varepsilon V_1 + \cdots \qquad V_0(\infty) = 1, \quad V_1(\infty) = 4/3.$$

THEN $V_0'' + V_0' = 0$, $V_1'' + V_1' = 0 \longrightarrow V_0 = 1 - e^{-y}$, $V_1 = \frac{4}{3}(1 - e^{-y})$

<u>OUTER</u> $u_{out} = e^{x^2/2}\left(1 + \varepsilon\left(\frac{4}{3} - x - x^3/3\right)\right)$, $u_{INN} = (1 - e^{-y})\left(1 + 4\varepsilon/3 + \cdots\right)$

THE COMPOSITE IS

$$U_c = e^{x^2/2}\left(1+\varepsilon\left(4/_3 - x - x^3/_3\right)\right) - \left(1 + 4\varepsilon/_3\right)e^{-x/\varepsilon}.$$

EXAMPLE (A NONLINEAR PROBLEM)

$$\varepsilon u'' + u' + u^2 = 0, \quad 0 \le x \le 1, \quad 0 < \varepsilon \ll 1$$

$$u'(0) = A, \quad u(1) = \frac{1}{2} + 2\varepsilon$$

SINCE $a(x) = 1 > 0$ WE EXPECT BL AT $x = 0$.

WE THEN EXPAND $u = u_0 + \varepsilon u_1 + \cdots$ $0 < x \le 1$ IN OUTER REGION.

THIS GIVES

$$u_0' + u_0^2 = 0, \qquad u_0(1) = 1/_2$$

$$u_1' + 2u_0 u_1 = -u_0'', \qquad u_1(1) = 2$$

WE SOLVE TO OBTAIN $u_0 = \dfrac{1}{x+1}$, $u_1 = \dfrac{2\log(x+1)}{(x+1)^2} + \dfrac{C_1}{(x+1)^2}$.

WE WANT $u_1(1) = 2$ SO THAT $C = 8 - 2\log 2$.

THUS TO TWO TERMS WE HAVE

$$u_{out}(x) = \frac{1}{x+1} + \varepsilon\left[\frac{2\log(x+1)}{(x+1)^2} + \frac{(8-2\log 2)}{(x+1)^2}\right]$$

IN THE INNER REGION WE LET $y = x/_\varepsilon$, $V(y) = u(\varepsilon y)$.

THEN WE OBTAIN $\dfrac{1}{\varepsilon}V'' + \dfrac{1}{\varepsilon}V' + V^2 = 0$, $V'(0) = \varepsilon A$

WE THEN EXPAND $V = V_0 + \varepsilon V_1 + \cdots$

$$V_0'' + V_0' = 0, \quad 0 \le y < \infty, \quad V_0'(0) = 0$$

$$V_1'' + V_1' = -V_0^2, \quad 0 \le y < \infty, \quad V_1'(0) = A$$

NOW LET $X = \varepsilon y$ IN OUTER EXPANSION AND LET $\varepsilon \to 0$.

$$U_{out} \sim \frac{1}{1+\varepsilon y} + \varepsilon\left[\frac{\log(1+\varepsilon y)}{(1+\varepsilon y)^2} + \frac{(8-2\log 2)}{(1+\varepsilon y)^2}\right] \sim 1 - \varepsilon y + \varepsilon(8-2\log 2) + O(\varepsilon^2)$$

WE OBTAIN $V_0(\infty) = 1$

$$V_1(y) \sim -y + 8 - 2\log 2 \quad \text{AS} \quad y \to \infty.$$

THUS $V_0 = C_0 + d_0 e^{-y}$ $\quad V_0'(0) = 0, \; V_0(\infty) = 1 \longrightarrow V_0(y) \equiv 1$

THEN $V_1'' + V_1' = -1$ HAS SOLUTION $V_1 = -y + C_1 + C_2 e^{-y}$.

$V_1'(0) = A \quad \to \quad -1 - C_2 = A$

$V_1 \sim -y + 8 - 2\log 2$ AS $y \to \infty$ IMPLIES $C_1 = 8 - 2\log 2$.

THUS $\quad V_1(y) = -y + 8 - 2\log 2 - (1+A)e^{-y}$

$$U_{INN} = 1 + \varepsilon\left(-y + 8 - 2\log 2 - (1+A)e^{-y}\right)$$

$$U_{out} = \frac{1}{X+1} + \varepsilon\left(\frac{2\log(X+1)}{(X+1)^2} + \frac{8 - 2\log 2}{(X+1)^2}\right)$$

$$U_{COMP} = U_{out}(X) - \varepsilon(1+A)e^{-X/\varepsilon} + O(\varepsilon^2).$$

EXAMPLE (ANOTHER NONLINEAR PROBLEM)

FIND A 1-TERM INNER, OUTER, AND COMPOSITE SOLUTION FOR

$$\varepsilon u'' - u' + e^{u} = 0 \qquad 0 \le x \le 1$$

$$u(0) = A, \quad u(1) = 0 \qquad A < 0.$$

SINCE $a(x) = -1 < 0$ WE EXPECT A BL NEAR $x = 1$.

IN THE OUTER REGION FOR $0 \le x < 1$ WE EXPAND

$$u = u_0 + \varepsilon u_1 + \cdots$$

THIS YIELDS $\quad u_0' = e^{u_0}, \quad u_0(0) = A$

$$u_0(x) = -\ln\left[e^{-A} - x\right]$$

SINCE $A < 0$ THE SINGULARITY AT $x = e^{-A}$ OCCURS FOR $x > 1$.

IN THE INNER REGION NEAR $x = 1$ WE PUT $y = \varepsilon^{-1}(1 - x)$, AND $V(y) = u(1 - \varepsilon y)$. THEN

$$\frac{1}{\varepsilon} V'' + \frac{1}{\varepsilon} V' + e^{V} = 0 \qquad y \ge 0$$

$V(0) = 0$, AND WE WRITE $V = V_0 + \varepsilon V_1 + \cdots$

THE MATCHING CONDITION IS $\quad V_0(\infty) = u_0(1) = -\ln\left(e^{-A} - 1\right)$.

THEN $\quad V_0'' + V_0' = 0$ SO THAT

$$V_0 = -\ln\left(e^{-A} - 1\right)\left(1 - e^{-y}\right)$$

leading order outer: $\quad u_{out} = -\ln\left(e^{-A} - x\right)$

" " INNER: $\quad u_{INN} = -\ln\left(e^{-A} - 1\right)\left(1 - e^{-(1-x)/\varepsilon}\right)$

" " COMPOSITE: $\quad u_{COMP} = -\ln\left(e^{-A} - x\right) + \ln\left(e^{-A} - x\right) e^{-(1-x)/\varepsilon}$

<u>REMARK</u> (i) IF $A > 0$ THEN u_0 HAS A SINGULARITY <u>INSIDE</u> THE DOMAIN $0 \le x \le 1$. IN THIS CASE THIS SUGGESTS NO SOLUTION TO THE BVP.

NOW IN THE BL REGION NEAR $X=0$ WE LET $y = X/\varepsilon^p$, $V(y) = U(\varepsilon^p X)$.

THIS YIELDS

$$\varepsilon^{1-2p} V'' + \frac{\varepsilon^p}{\varepsilon^p} y V' - \varepsilon^p y V = 0, \quad 0 \le y < \infty$$

$$V(0) = 0.$$

TO BALANCE TERMS WE TAKE $\varepsilon^{1-2p} = \varepsilon^0 \rightarrow p = 1/2$.

THIS GIVES
$$\begin{cases} V'' + yV' - \varepsilon^{1/2} y V = 0 & 0 \le y < \infty \\ V(0) = 0 \end{cases}$$

NOW WE TAKE THE OUTER SOLUTION AND LET $X = \varepsilon^{1/2} y$.

THIS LEADS TO
$$U_{out} = e^{\varepsilon^{1/2} y} - \varepsilon e^{\varepsilon^{1/2} y} \left[\tfrac{1}{2}\log\varepsilon + \log y\right]$$

$$U_{out} = 1 + \varepsilon^{1/2}(y) - \frac{1}{2}\varepsilon\log\varepsilon - \varepsilon\left(\log y + y^2/2\right) + \cdots$$

THIS SUGGESTS THE INNER EXPANSION

$$V = V_0 + \varepsilon^{1/2} V_1 + (-\varepsilon\log\varepsilon) V_2 + \cdots$$

WE OBTAIN
$$\begin{cases} V_0'' + y V_0' = 0 \\ V_0(0) = 0, \quad V_0(\infty) = 1 \end{cases}$$

$$\begin{cases} V_1'' + y V_1' = y V_0 \\ V_1(0) = 0, \quad V_1 \sim y \text{ AS } y \to \infty \end{cases}$$

$$\begin{cases} V_2'' + y V_2' = 0 \\ V_2(0) = 0, \quad V_2(\infty) = +\frac{1}{2} \end{cases}$$

REMARK (i) SO FAR WE HAVE ONLY CONSIDERED SECOND ORDER BVP OF THE FORM $u'' + a(x)u'$ WHERE EITHER $a(x) > 0$ OR $a(x) < 0$ ON $0 \leq x \leq 1$.

(ii) WHAT HAPPENS IF $a(x)$ VANISHES AT AN ENDPOINT? (BL WIDTH WILL NOT BE $O(\varepsilon)$).

(iii) WHAT HAPPENS IF $a(x_0) = 0$ FOR SOME UNIQUE x_0 IN $(0,1)$? THESE POINTS ARE CALLED TURNING POINTS.

EXAMPLE (OF (ii) ABOVE).

CONSIDER
$$\varepsilon u'' + x u' - x u = 0 \,, \quad 0 \leq x \leq 1, \quad \varepsilon \to 0.$$
$$u(0) = 0, \quad u(1) = e$$

IT IS STILL CLEAR THAT SINCE $a(x) = x > 0$ NEAR $x = 1$ WE CANNOT HAVE A BL NEAR $x = 1$.

IN THE OUTER REGION WE EXPAND $u = u_0 + \varepsilon u_1 + \dots$ FOR $0 < x \leq 1$. THIS YIELDS THAT

$$\left.\begin{array}{l} u_0' - u_0 = 0 \,, \quad 0 < x \leq 1 \\ u_0(1) = e \end{array}\right\} \longrightarrow u_0 = e^x$$

$$\left.\begin{array}{l} u_1' - u_1 = -\dfrac{u_0''}{x} = \dfrac{-e^x}{x} \\ u_1(1) = 0 \end{array}\right\} \longrightarrow u_1 = -e^x \log x$$

THE TWO-TERM OUTER SOLUTION IS
$$u_{out} = e^x - \varepsilon e^x \log x + \dots$$

WE SOLVE TO GET

$$V_0' e^{y^2/2} = A_0$$

$$V_0(y) = A_0 \int_0^y e^{-\Lambda^2/2}\, d\Lambda .$$

BUT $V_0(\infty) = 1 \longrightarrow A_0 \int_0^\infty e^{-\Lambda^2/2}\, d\Lambda = 1 \longrightarrow A_0 = \left(2/\pi\right)^{1/2}$.

THIS YIELDS $V_0(y) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^y e^{-\Lambda^2/2}\, d\Lambda .$

THEN $\left(V_1' e^{y^2/2} \right)' = y e^{y^2/2} V_0$

$$V_1' e^{y^2/2} = \int_0^y \Lambda e^{\Lambda^2/2} V_0(\Lambda)\, d\Lambda$$

$$V_1 = \int_0^y e^{-t^2/2} \left(\int_0^t \Lambda e^{\Lambda^2/2} V_0(\Lambda)\, d\Lambda \right) dt .$$

INDEED $V_1 \sim y$ AS $y \to \infty$.

FINALLY $V_2 = \frac{1}{2} \left(\frac{2}{\pi}\right)^{1/2} \int_0^y e^{-\Lambda^2/2}\, d\Lambda .$

EXAMPLE (INTERNAL LAYER)

WHEN $a(x)$ CHANGES SIGN INSIDE THE DOMAIN WE COULD HAVE AN INTERNAL LAYER WHERE u CHANGES RAPIDLY IN A NARROW REGION NEAR WHERE $a(x)$ CROSSES THROUGH ZERO.

CONSIDER $$\varepsilon u'' + (x-1/2)u' - (x-1/2)u = 0 \qquad 0 \le x \le 1$$

$$u(0) = \alpha, \quad u(1) = \beta \qquad a(x) = x - 1/2.$$

SINCE $a(0)<0$ AND $a(1)>0$ THERE ARE NO BL NEAR $x=0$ OR $x=1$.

IN THE OUTER REGION WE EXPAND

$$u = u_0 + \varepsilon u_1 + \ldots$$

$$(x-1/2)u_0' - (x-1/2)u_0 = 0$$

WHERE $u_0(0) = \alpha$ AND $u_0(1) = \beta$ BUT u_0 DISCONTINUOUS ACROSS $x = 1/2$.

WE WRITE
$$u_0(x) = \begin{cases} C_L e^x, & 0 \le x < 1/2 \\ C_R e^x, & 1/2 < x \le 1 \end{cases}$$

SATISFYING THE B.C. WE GET $$u_0 = \begin{cases} \alpha e^x, & 0 \le x < 1/2 \\ \beta e^{x-1}, & 1/2 < x \le 1. \end{cases}$$

NOW WE INSERT AN INTERNAL LAYER NEAR $x = 1/2$. WE LET

$$y = \varepsilon^{-p}(x - 1/2), \quad w(y) = u\left[\frac{1}{2} + \varepsilon^p y\right].$$

THIS YIELDS, $$\varepsilon^{1-2p} w'' + y w' - \varepsilon^p y w = 0.$$

WE BALANCE WITH $p = 1/2$ AND EXPAND $w = w_0 + \varepsilon^{1/2} w_1 + \ldots$

$$\begin{cases} w_0'' + y w_0' = 0, & -\infty < y < \infty \\ w_0(\infty) = \beta e^{-1/2} & w_0(-\infty) = \alpha e^{1/2} \end{cases}$$

(FROM MATCHING CONDITION)

WE SOLVE TO OBTAIN $W_0(y) = C_2 + C_1 \int_0^y e^{-s^2/2}\, ds$.

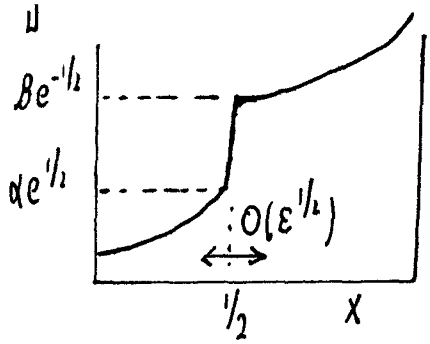
THEN $C_2 + C_1 \left(\pi/2\right)^{1/2} = \beta e^{-1/2}$ $\qquad \int_0^\infty e^{-s^2/2}\, ds = \left(\pi/2\right)^{1/2}$

$$C_2 - C_1 \left(\pi/2\right)^{1/2} = \alpha e^{1/2}$$

THIS YIELDS $C_2 = \frac{1}{2}\left(\beta e^{-1/2} + \alpha e^{1/2}\right)$, $\quad C_1 = \frac{1}{(2\pi)^{1/2}}\left(\beta e^{-1/2} - \alpha e^{1/2}\right)$

WHICH COMPLETELY DETERMINES $W_0(y)$.

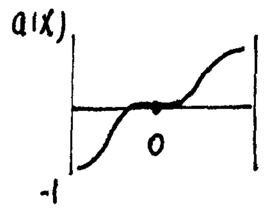
WHEN $\alpha e^{+1/2} < \beta e^{-1/2}$ WE GET



<u>EXAMPLE</u> <u>(INTERNAL LAYER)</u>

CONSIDER $\quad \varepsilon u'' + a(x) u' = 0 \qquad -1 \le x \le 1$

$$u(-1) = \alpha, \quad u(1) = \beta.$$



SUPPOSE THAT $a(0) = 0$, $a(x) > 0$ FOR $x > 0$ AND $a(x) < 0$ FOR $x < 0$ AND ASSUME THAT $a(x) \sim \gamma x^{2m+1}$ AS $x \to 0$ WHERE $m > 0$ IS AN INTEGER AND $\gamma > 0$.

IN THE OUTER REGION $u = u_0 + \varepsilon u_1 + \ldots$ SO THAT $a(x) u_0' = 0$. THERE WILL BE NO BL NEAR $x = \pm 1$ SO THAT u_0 MUST SATISFY THE BOUNDARY CONDITIONS. THE OUTER SOLUTION IS

$$u_0 = \begin{cases} \alpha & -1 \le x < 0 \\ \beta & 0 < x \le 1 \end{cases}$$

NOW CONSTRUCT AN INTERNAL LAYER NEAR $x = 0$ WITH

$$y = x/\varepsilon^p \quad \text{AND} \quad W(y) = u(\varepsilon^p y).$$

THIS YIELDS THAT $\varepsilon^{1-2p} w'' + \varepsilon^{-p} a(\varepsilon^p y) w' = 0$.

NOW $a(\varepsilon^p y) = \gamma \varepsilon^{(2m+1)p} y^{2m+1}$. WE BALANCE $\varepsilon^{1-2p} = \varepsilon^{-p+(2m+1)p}$.

THIS YIELDS $1 - 2p = 2mp$ SO $p = 1/2m+2$

WE THEN WRITE $W = W_0 + \ldots$

$$W_0'' + \gamma y^{2m+1} W_0' = 0 \qquad -\infty < y < \infty$$

$$W_0(\infty) = \beta, \quad W_0(-\infty) = \alpha.$$

THE INTERNAL LAYER WIDTH IS $O\left(\varepsilon^{1/(2m+2)}\right)$ WHICH WIDENS AS m INCREASES. WE SOLVE TO OBTAIN

$$W_0(y) = C_1 + C_2 \int_0^y e^{-\gamma s^{2m+2}/(2m+2)}\, ds$$

NOW $$\int_0^\infty e^{-\gamma s^{2m+2}/(2m+2)}\, ds = \left(\frac{2m+2}{\gamma}\right)^{1/2m+2} \frac{1}{2m+2} \int_0^\infty r^{1/(2m+2)-1} e^{-r}\, dr$$

RECALLING $\int_0^\infty t^{z-1} e^{-t}\, dt = \Gamma(z)$, Γ = gamma FUNCTION.

THEN $$\int_0^\infty e^{-\gamma s^{2m+2}/(2m+2)}\, ds = \left(\frac{2m+2}{\gamma}\right)^{1/2m+2} \frac{1}{2m+2} \Gamma\left(1/2m+2\right) \doteq I.$$

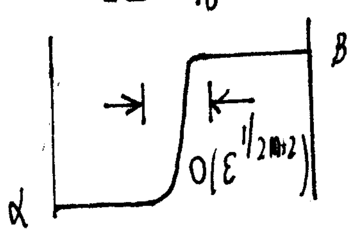
THEN $$C_1 + C_2 I = \beta \qquad (W_0(\infty) = \beta)$$

$$C_1 - C_2 I = \alpha \qquad (W_0(-\infty) = \alpha)$$

THIS YIELDS $C_1 = (\beta + \alpha)/2$, $C_2 = (\beta - \alpha)/2I$

$$W_0 = \frac{\beta + \alpha}{2} + \frac{(\beta - \alpha)}{2I} \int_0^y e^{-\gamma s^{2m+2}/(2m+2)}\, ds.$$

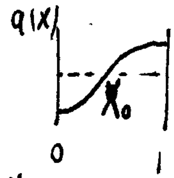
WITH $y = x/\varepsilon^{1/(2m+2)}$



REMARKS ON INTERNAL LAYERS

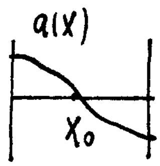
$$\varepsilon u'' + a(x)u' + b(x)u = 0 \qquad 0 \le x \le 1$$

SUPPOSE THAT FOR SOME UNIQUE $x_0 \in (0,1)$ THAT



(i) $a(x_0) = 0$, $a(x) < 0$ FOR $x < x_0$, $a(x) > 0$ FOR $x > x_0$ WITH $a'(x_0) > 0$. THEN WE HAVE OUTER SOLUTIONS ON EITHER SIDE OF x_0 AND AN INTERNAL LAYER OF WIDTH $O(\varepsilon^{1/2})$ NEAR x_0. THERE ARE NO BOUNDARY LAYERS NEAR $x = 0, 1$.

(ii) IF $a(x_0) = 0$, $a(x) > 0$ FOR $x < x_0$, $a(x) < 0$ FOR $x > x_0$ WITH $a'(x_0) < 0$. THEN $u = 0$ IS THE OUTER SOLUTION AND THERE IS NO INTERNAL LAYER NEAR $x = x_0$. INSTEAD THERE ARE BOUNDARY LAYERS NEAR $x = 0, 1$. THERE ARE SOME EXCEPTIONAL CASES TO THE STATEMENT THAT $u = 0$ IS OUTER SOLUTION



EXAMPLE (A CORNER LAYER: BOUNDARY LAYER IN THE DERIVATIVE)

CONSIDER $$\varepsilon u'' + x u' - u = 0 \qquad -1 \le x \le 1 \qquad \varepsilon \to 0$$
$$u(-1) = 1, \quad u(1) = 2.$$

SINCE $a(x) = x$ WITH $a(-1) < 0$ AND $a(1) > 0$ THERE ARE NO BOUNDARY LAYERS NEAR $x = 0$ AND $x = 1$.

THE OUTER SOLUTION IS EXPANDED AS

$$u = u_0 + \varepsilon u_1 + \dots$$

SO THAT $$x u_0' - u_0 = 0 \quad \text{IN } [-1, 0) \text{ AND } (0, 1]$$

WE OBTAIN $u_0 = cx$ IS THE SOLUTION.

THEREFORE $$u_0(x) = \begin{cases} 2x, & 0 < x \le 1 \\ -x, & -1 \le x < 0 \end{cases}$$

NOTICE THAT u_0 IS CONTINUOUS BUT u_0' IS NOT.

WE MUST CONSTRUCT AN INTERNAL LAYER TO SMOOTHE u_0'.
SUCH A LAYER IS CALLED A "<u>CORNER</u>" LAYER

WE LET $y = x/\varepsilon^p$, $w(y) = u(\varepsilon^p y)$.

THUS $$\varepsilon^{1-2p} w'' + y w' - w = 0 \qquad \text{CHOOSE } p = 1/2.$$

THIS GIVES $$w'' + y w' - w = 0, \quad -\infty < y < \infty.$$

NOW WRITING THE OUTER EXPANSION u_0 IN TERMS OF y TO OBTAIN
$$u_0 \sim \begin{cases} 2\varepsilon^{1/2} y, & y > 0 \\ -\varepsilon^{1/2} y, & y < 0 \end{cases}$$

THUS WE SET $w = \varepsilon^{1/2} w_0 + \ldots$ WHICH YIELDS
$$w_0'' + y w_0' - w_0 = 0 \qquad -\infty < y < \infty$$
$$w_0 \sim 2y \quad y \to +\infty; \qquad w_0 \sim -y \text{ AS } y \to -\infty.$$

ONE SOLUTION IS $w_0 = y$. TO FIND THE SECOND SOLUTION WE CAN USE REDUCTION OF ORDER BY SETTING $w_0 = yv$.

THEN $w_0' = v + yv'$, $w_0'' = yv'' + 2v'$. THIS GIVES,
$$yv'' + 2v' + yv + y^2 v' - yv = yv'' + v'(2 + y^2) = 0$$
$$v'' + \left(\frac{2}{y} + y\right) v' = 0$$

WE SOLVE TO OBTAIN $w_0 = e^{-y^2/2} + y \int_{-\infty}^{y} e^{-s^2/2}\, ds.$

THIS GIVES THE SOLUTION

$$W_0 = C_1 y + C_2 \left(e^{-y^2/2} + y \int_{-\infty}^{y} e^{-s^2/2}\, ds \right) \qquad \int_{-\infty}^{\infty} e^{-s^2/2}\, ds = (2\pi)^{1/2}$$

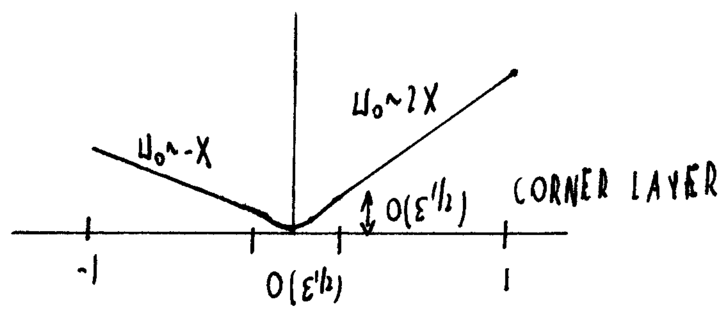
NOW $\quad W_0 \to 2y$ AS $y \to +\infty$ GIVES $C_1 + C_2 (2\pi)^{1/2} = 2$

$W_0 \to -y$ AS $y \to -\infty$ GIVES $C_1 = -1$.

HENCE $\quad C_2 = 3/(2\pi)^{1/2}$

$$W_0 = -y + \frac{3}{(2\pi)^{1/2}} \left(e^{-y^2/2} + y \int_{-\infty}^{y} e^{-s^2/2}\, ds \right)$$

AND $\quad W = \varepsilon^{1/2} W_0 + \ldots$



<u>EXAMPLE</u> <u>(INDETERMINACY: CASE OF (ii))</u>.

THIS EXAMPLE SHOWS AN APPARENT FAILURE OF BOUNDARY LAYER THEORY.

$$\varepsilon u'' - x u' + u = 0 \qquad -1 \leq x \leq 1$$

$$u(-1) = 1, \quad u(1) = 2$$

SINCE $a(x) = -x$ THERE IS NO INTERNAL LAYER NEAR $x = 0$. INSTEAD WE HAVE BOUNDARY LAYERS NEAR $x = \pm 1$.

IN THE OUTER REGION WE EXPAND $u = u_0 + \varepsilon u_1 + \cdots$

THIS YIELDS THAT $\quad -x u_0' + u_0 = 0 \longrightarrow u_0 = \alpha x \quad \alpha$ ARBITRARY.

NOTICE THAT $u_0'' = 0$ SO THAT $u_0 = \alpha x$ SATISFIES THE FULL EQUATION.

WE NOW CONSTRUCT A BL NEAR $X=-1$.

WE LET $y=(X+1)/\varepsilon$, $W(y)=U(-1+\varepsilon y)$ AND EXPAND $W=W_0+\varepsilon W_1+\dots$

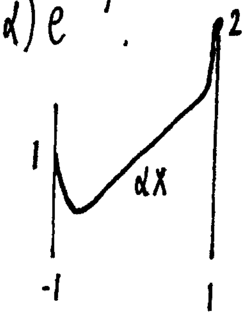
SO THAT

$$\left.\begin{array}{l} W_0''+W_0'=0 \\ W_0(0)=1,\quad W_0(\infty)=-\alpha \end{array}\right\} \Rightarrow W_0=-\alpha+(\alpha+1)e^{-y}.$$

SIMILARLY NEAR $X=1$ WE LET $y=(1-X)/\varepsilon$, $V(y)=U(1-\varepsilon y)$

AND WE EXPAND $V=V_0+\varepsilon V_1+\dots$ SO THAT

$$\left.\begin{array}{l} V_0''+V_0'=0 \\ V_0(0)=2,\quad V_0(\infty)=\alpha \end{array}\right\} \rightarrow V_0(y)=\alpha+(2-\alpha)e^{-y}.$$



THIS GIVES A COMPOSITE OR UNIFORM SOLUTION

$$U_{COMP}=\alpha X+(\alpha+1)e^{-(X+1)/\varepsilon}+(2-\alpha)e^{-(1-X)/\varepsilon}$$

A FUNDAMENTAL DIFFICULTY IS THAT WE CANNOT DETERMINE α.

<u>REMARKS</u> (i) A HIGHER ORDER ANALYSIS TO INCORPORATE TERMS OF ORDER $O(\varepsilon^{q})$ FOR SOME q DOES NOT DETERMINE α.

(ii) THIS PHENOMENON WAS OBSERVED BY ACKBERG-O'MALLEY (1970), AND GENERATED $1/\varepsilon$ PAPERS OVER THE YEARS.

(iii) SIMILAR DIFFICULTIES OCCUR FOR

$$\varepsilon U''+XU'=0$$
$$U(-1)=1,\qquad U(1)=2.$$

THE "SOLUTION" IS BY BL THEORY:

$$U_{COMP}=\alpha+(1-\alpha)e^{-(X+1)/\varepsilon^{1/2}}+(2-\alpha)e^{-(1-X)/\varepsilon^{1/2}}$$