EXAMPLE POTENTIAL OUTSIDE A SPHERICAL PARTICLE IN AN ELECTROLYTIC FLUID SATISFIES

$$\Delta \Phi = -\frac{q}{\varepsilon_s}\left(N_+ - N_- \right)$$

N_+ : positive ion density, N_- : negative ion density, ε_s : dielectric constant

NOW $N_+ = N_0 e^{-q\Phi/K_BT}$ AND $N_- = N_0 e^{q\Phi/K_BT}$. $N_\pm \to N_0$ AS $\Phi \to 0$.

WE LET $\psi = \frac{q}{K_BT}\Phi$ TO OBTAIN

$$(1) \quad \Delta \psi = \frac{1}{\delta^2}\sinh(\psi) \qquad \delta^2 = \frac{K_BT\varepsilon_s}{2q^2N_0} \qquad \delta = \sqrt{\frac{K_BT\varepsilon_s}{2N_0q^2}}$$

(POISSON-BOLTZMAN EQUATION) (Debye Length)

WE WANT TO SOLVE (1) WITH $\psi = \Psi_0$ AT $r = a$ AND $\psi \to 0$ AS $r \to \infty$. WE LET $\rho = r/a$ TO NON-DIMENSIONALIZE.

WE OBTAIN

$$\varepsilon^2\left(\psi_{\rho\rho} + \frac{2}{\rho}\psi_\rho \right) = \sinh\psi \qquad \varepsilon = \delta/a \ll 1 \text{ is assumed}$$

$$\psi(1) = \Psi_0, \quad \psi(\infty) = 0.$$

IN THE INNER REGION NEAR THE SPHERE, WE LET

$$X = \frac{\rho - 1}{\varepsilon} \quad \text{AND} \quad \psi = W_0 + \varepsilon^{0} W_1 + \dots \qquad \psi(1+\varepsilon X) = W(X).$$

THIS GIVES

$$W_{XX} + \frac{2\varepsilon}{(1+\varepsilon X)} W_X = \sinh(W)$$

NOW

$$W_{0XX} + \varepsilon W_{1XX} + 2\varepsilon W_{0X} + \dots = \sinh(W_0) + \varepsilon \cosh(W_0) W_1 + \dots$$

THIS YIELDS THAT

$$\begin{cases} W_{0XX} = \sinh(W_0), & 0 < X < \infty \\ W_0(0) = \Psi_0, & W_0(\infty) = 0 \end{cases}$$

$$\begin{cases} W_{1xx} - (\cosh W_0)\, W_1 = -2\, W_{0x} \\ W_1(0) = 0, \quad W_1(\infty) = 0. \end{cases}$$

THE SOLUTIONS FOR W_0 AND W_1 ARE

$$W_0(x) = 2\log\left(\frac{1+te^{-x}}{1-te^{-x}}\right) \qquad t = \tanh(\psi_0/4)$$

$$W_1(x) = \frac{2te^{-x}}{(1-t^2e^{-2x})}\left(t^2(1-e^{-2x}) - 2x\right)$$

AND SO $\quad \psi = W_0 + \varepsilon W_1 + \dots \qquad \varepsilon = \delta/a \ll 1$

NOW FOR $x \gg 1$ WE CALCULATE $\log(1+te^{-x}) - \log(1-te^{-x}) \approx 2te^{-x}$,

SO THAT $\quad W_0(x) \sim 4te^{-x} \qquad W_1(x) \sim 2t(t^2-2x)e^{-x}$

THUS FOR $x \gg 1 \qquad \psi \sim e^{-x}\left(4t + \varepsilon\, 2t\,(t^2 - 2x)\right) \qquad (*)$

NOW IN THE OUTER REGION THE SOLUTION IS $\psi_{out} = 0$.

TO GET A BETTER APPROXIMATION WE LINEARIZE $\sinh\psi \approx \psi$, WHICH GIVES

$$\varepsilon^2\left(\psi_{\rho\rho} + \frac{2}{\rho}\psi_\rho\right) - \psi = 0 \qquad \psi = \frac{F(\rho)}{\rho}, \longrightarrow \varepsilon^2 F'' - F = 0$$

THIS PROBLEM HAS AN EXACT SOLUTION

$$\psi_{out} \sim \frac{A}{\rho}\, e^{-(\rho-1)/\varepsilon} \quad \text{FOR} \quad \rho - 1 \gg O(\varepsilon) \qquad A = A_0 + \varepsilon A_1 + \dots$$

NOW WRITE ψ_{out} IN TERMS OF LOCAL VARIABLE $\rho - 1 = \varepsilon x$ TO GET

$$\psi_{out} \sim \frac{A}{1+\varepsilon x}\, e^{-x} \cong A(1-\varepsilon x)e^{-x}.$$

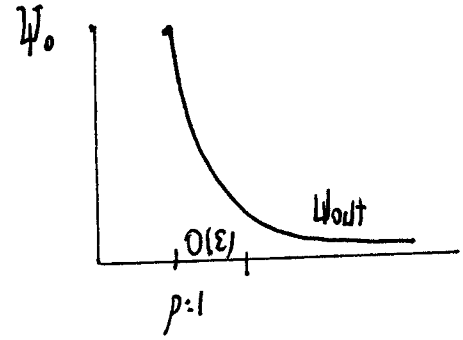
COMPARE WITH $(*)$ TO GET $\quad A - A\varepsilon x \sim 4t - 4\varepsilon x t + 2\varepsilon t^3$.

THUS YIELDS THAT $\qquad A = 4t + 2\varepsilon t^3 + \dots$

THIS YIELDS THE OUTER APPROXIMATION

$$\psi_{out} \sim \frac{A}{\rho} e^{-(\rho-1)/\varepsilon} \qquad A = 4t + 2\varepsilon t^3 \qquad \varepsilon = \delta/a \ll 1$$

$$t = \tanh(\psi_0/4) \qquad \rho = r/a$$



NOTICE THAT AS $\Psi_0 \to \infty$ $\quad A \to 4 + 2\varepsilon$ INDEPENDENT OF Ψ_0.

<u>REMARK</u> FOR A LINEAR PDE WE HAVE

$$\varepsilon^2\left(\psi_{\rho\rho} + \frac{2}{\rho}\psi_\rho\right) - \psi = 0 \qquad \rho \geq 1$$

$$\psi(1) = \Psi_0 \qquad \longrightarrow \qquad \psi = \frac{C}{\rho} e^{-(\rho-1)/\varepsilon} \qquad C = \Psi_0 \text{ IS CAPACITANCE.}$$

EXAMPLE (NONLINEAR SYSTEM BVP)

CONSIDER $\quad U'' = V, \quad U(0) = U(1) = 0$

$$\varepsilon V'' + U'V' = 0, \quad V(0) = V_L, \quad V(1) = V_R \qquad 0 < V_L < V_R$$

NOW WE CALCULATE $V' = A e^{-U/\varepsilon}$ WITH A CONSTANT. NOW V' DOES NOT CHANGE SIGN ON $0 \le X \le 1$ SO THAT $V' > 0$ ON $0 < X < 1$ IN ORDER TO SATISFY $V(0) = V_L$ AND $V(1) = V_R$.

HENCE $U'' > 0$ SO, U IS CONVEX

NOW $U' < 0$ AT $X = 0$ AND $U' > 0$ AT $X = 1$. THIS MEANS THAT V DOES NOT HAVE BOUNDARY LAYERS AT $X = 0, 1$.

WE THEN CALCULATE $\quad V(X) = V_L + A \int_0^X e^{-U/\varepsilon}\, d\eta \qquad A = \dfrac{(V_R - V_L)}{\int_0^1 e^{-U/\varepsilon}\, d\eta}$.

NOW IT IS CLEAR THAT $V(X)$ CHANGES RAPIDLY NEAR THE POINT WHERE U HAS ITS MINIMUM. THUS V HAS AN INTERNAL LAYER NEAR THE POINT X_0 WHERE $U'(X_0) = 0$.

WE EXPAND IN THE OUTER REGION $U = U_0 + \varepsilon U_1 + \cdots, \; V = V_0 + \varepsilon V_1 + \cdots$ TO OBTAIN

$$U_0'' = V_0$$

$$U_0' V_0' = 0$$

THIS YIELDS THAT $\quad V_0 = \begin{cases} V_L, & 0 < X < X_0 \\ V_R, & X_0 < X < 1 \end{cases}$

AND

$$U_0 = \begin{cases} U_L = \frac{1}{2} V_L X^2 + C_L X, & 0 \le X < X_0 \\ U_R = \frac{1}{2} V_R (X-1)^2 + C_R (X-1), & X_0 < X \le 1 \end{cases}$$

NOW IN A LAYER NEAR $X = X_0$ WE OBTAIN $\quad U'' = \varepsilon^2 V$.

NOW WE MUST HAVE THAT $U_L(X_0) = U_R(X_0)$ AND THAT

$$U_L'(X_0) = 0 = U_R'(X_0)$$

THIS YIELDS THAT

$$\left(\tfrac{1}{2} V_L X_0\right) 2 + C_L = 0 \longrightarrow C_L = -V_L X_0 .$$

$$V_R (X_0 - 1) + C_R = 0 \longrightarrow C_R = V_R (1 - X_0)$$

$$, \quad \tfrac{1}{2} V_L X_0^2 + C_L X_0 = \tfrac{1}{2} V_R (X_0 - 1)^2 + C_R (X_0 - 1)$$

$$\longrightarrow -\tfrac{1}{2} V_L X_0^2 = -\tfrac{1}{2} V_R (1 - X_0)^2 .$$

THIS IMPLIES THAT

$$\frac{X_0^2}{(1-X_0)^2} = \frac{V_R}{V_L} \longrightarrow X_0 = (1 - X_0)\sqrt{\frac{V_R}{V_L}}$$

THIS GIVES,

$$X_0 = \frac{1}{1 + \sqrt{V_L / V_R}}$$

THE EQUATION FOR U_0 BECOMES,

$$U_0 = \begin{cases} \tfrac{1}{2} V_L X^2 - V_L X X_0 , & 0 \le X < X_0 \\ \tfrac{1}{2} V_R (X-1)^2 + V_R (1 - X_0)(X - 1), & X_0 < X \le 1 . \end{cases}$$

THEN, WE CALCULATE

$$U_0' = \begin{cases} V_L (X - X_0) , & 0 \le X < X_0 \\ V_R (X - X_0) , & X_0 < X \le 1 \end{cases}$$

NOW IN THE INTERNAL LAYER LET

$$y = \varepsilon^{-P} (X - X_0) \quad \text{AND} \quad V(y) = V(X_0 + \varepsilon^P y) .$$

THIS YIELDS THAT

$$\varepsilon^{1-2P} V'' + V_L \varepsilon^P y \, \varepsilon^{-P} V' = 0 , \quad -\infty < y < 0 \quad V(-\infty) = V_L$$

$$\varepsilon^{1-2P} V'' + V_R \varepsilon^P y \, \varepsilon^{-P} V' = 0 , \quad 0 < y < \infty , \quad V(+\infty) = V_R .$$

NOW LET $P = 1/2$ AND ENSURE THAT V IS CONTINUOUS AND SMOOTH AT $X = X_0$. THIS YIELDS

$$V'' + V_L y V' = 0 , \quad -\infty < y < 0 , \quad V(-\infty) = V_L$$

$$V'' + V_R y V' = 0 , \quad 0 < y < \infty , \quad V(+\infty) = V_R$$

WITH $\quad V(0^-) = V(0^+), \quad V'(0^-) = V'(0^+) .$

NOW FOR $y < 0$ WE CAN SOLVE

$$\left(e^{V_L y^2/2} v' \right)' = 0 \longrightarrow v = c_1 \int_0^y e^{-V_L \lambda^2/2} \, d\lambda + c_2$$

WE CHANGE VARIABLES TO OBTAIN

$$v = c_1 \sqrt{\frac{2}{V_L}} \int_0^{\sqrt{V_L/2}\, y} e^{-\lambda^2} \, d\lambda + c_2 .$$

WE CAN WRITE THIS AS UPON PUTTING $v(0^-) = v(0^+)$.

$$v = \begin{cases} d_1 \, \mathrm{ERF}\left(\sqrt{\frac{V_L}{2}}\, y \right) + d_2 , & y < 0 \\ d_3 \, \mathrm{ERF}\left(\sqrt{\frac{V_R}{2}}\, y \right) + d_2 , & y > 0 \end{cases}$$

NOW WE HAVE $\mathrm{ERF}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\lambda^2} \, d\lambda$ WITH $\mathrm{ERF}(\infty) = 1$, $\mathrm{ERF}(-\infty) = -1$.

WE THEN USE $v(-\infty) = V_L$, $v(+\infty) = V_R$ AND $v'(0^-) = v'(0^+)$ TO GET

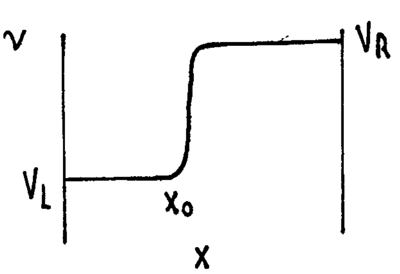
$$-d_1 + d_2 = V_L$$
$$d_3 + d_2 = V_R$$
$$d_1 \sqrt{V_L} = d_3 \sqrt{V_R}$$

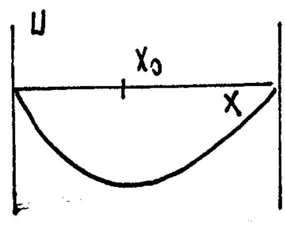
WE CAN THEN SOLVE TO OBTAIN

$$d_2 = \frac{V_R + V_L \sqrt{V_L/V_R}}{1 + \sqrt{V_L/V_R}} , \quad d_1 = \frac{V_R - V_L}{1 + \sqrt{V_L/V_R}} , \quad d_3 = \frac{V_R - V_L}{1 + \sqrt{V_R/V_L}} .$$

THIS COMPLETELY DETERMINES $v(y)$, THE INNER SOLUTION.

THE PLOT OF THE SOLUTION IS





EXAMPLE (HOT-SPOTS FOR BRATU'S EQUATION)

WE RECALL THAT $u_{rr} + \frac{1}{r} u_r + \lambda e^u = 0, \quad 0 < r \leq 1$

$$u(1) = 0, \quad u(0) \text{ FINITE.}$$

THE SOLUTION IS $u = 2 \log \left(\frac{1+\alpha}{1+\alpha r^2} \right)$, $\lambda = \frac{8\alpha}{(1+\alpha)^2}$. $u_r\big|_{r=1} = \frac{-4\alpha}{(1+\alpha)^{\bullet}}$

NOW LABEL $u(0) = 2\log(1+\alpha)$ AND FOR $\alpha \gg 1$ WE OBTAIN,

$$\lambda = \frac{8\alpha}{\alpha^2 (1+1/\alpha)^2} = \frac{8}{\alpha}\left(1+1/\alpha\right)^{-2} \sim \frac{8}{\alpha}\left(1 - 2/\alpha + \cdots\right).$$

WE CAN SOLVE TO OBTAIN $\alpha \sim 8/\lambda - 2 + \cdots$ FOR $\lambda \ll 1$, $\alpha \gg 1$.

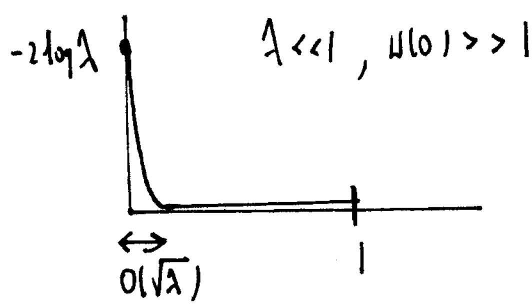
THIS YIELDS $u \sim 2\log\left(1 + \frac{8}{\lambda} - 2\right) - 2\log\left(1 + 8r^2/\lambda\right)$

THIS GIVES, $u \sim 2\log\left(\frac{8}{\lambda}\left(1 - \frac{\lambda}{8}\right)\right) - 2\log\left(1 + 8r^2/\lambda\right)$

THIS YIELDS

$$u \sim -2\log\lambda + 2\log 8 - 2\log\left(1 + 8r^2/\lambda\right)$$

THE PICTURE OF THE SOLUTION IS



WE PROCEED BY CONSTRUCTING THE SOLUTION BY BOUNDARY-LAYER THEORY: WE LET $u = -2\log\lambda + v$.

WE SUBSTITUTE TO OBTAIN $\Delta v + \frac{1}{\lambda} e^v = 0$, $\lambda \ll 1$. $\Delta = \partial_{rr} + \frac{1}{r}\partial_r$.

THIS SUGGESTS WRITING $\rho = r/\sqrt{\lambda}$ SO THAT

$$v_{\rho\rho} + \frac{1}{\rho} v_\rho + e^v = 0, \quad 0 < \rho < \infty$$

$$v(0) = \alpha_0 .$$

WE SOLVE TO OBTAIN

$$V = \alpha_0 - 2\log(1+A\rho^2) \quad \text{where} \quad A = \frac{1}{8}e^{\alpha_0}.$$

WE VERIFY BY CALCULATING AS FOLLOWS:

$$V_\rho = -\frac{4A\rho}{(1+A\rho^2)}, \quad V_{\rho\rho} = -\frac{4A}{(1+A\rho^2)} + \frac{8A^2\rho^2}{(1+A\rho^2)^2}.$$

WE SUBSTITUTE TO OBTAIN $V_{\rho\rho} + \frac{1}{\rho}V_\rho + e^V = 0 \rightarrow \frac{8A^2\rho^2}{(1+A\rho^2)^2} - \frac{8A}{(1+A\rho^2)} + \frac{e^{\alpha_0}}{(1+A\rho^2)^2} = 0$

THIS YIELDS THAT $-8A + e^{\alpha_0} = 0$ OR $A = \frac{1}{8}e^{\alpha_0}$.

NOW EXPAND V FOR $\rho \gg 1$

$$V \sim \alpha_0 - 2\log A - 4\log\rho.$$

HENCE FOR $\rho \gg 1$ AND WITH $\rho = r/\sqrt{\lambda}$ WE OBTAIN

$$u \sim -2\log\lambda + \alpha_0 - 2\log\left(\frac{1}{8}e^{\alpha_0}\right) - 4\log r + 2\log\lambda.$$

THIS YIELDS $\quad u \sim -\alpha_0 + 2\log 8 - 4\log r.$

IN THE OUTER REGION FOR $r \gg O(\lambda^{1/2})$ WE EXPAND $u = u_0 + \lambda u_1 + \cdots$

TO OBTAIN $\quad \Delta u_0 = 0$

$$u_0(1) = 0, \quad u_0 \sim -\alpha_0 + 2\log 8 - 4\log r \quad \text{AS } r \to 0.$$

THE SOLUTION IS $u_0 = c_0 + d_0\log r$. WITH $u_0(1) = 0 \rightarrow c_0 = 0$.

MATCHING YIELDS THAT $d_0 = -4, \quad \alpha_0 = 2\log 8$.

THIS DETERMINES THE INNER SOLUTION AS

$$u_{INN} \sim -2\log\lambda + 2\log 8 - 2\log\left(1 + 8r^2/\lambda\right) \qquad 0 \le r \ll O(\sqrt{\lambda})$$

$$u_{out} \sim -4\log r.$$

APPARENT FAILURE OF BOUNDARY-LAYER THEORY

CONSIDER $\quad \varepsilon u'' - x u' = 0, \quad -1 < x < b$

$$u(-1) = 0, \quad u(b) = 1 \qquad b > 0$$

WE EXPECT A BOUNDARY LAYER NEAR $x = -1$ AND $x = b$. THE OUTER SOLUTION IS $u_0 = \alpha$

THE LEFT BL IS

$$y = (x+1)/\varepsilon, \quad V(y) = u[-1+\varepsilon y] \quad \text{SO THAT}$$

$$\frac{\varepsilon}{\varepsilon^2} V'' - [-1+\varepsilon y]\frac{V'}{\varepsilon} = 0 \longrightarrow V_0'' + V_0' = 0$$

$$V_0(0) = 0, \quad V_0(\infty) = \alpha.$$

WE SOLVE TO OBTAIN

$$V_0 = \alpha\left(1 - e^{-y}\right) = \alpha\left(1 - e^{-\varepsilon^{-1}(x+1)}\right)$$

THE RIGHT BL IS

$$y = \varepsilon^{-1}(b-x), \quad V(y) = u[b - \varepsilon y].$$

$$\frac{\varepsilon}{\varepsilon^2} V'' + (b - \varepsilon y)\frac{V'}{\varepsilon} = 0 \longrightarrow V_0'' + b V_0' = 0$$

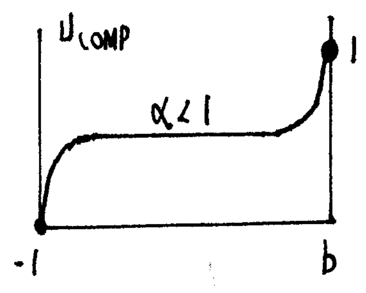
$$V_0(\infty) = \alpha, \quad V_0(0) = 1.$$

THIS YIELDS $\quad V_0(y) = \alpha + [1-\alpha] e^{-by}.$

IN SUMMARY, THE COMPOSITE OR UNIFORM EXPANSION IS

$$u_{comp} \sim \alpha - \alpha e^{-\varepsilon^{-1}(x+1)} + (1-\alpha) e^{-b\varepsilon^{-1}(b-x)}$$

FOR ANY α!



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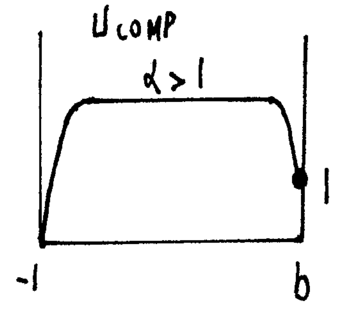


EVEN BY RETAINING HIGHER ORDER ALGEBRAIC TERMS IN ε FROM THE BOUNDARY LAYER REGIONS WE STILL CANNOT DETERMINE α.

QUESTION (i) HAS BOUNDARY LAYER THEORY FAILED? IS THIS METHOD NOT RELIABLE?

(ii) IS THE PROBLEM BAD? IS IT VERY SENSITIVE AND ALMOST ILL-POSED?

IT TURNS OUT THAT (ii) IS THE ANSWER.

WE CAN SOLVE $u'' - \frac{x}{\varepsilon} u' = 0 \longrightarrow (u' e^{-x^2/2\varepsilon})' = 0.$

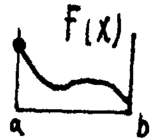
THIS YIELDS $u(x) = C_0 \int_{-1}^{x} e^{\Lambda^2/2\varepsilon}\, d\Lambda$ SATISFIES $u(-1) = 0$.

NOW $u(b) = 1$ YIELDS THAT

$$u(x) = \frac{\int_{-1}^{x} e^{\Lambda^2/2\varepsilon}\, d\Lambda}{\int_{-1}^{b} e^{\Lambda^2/2\varepsilon}\, d\Lambda}$$

IS <u>EXACT</u> SOLUTION.

WE COULD USE A RESULT FROM ASYMPTOTIC EVALUATION OF INTEGRALS: LET $\lambda \longrightarrow +\infty$



$$I(\lambda) \equiv \int_a^b e^{\lambda F(x)} g(x)\, dx \sim \frac{e^{\lambda F(a)} g(a)}{-\lambda F'(a)}$$

WHEN $F(a) = \max\limits_{[a,b]} F(x)$ AND $F'(a) < 0$

ALTERNATIVELY

$$I(\lambda) \sim \frac{e^{\lambda F(b)} g(b)}{\lambda F'(b)}$$

WHEN $F(b) = \max\limits_{[a,b]} F(x)$, $F'(b) > 0$

A ROUGH DERIVATION FOR THE CASE WHERE $f(a) = \max_{[a,b]} f(x)$

AND $f'(a) < 0$ IS TO USE TAYLOR EXPANSION NEAR THE ENDPOINT

$x = a$ AS FOLLOWS

$$I(\lambda) \sim \int_a^{a+\delta} e^{\lambda f(a) + \lambda f'(a)(x-a)} (g(a) + \dots)\, dx$$

HENCE $$I(\lambda) \sim \left(\int_a^{a+\delta} e^{\lambda f'(a)(x-a)}\, dx \right) e^{\lambda f(a)} g(a).$$

THIS INTEGRATES TO $$I(\lambda) \sim \frac{e^{\lambda f(a)} g(a)}{\lambda f'(a)} \, e^{\lambda f'(a)(x-a)} \Big|_a^{a+\delta}.$$

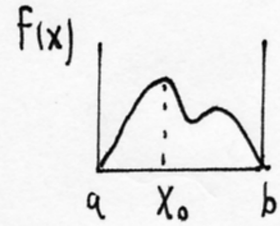
SINCE $f'(a) < 0$ THE CONTRIBUTION FROM UPPER ENDPOINT

IS SMALL AND IS NEGLECTED SO THAT

$$I(\lambda) \sim \frac{e^{\lambda f(a)} g(a)}{-\lambda f'(a)}$$

A SIMILAR (ROUGH) DERIVATION ALSO CAN BE DONE

WHEN $f(b) = \max_{[a,b]} f(x)$.



IF $f(x_0) = \max_{[a,b]} f(x)$ WITH $a < x_0 < b$

THEN $$I(\lambda) = \int_a^b e^{\lambda f(x)} g(x)\, dx \sim \left(\frac{2\pi}{\lambda |f''(x_0)|} \right)^{1/2} e^{\lambda f(x_0)} g(x_0).$$
$\lambda >> 1$.

ROUGH DERIVATION IS TO CALCULATE:

$$I(\lambda) = \int_{x_0-\delta}^{x_0+\delta} e^{\lambda [f(x_0) + (x-x_0)^2 f''(x_0)/2]} g(x_0)\, dx$$

ANYWAYS, WE USE THIS TO CALCULATE

$$u(x) = \frac{\int_{-1}^{x} e^{\Lambda^2/2\varepsilon}\, d\Lambda}{\int_{-1}^{b} e^{\Lambda^2/2\varepsilon}\, d\Lambda} \qquad \text{FOR } \lambda = \frac{1}{2\varepsilon} >> 1.$$

<u>SUPPOSE $b = 1$</u> THEN

$$\int_{-1}^{1} e^{\Lambda^2/2\varepsilon}\, d\Lambda \sim \frac{e^{1/2\varepsilon}}{\left(\frac{1}{2\varepsilon}\, 2\right)} \cdot \boxed{2} \sim 2\varepsilon e^{1/2\varepsilon}$$

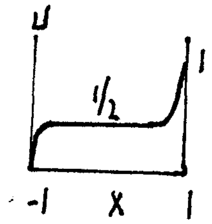
FROM NOTING THAT $f(\Lambda) = \Lambda^2$ HAS A MAXIMUM AT $x = -1$ AND $x = 1$. HENCE WE NEED THE FACTOR OF $\boxed{2}$ BY SYMMETRY. NOTICE $f(\pm 1) = 1$, $\lambda = 1/2\varepsilon$ AND $f'(-1) = -2 = -f'(1)$.

NOW SUPPOSE $-1 < x < 1$ WITH x <u>NOT</u> $O(\varepsilon)$ CLOSE TO EITHER ENDPOINT. THEN

$$u(x) \sim \frac{\int_{-1}^{x} e^{\Lambda^2/2\varepsilon}\, d\Lambda}{2\varepsilon e^{1/2\varepsilon}} \sim \frac{\varepsilon e^{1/2\varepsilon}}{2\varepsilon e^{1/2\varepsilon}} \simeq \frac{1}{2}.$$

THUS WE EXPECT $\alpha = 1/2$ AND

$$u_{comp} = \frac{1}{2} - \frac{1}{2} e^{-\varepsilon^{-1}(1+x)} + \frac{1}{2} e^{-\varepsilon^{-1}(1-x)}$$



<u>SUPPOSE $b = 1 + O(\varepsilon)$</u>. THEN WE CALCULATE BY ADDING BOTH BOUNDARY CONTRIBUTIONS

$$\int_{-1}^{b} e^{\Lambda^2/2\varepsilon}\, d\Lambda \sim \frac{e^{1/2\varepsilon}}{\left(\frac{1}{2\varepsilon}\right)(2)} + \frac{e^{b^2/2\varepsilon}}{\left(\frac{1}{2\varepsilon}\right) 2b}$$

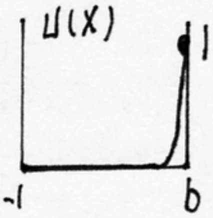
$$\int_{-1}^{b} e^{\Lambda^2/2\varepsilon}\, d\Lambda \sim \underline{\varepsilon e}^{1/2\varepsilon} \left(1 + \frac{e^{(b^2-1)/2\varepsilon}}{b} \right).$$

THUS
$$u(x) = \frac{\int_{-1}^{x} e^{\lambda^2/2\varepsilon}\, d\lambda}{\varepsilon e^{1/2\varepsilon}\left(1 + \frac{e^{(b^2-1)/2\varepsilon}}{b}\right)}$$

NOW FOR $-1 < x < b$ WITH x <u>NOT</u> $O(\varepsilon)$ CLOSE TO EITHER ENDPOINT (i.e. IN THE OUTER REGION) THEN
$$\int_{-1}^{x} e^{\lambda^2/2\varepsilon}\, d\lambda \sim \varepsilon e^{1/2\varepsilon}$$

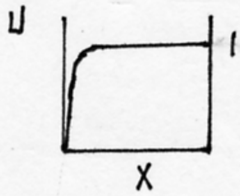
THIS GIVES
$$u_{out} = \alpha = \frac{1}{1 + \frac{1}{b} e^{(b^2-1)/2\varepsilon}} \qquad b > 0$$

<u>NOTICE IF $b > 1$</u> WITH $b - 1 = O(1)$ THEN FOR $\varepsilon \ll 1$ WE ESTIMATE $\alpha \approx 0$ (EXPONENTIALLY SMALL)



<u>NOTICE IF $0 < b < 1$</u> THEN $b^2 - 1 < 0$ AND SO $\alpha \approx 1$.

THIS GIVES A SOLUTION $\alpha \approx 1$ AND



IF $b = 1$ THEN $\alpha = 1/2$ AS WE SAW BEFORE.

IF WE WRITE $b - 1 = b_0 \varepsilon$ THEN $b^2 - 1 = (b-1)(b+1) \approx 2 b_0 \varepsilon + O(\varepsilon^2)$

AND SO
$$\alpha \sim \frac{1}{1 + \frac{1}{1+O(\varepsilon)} e^{b_0 + O(\varepsilon)}} \sim \frac{1}{1 + e^{b_0}}.$$

THUS $O(\varepsilon)$ CHANGES IN b MAKE $O(1)$ CHANGES IN α!!

EXPONENTIALLY ILL-CONDITIONED

WE NOW SHOW THAT

$$(SL)\quad \begin{cases} \varepsilon \phi'' - x\phi' + \lambda\phi = 0 & -1<x<b \\ \phi(-1)=0,\ \phi(b)=0 & , b>0 \end{cases}$$

HAS AN EIGENVALUE λ_0 WHICH IS EXPONENTIALLY SMALL AS $\varepsilon \to 0$.

THIS SUGGESTS THAT THE SOLUTION TO

$$\varepsilon u'' - x u' = 0 \qquad -1<x<b$$

$$u(-1)=0,\quad u(b)=1$$

WILL BE EXTREMELY ILL-CONDITIONED AND THAT THE PROBLEM IS ESSENTIALLY ILL-POSED AS $\varepsilon \to 0^+$.

NOW WE WRITE (SL) IN CANONICAL STURM-LIOUVILLE FORM AS

$$\left(e^{-x^2/2\varepsilon}\phi'\right)' + \frac{\lambda}{\varepsilon} e^{-x^2/2\varepsilon}\phi = 0 \qquad -1<x<b$$

$$\phi(-1)=0,\quad \phi(b)=0$$

THUS THE WEIGHT FUNCTION IS $w(x) = e^{-x^2/2\varepsilon}$.

IF WE INTEGRATE FROM $-1<x<b$ WE OBTAIN

$$\frac{\lambda}{\varepsilon}\int_{-1}^{b} e^{-x^2/2\varepsilon}\phi\, dx = -e^{-x^2/2\varepsilon}\phi'\Big|_{-1}^{b}.$$

HENCE, $(*)$

$$\lambda\int_{-1}^{b} e^{-x^2/2\varepsilon}\phi\, dx = -\varepsilon e^{-x^2/2\varepsilon}\phi'\Big|_{-1}^{b}.$$

NOW NOTICE THAT WLOG WE CAN ASSUME $\phi = O(1)$.

THEN SINCE $\int_{-1}^{b} e^{-x^2/2\varepsilon}\phi\, dx$ HAS DOMINANT CONTRIBUTION NEAR $x=0$

WHICH WILL BE ALGEBRAIC AS $\varepsilon \to 0$ WHILE THE BOUNDARY LAYER TERM IS EXPONENTIALLY SMALL $\longrightarrow \exists\ \lambda$ WHICH IS EXPONENTIALLY SMALL AS $\varepsilon \to 0$.

THEN IF WE SET $\lambda = 0$ TO FIND <u>APPROXIMATE</u> EIGENFUNCTION, IT IS EQUIVALENT TO SOLVING

$$\varepsilon \phi'' - x\phi' \approx 0$$
$$\phi(-1) = 0, \quad \phi(1) = b$$

WHICH GIVES $\quad \phi_{COMP} = 1 - e^{-\varepsilon^{-1}(1+x)} - e^{-b\varepsilon^{-1}(b-x)}$

USING THIS IN INTEGRAL IDENTITY $(*)$ WE GET

$$\phi_{COMP}'(-1) = \varepsilon^{-1}, \quad \phi_{COMP}'(b) = -b\varepsilon^{-1}$$

AND $\quad \displaystyle\int_{-1}^{b} e^{-x^2/2\varepsilon}\phi_{COMP}\,dx \sim \int_{-1}^{b} e^{-x^2/2\varepsilon}\,dx \approx \sqrt{\frac{2\pi}{(\frac{1}{2\varepsilon})\,2}} = \sqrt{2\pi\varepsilon}.$

(RECALL WITH $\lambda = 1/2\varepsilon$, $f(x) = -x^2$, f HAS GLOBAL MAX AT $x = 0$ AND
$\displaystyle\int_{-1}^{b} e^{\lambda f(x)}\,dx \sim \sqrt{\frac{2\pi}{\lambda|f''(0)|}}\; e^{\lambda f(0)}$)

THIS YIELDS THAT $\quad \displaystyle\lambda\int_{-1}^{b} e^{-x^2/2\varepsilon}\phi_{COMP}\,dx \sim -\varepsilon e^{-b^2/2\varepsilon}\phi_{COMP}'(b) + \varepsilon e^{-1/2\varepsilon^2}\phi_{COMP}'(-1)$

WE GET $\quad \lambda\sqrt{2\pi\varepsilon} \sim b e^{-b^2/2\varepsilon} + e^{-1/2\varepsilon}$

HENCE $\quad \displaystyle\lambda \sim \frac{e^{-1/2\varepsilon}}{\sqrt{2\pi\varepsilon}}\left(1 + b e^{-(b^2-1)/2\varepsilon}\right).$

NOTICE THAT WHEN $b=1$ THEN

$$\lambda \sim \sqrt{\frac{2}{\pi \varepsilon}}\ e^{-1/2\varepsilon}$$ WHICH IS AN EXPONENTIALLY SMALL EIGENVALUE AS $\varepsilon \to 0^+$.

A HIGHER-ORDER CALCULATION GIVES THE FOLLOWING IMPROVED RESULT

$$\lambda \sim \sqrt{\frac{2}{\pi \varepsilon}} \left(1 - \varepsilon + O(\varepsilon^2) \right) e^{-1/2\varepsilon}$$

THEREFORE OUR PROBLEM IS SIMILAR TO SOLVING

$A\underline{x} = \underline{b}$ WITH THE MATRIX A HAVING ALMOST A ZERO EIGENVALUE AS $\varepsilon \to 0$ WITH Eigenvector $AV_0 = \lambda_0 V_0$

$\lambda_0 = O(e^{-c/\varepsilon})$. HENCE IN THE LIMIT $\varepsilon \to 0$

WE NEED $V_0^T b \to 0$

FOR THE DIFFERENTIAL EQUATION PROBLEM WE WRITE

$$L_\varepsilon u = \varepsilon u'' - x u'$$

NOW let $u = u_{COMP}(x;\alpha) + V$

$$L_\varepsilon V = - L_\varepsilon u_{COMP} \qquad -1 < x < 1$$

$$V = - u_{COMP} \quad \text{ON} \quad x = \pm 1$$

IF WE DO A PROJECTION AGAINST THE EIGENFUNCTION CORRESPONDING TO $\lambda = O(\varepsilon^{-1/2} e^{-1/2\varepsilon})$ WE CAN OBTAIN THE CORRECT α.

TRANSITION LAYERS: GEOMETRIC SINGULAR PERTURBATIONS

WE BEGIN WITH $\varepsilon u'' + u - u^3 = 0 \qquad u(0) = 0, \; u(1) = 0$

IN THE OUTER REGION WE EXPAND $u = u_0 + \ldots$ SO THAT $u_0 = 0$ OR $u_0 = \pm 1$ ARE OUTER SOLUTIONS. IF WE TAKE $u_0 = 0$ THEN $u \equiv 0$ IS A SOLUTION. LET'S CONSTRUCT OTHER SOLUTIONS.

TAKE $u_0 = 1$ AS AN OUTER SOLUTION. NEAR $x = 0$ WE CONSTRUCT A BL BY LETTING $y = x/\varepsilon^p$, $w(y) = u[\varepsilon^p y]$. THEN,

$$\varepsilon^{1-2p} w'' + w - w^3 = 0 \qquad \text{CHOOSE } p = 1/2.$$

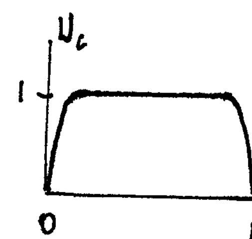
WE THEN OBTAIN

$$w'' + w - w^3 = 0, \quad 0 \le y < \infty$$
$$w(0) = 0, \quad w(\infty) = 1$$

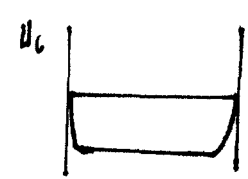
THE EXACT SOLUTION IS $w(y) = \tanh[y/\sqrt{2}]$. TO VERIFY LET $w = \tanh(\alpha y)$ SO $w' = \alpha \,\mathrm{sech}^2(\alpha y)$, $w'' = -2\alpha^2 \,\mathrm{sech}^2 \tanh$. THIS GIVES $-2\alpha^2 \mathrm{sech}^2 \tanh + \tanh(1 - \tanh^2) = -2\alpha^2 \mathrm{sech}^2 \tanh + \mathrm{sech}^2 \tanh = 0$ SO $2\alpha^2 = 1$ OR $\alpha = 1/\sqrt{2}$. HERE WE USE $1 - \tanh^2 = \mathrm{sech}^2$.

IN A SIMILAR WAY WE CAN CONSTRUCT A BL NEAR $x = 1$. THE COMPOSITE EXPANSION IS

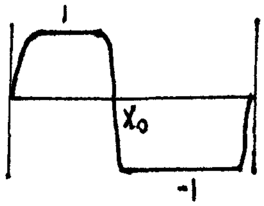
$$u_c(x) = \tanh\left[\frac{x}{\sqrt{2\varepsilon}}\right] + \tanh\left[\frac{1-x}{\sqrt{2\varepsilon}}\right] - 1$$



NOTICE THAT THE ORIGINAL PROBLEM HAS MORE SOLUTIONS. IN PARTICULAR $-u_c(x)$ WOULD BE A COMPOSITE EXPANSION FOR AN OUTER SOLUTION WITH $u_0 = -1$.



CAN WE CONSTRUCT OTHER SOLUTIONS? IN PARTICULAR CAN WE FIND A SOLUTION OF THE FORM



WITH AN INTERIOR LAYER?

SUPPOSE WE LET $y = \varepsilon^{-1/2}(X - X_0)$, $V(y) = U[X_0 + \varepsilon^{1/2} y]$. THEN WE OBTAIN

$$V'' + V - V^3 = 0, \quad V(-\infty) = 1, \quad V(+\infty) = -1$$

WE THEN OBTAIN $V(y) = -\tanh\left[\dfrac{y - y_0}{\sqrt{2}}\right]$. WE CAN CHOOSE $y_0 = 0$ SO THAT $V = 0$ WHEN $y = 0$ OR EQUIVALENTLY $X = X_0$.

THIS GIVES $U \sim -\tanh\left[\dfrac{X - X_0}{\sqrt{2\varepsilon}}\right]$ BUT WITH X_0 IN $0 < X_0 < 1$ APPARENTLY UNDETERMINED. THE CORRECT VALUE OF X_0 CAN ONLY BE FOUND BY MATCHING EXPONENTIALLY SMALL TERMS AS $\varepsilon \to 0$ (HARD!).

NOW LET'S TAKE A MORE GEOMETRIC VIEWPOINT BY LOOKING AT THE ORIGINAL PROBLEM IN THE PHASE-PLANE.

LET $Z = X/\varepsilon^{1/2}$ SO THAT
$$U''(Z) + U(Z) - U^3(Z) = 0$$
$$U(0) = 0, \quad U\left(1/\sqrt{\varepsilon}\right) = 0.$$

NOW LET $W = U'$ SO THAT

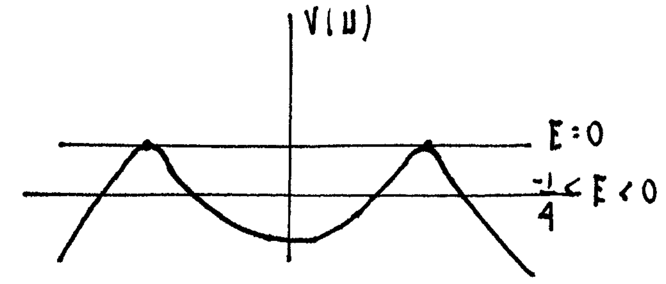
$$W' = -U + U^3$$
$$U' = W$$

WE GET THE FIRST INTEGRAL

$$U'U'' + UU' - U'U^3 = 0$$
$$\frac{d}{dZ}\left(\frac{1}{2}U'^2 + \frac{U^2}{2} - \frac{U^4}{4}\right) = 0$$
OR $$\frac{1}{2}U'^2 - \frac{1}{4}(1 - U^2)^2 = E$$

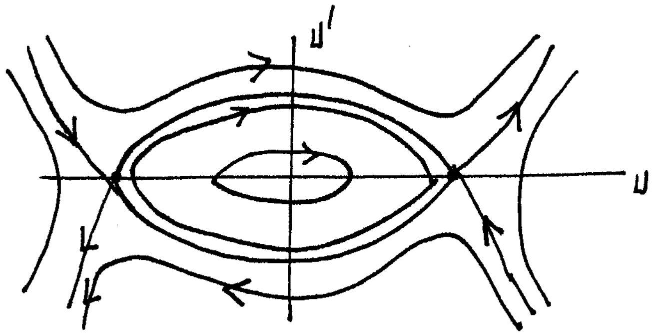
WE WRITE THIS AS

$$\frac{1}{2}U'^2 + V(U) = E$$
$$V(U) = -\frac{1}{4}(1 - U^2)^2.$$



$$\frac{1}{2} u'^2 = E - V(u)$$

WE NEED $E > -1/4$ AND $E < 0$ FOR "BOUND STATES" AND PERIODIC SOLUTIONS



NOTICE:

(i) $u = \pm 1$ ARE SADDLE POINTS

$u = 0$ IS A CENTER

(ii) THE SEPARATRIX $E = 0$ IS THE HETEROCLINIC ORBIT CONNECTING $u = -1$ TO $u = 1$ IN INFINITE TIME.

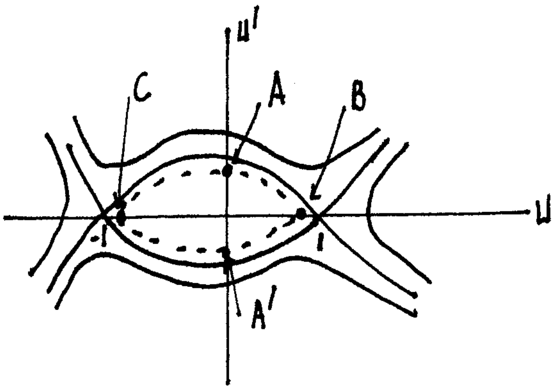
i.e. $u(-\infty) = -1$, $u(\infty) = 1$

OR $u(\infty) = -1$ AND $u(-\infty) = 1$.

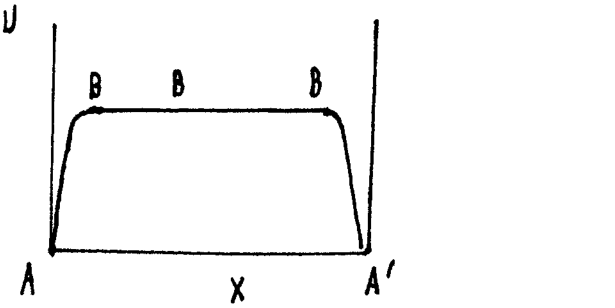
$\longrightarrow$ $u = \tanh(z/\sqrt{2})$ OR $u = -\tanh(z/\sqrt{2})$

(iii) IN SOLVING OUR ORIGINAL PROBLEM WE WANT $u(0) = 0$ AND $u(1/\sqrt{\varepsilon}) = 0$. HENCE WE MUST START AND END ON y-AXIS AND WE NEED A "LONG-TIME" SOLUTION THAT IS NEAR THE SEPARATRIX. WE MUST START AND END INSIDE THE LOBE.

(iv) MOST OF THE "TIME" IS TAKEN IN PASSING NEAR SADDLE POINTS. $\longrightarrow$ outer solutions



$\longrightarrow$



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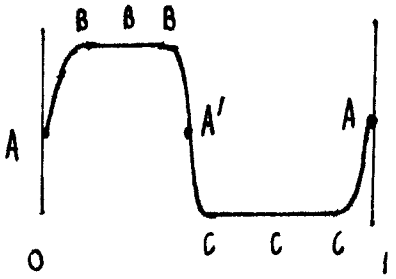
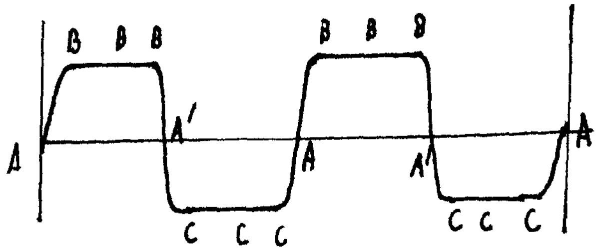


THIS IS THE INTERNAL LAYER PROFILE. SINCE BY SYMMETRY THE "TIME" SPENT NEAR C AND B ARE THE SAME THEN A' SHOULD BE LOCATED AT THE MIDPOINT

i.e. $x_0 = 1/2$.

IF WE LOOP SEVERAL TIMES WE CAN GET SOLUTIONS LIKE



<u>DIFFICULT</u> <u>QUESTIONS</u> (i) DETERMINE ANALYTICALLY THE LOCATION OF THE INTERNAL LAYERS.

(ii) CALCULATE THE NUMBER $N(\varepsilon)$ OF SOLUTIONS FOR $\varepsilon \ll 1$

A SIMILAR QUALITATIVE ANALYSIS CAN BE DONE WHENEVER WE HAVE A HETEROCLINIC CONNECTION.

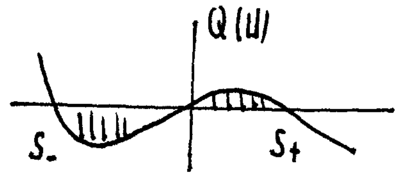
IN THE GENERAL CASE SUPPOSE THAT

$$\varepsilon^2 u'' + Q(u) = 0 \quad \text{IN } -1 < x < 1$$

$$u'(\pm 1) = 0$$

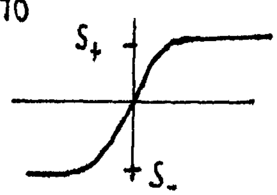
SUPPOSE THAT $Q(S_-) = Q(0) = Q(S_+) = 0$ WITH $S_- < 0 < S_+$ (WLOG) WITH $Q'(S_\pm) < 0$ AND $Q'(0) > 0$. FURTHER SUPPOSE THAT

$$\int_{S_-}^{S_+} Q(u)\, du = 0$$



SHADED AREAS EQUAL BY ASSUMPTION

THEN $\exists$ A UNIQUE SOLUTION $w(y)$ (heteroclinic connection) to



$$w'' + Q(w) = 0 \qquad -\infty < y < \infty$$

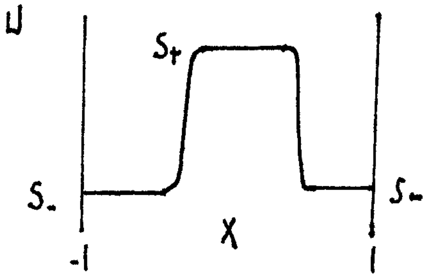
$$w \sim S_+ \quad \text{AS } y \to +\infty \qquad (w \sim S_+ - a_+ e^{-\nu_+ y} \quad y \gg 1)$$

$$w \sim S_- \quad \text{AS } y \to -\infty \qquad (w \sim S_- + a_- e^{\nu_- y}, \quad y \to -\infty)$$

AND $w(0) = 0$. THE DECAY RATES ARE $\nu_\pm = [-Q'(S_\pm)]^{1/2}$

IF $Q(u) = 2(u - u^3)$, THEN $S_\pm = \pm 1$, AND $w(y) = \tanh y$ $\quad w \sim 1 - 2e^{-2y}$ AS $y \to +\infty$.

WE THEN CAN CONSTRUCT SOLUTIONS LIKE



TO SHOW THE EXISTENCE OF HETEROCLINIC, WE WRITE

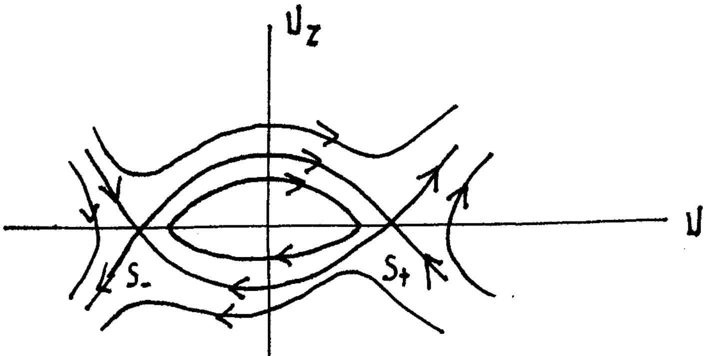
$$w'w'' + w'Q(w) = 0 \longrightarrow \frac{d}{dy}\left[\frac{1}{2}w'^2 + V(w)\right] = 0$$

WHERE $V(w) = \int_{s_-}^{w} Q(\lambda)\,d\lambda$. $\longrightarrow$ $\frac{1}{2}w'^2 + V(w) = E$.

THE HETEROCLINIC IS $E = 0$. $\longrightarrow$ $\frac{1}{2}w'^2 + V(w) = 0$.

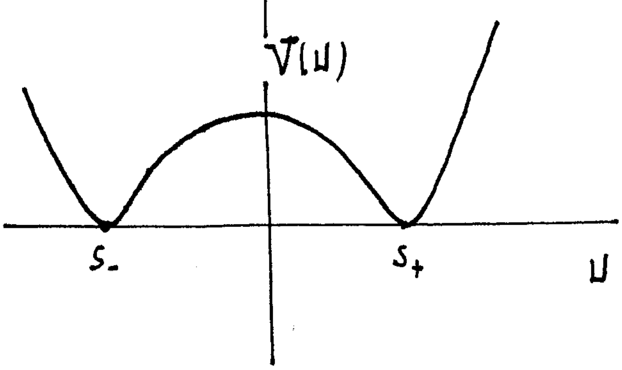
NOTICE $w = s_+$ WITH $w' = 0$ IS SATISFIED. THEN $w = s_+$ WITH $w' = 0$ IS SATISFIED ONLY WHEN $V(w_+) = \int_{s_-}^{s_+} Q(\lambda)\,d\lambda = 0$.

THIS GIVES THE EQUAL AREA CONDITION. NOTICE $u = s_\pm$ ARE SADDLE POINTS AND $u = 0$ IS THE CENTER. THE PHASE PLANE IS



let $z = (x+1)/\sqrt{\varepsilon}$

$$\begin{cases} u' = v \\ v' = -Q(u) \end{cases}$$



NOW CONSIDER A PROBLEM WITH A SPIKES GIVEN BY

$$\varepsilon u'' - u + u^2 = 0, \quad -1 < x < 1$$
$$u'(-1) = u'(1) = 0$$

IF WE LET $Q(u) = -u + u^2$ THEN $Q'(0) = -1$ AND $Q'(1) = 1 > 0$.

THEREFORE, THE ONLY MATCHABLE OUTER SOLUTION IS $u = 0$.

WE THEN TRY TO CONSTRUCT AN INTERNAL LAYER AS A SPIKE ABOUT SOME POINT $x_0 \in (-1, 1)$. LET $y = \varepsilon^{-1/2}(x - x_0)$ AND $w(y) = u(x_0 + \varepsilon^{1/2} y)$.

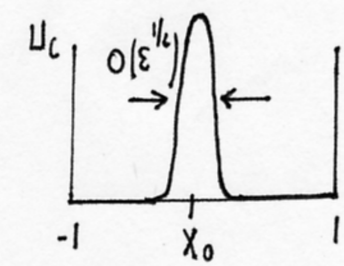
THEN WE OBTAIN

$$w'' - w + w^2 = 0$$
$$w(y) \to 0 \text{ AS } y \to \pm\infty,$$
$$w(0) > 0 \text{ FOR A SPIKE}, \quad w'(0) = 0 \to \text{EVEN SOLUTION}.$$

THIS PROBLEM HAS EXACT SOLUTION $w(y) = \frac{3}{2}\,\text{SECH}^2(y/2)$.

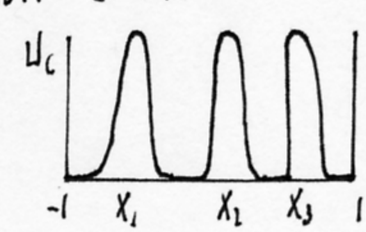
THIS GIVES A COMPOSITE EXPANSION AS

$$u_c \sim \frac{3}{2}\,\text{SECH}^2\left(\frac{x - x_0}{2\sqrt{\varepsilon}}\right)$$



SIMILARLY, WE CAN FORMALLY CONSTRUCT MULTI-SPIKE SOLUTIONS AS

$$u_c \sim \frac{3}{2}\sum_{j=1}^{N} \text{SECH}^2\left(\frac{x - x_j}{2\sqrt{\varepsilon}}\right)$$



DIFFICULT QUESTION

(i) WHERE ARE THE SPIKES LOCATED?

(ii) HOW MANY SOLUTIONS $N(\varepsilon)$ AS $\varepsilon \to 0^+$.

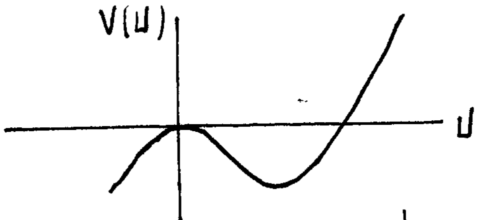
NOW LET $Z = \varepsilon^{-1/2}(X+1)$ WITH $U = U(Z)$ AND SO

$$U'' - U + U^2 = 0$$

$$U'(0) = 0, \quad U'(2/\sqrt{\varepsilon}) = 0$$

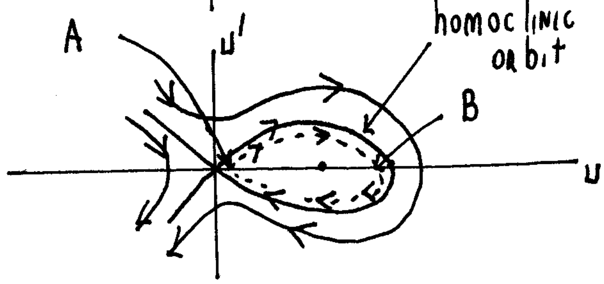
MULTIPLY BY U' TO OBTAIN $\dfrac{U'^2}{2} - \dfrac{U^2}{2} + \dfrac{U^3}{3} = \dfrac{E}{2}.$

THIS GIVES $U'^2 + V(U) = E$ WITH $V(U) = -U^2 + \frac{2}{3}U^3$.

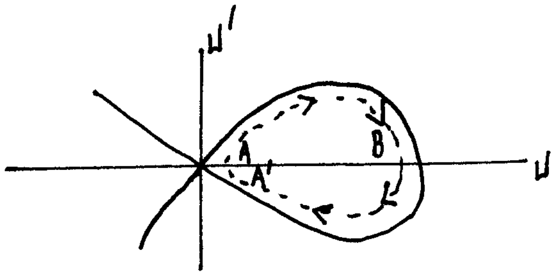


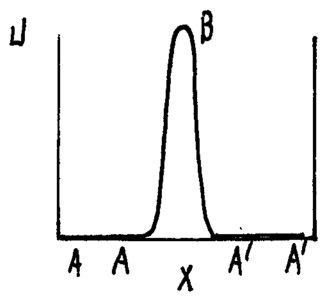
NOTICE $V(3/2) = 0$.

- ALSO $E = 0$ IS HOMOCLINIC TRAJECTORY WITH $W'' - W + W^2 = 0$, $W(\pm\infty) = 0$.
- AGAIN WE ARE LOOKING FOR "LONG-TIME" ORBITS WITH $U'(0) = 0$ AND $U'(2/\sqrt{\varepsilon}) = 0$.



WE NEED A SOLUTION NEAR THE HOMOCLINIC WITH $U'(0) = U'(2/\sqrt{\varepsilon}) = 0$





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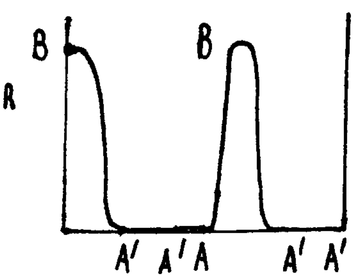


REMARK: IF WE SET $U(0) = U(2/\sqrt{\varepsilon}) = 0$ THEN WE CAN ONLY HAVE A 1-SPIKE SOL'N

IN GENERAL UNDER WHAT CONDITIONS DO WE HAVE A SPIKE-LAYER SOLUTION FOR

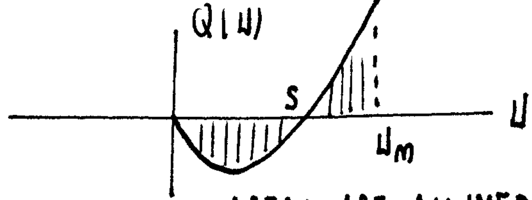
$$\varepsilon U'' + Q(U) = 0, \quad -1 < X < 1$$

$$U'(\pm 1) = 0$$

WE WILL ASSUME THAT $Q(0) = 0$, $Q(S) = 0$ WITH $S > 0$

WITH $Q'(0) < 0$ AND $Q'(S) > 0$ AND THAT THERE EXISTS A

VALUE $u_m > s$ WITH $\int_0^{u_m} Q(u)\,du = 0$



AREAS ARE ASSUMED EQUAL.

NOW WE CAN INTEGRATE TO OBTAIN

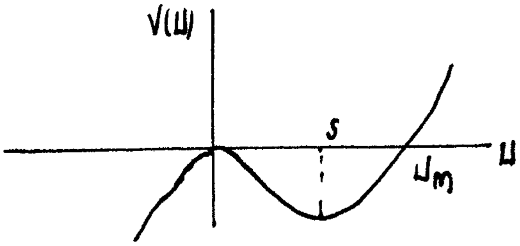
$$\varepsilon \frac{u'^2}{2} + V(u) = E \qquad \text{WITH} \qquad V(u) = \int_0^u Q(\lambda)\,d\lambda .$$

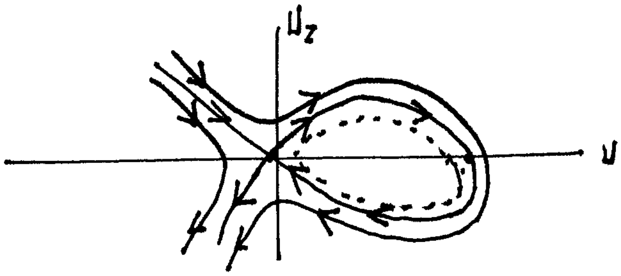
NOW IF WE LET $z = \varepsilon^{-1/2}(x+1)$ TO OBTAIN

$$\frac{1}{2}u_z^2 + V(u) = E$$

$u = s$ IS A CENTER

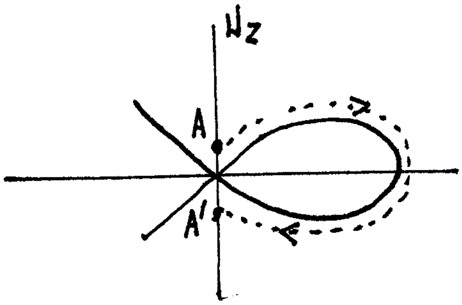
$u = 0$ IS A SADDLE POINT





WHEN $u' = 0$ AT $x = \pm 1$ WE OBTAIN SPIKE SOLUTIONS BY FOLLOWING THE DOTTED LINE INSIDE THE TRAJECTORY OF THE HOMOCLINIC.

NOTICE WITH THE BOUNDARY CONDITION $u(-1) = 0$ AND $u(1) = 0$ WE HAVE TO FOLLOW A TRAJECTORY OUTSIDE THE HOMOCLINIC.



WE HAVE TO START AND STOP AT A AND A', RESPECTIVELY. THIS GIVES A <u>ONE-SPIKE SOLUTION</u>. HOWEVER THERE CANNOT BE MULTI-SPIKE SOLUTIONS WITH THESE BOUNDARY CONDITIONS.

STABILITY OF SPIKE AND TRANSITION SOLUTIONS

SPIKE SOLUTION CONSIDER $u_t = \varepsilon^2 u_{xx} - u + u^2 \quad -1<x<1 \; ; \; u_x(\pm 1, t) = 0.$

THE SPIKE SOLUTION FOR THE EQUILIBRIUM PROBLEM IS

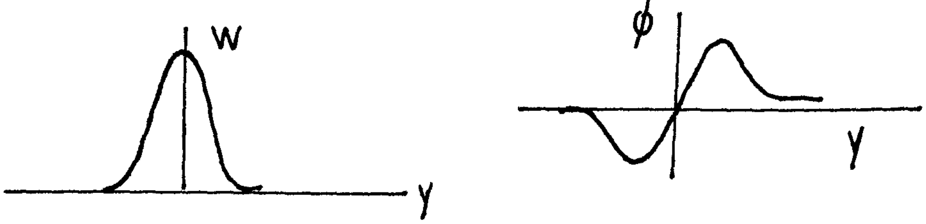
$$u = w\left[\frac{x - x_0}{\varepsilon}\right] \qquad w(y) = \frac{3}{2}\,\mathrm{sech}^2(y/2)$$

LINEARIZE: let $u = w + e^{\lambda t}\phi$ $\qquad w'' - w + w^2 = 0$

THEN $\varepsilon^2 \phi_{xx} - \phi + 2w\phi = \lambda\phi$

$$\phi_x(\pm 1) = 0$$

NOTICE: ON INFINITE LINE $\phi = w'$, $\lambda = 0$ IS AN EIGENPAIR.



THUS ϕ HAS ONE NODAL POINT. HENCE ON INFINITE LINE $\lambda = 0$ IS SECOND EIGENVALUE.

since eigenvalues are NON-degenerate, $\exists$ A $\lambda_0 > 0$ WITH $\phi_0 > 0$ corresponding to FIRST Eigenvalue ON INFinite LINE. ON infinite LINE, ε CAN BE scaled out AND so $\lambda_0 = O(1)$ $\lambda_0 > 0$. The effect OF "FAR AWAY BOUNDARIES" PERTURBS λ_0 BY EXPONENTIALLY SMALL TERMS AS $\varepsilon \to 0 \to \lambda_0 > 0$ UNSTABLE.

TRANSITION LAYER CONSIDER $u_t = \varepsilon^2 u_{xx} + u - u^3 \quad -1<x<1, \; u_x(\pm 1, t) = 0$

THUS $w'' + w - w^3 = 0$ AND $w = \tanh(y/\sqrt{2})$. LET $u = \tanh\left[\frac{\varepsilon^{-1}(x - x_0)}{\sqrt{2}}\right] + e^{\lambda t}\phi$

THEN $\varepsilon^2\phi'' + \phi - 3w^2\phi = \lambda\phi$ WITH $\lambda = 0$ AND $\phi = w'$ AN Eigenpair ON INFinite LINE. SINCE $w' > 0 \to \lambda_0 = 0$, $\phi_0 = w'$ IS FIRST EIGENPAIR ON INFinite LINE. PResence OF "FAR AWAY BOUNDARIES" MAKE $\lambda_0 = O(e^{-C/\varepsilon})$

$\to$ TRANSITION LAYER IS METASTABLE.

EXAMPLE (BURGER'S EQUATION)

CONSIDER BURGERS EQUATION WITH SMALL DIFFUSIVITY $\varepsilon \ll 1$ GIVEN BY

$$U_t + U U_x = \varepsilon U_{xx} \qquad -\infty < x < \infty,\ t > 0.$$

WE WILL LOOK FOR A TRAVELLING WAVE SOLUTION TO THIS PROBLEM IN THE FORM $U = V(x - ct)$ WITH $z = x - ct$

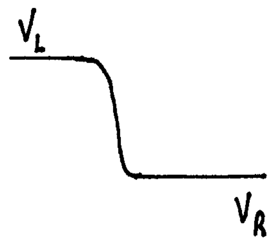
AND $V(-\infty) = V_L$ WITH $V(+\infty) = V_R$.

WE SUBSTITUTE TO OBTAIN $U_t = -cV'$ $U_x = V'$,

WHICH YIELDS THAT

$$-cV' + VV' = \varepsilon V'' \qquad -\infty < z < \infty$$

$$V(-\infty) = V_L, \quad V(+\infty) = V_R$$



TO MAKE V UNIQUE WE PIN THE PHASE SHIFT AS

$$V(0) = (V_R + V_L)/2.$$

WE ASSUME $V_L > V_R$.

NOW WE INTEGRATE TO OBTAIN, UP TO A CONSTANT A

$$-cV + \frac{1}{2}V^2 + A = \varepsilon V'$$

THEN $V(-\infty) = V_L$ WITH $\quad -cV_L + \frac{1}{2}V_L^2 + A = 0$

$V(+\infty) = V_R$ $\quad -cV_R + \frac{1}{2}V_R^2 + A = 0.$

THIS YIELDS $\quad -c(V_R - V_L) + \frac{1}{2}(V_R^2 - V_L^2) = 0 \longrightarrow c = \frac{1}{2}(V_R + V_L)$

THEN $\quad -\frac{1}{2}(V_R + V_L)V_L + \frac{1}{2}V_L^2 + A = -\frac{1}{2}V_L V_R + A = 0 \longrightarrow A = \frac{1}{2}V_R V_L.$

THEN, WE OBTAIN $\quad \varepsilon V' = \frac{1}{2}(V - V_R)(V - V_L)$

NOW WE LET $\quad V = \dfrac{V_R + V_L}{2} + W$

THEN WE OBTAIN

$$\varepsilon w' = \frac{1}{2}(w^2 - \alpha^2), \qquad \alpha = \frac{V_L - V_R}{2} > 0$$

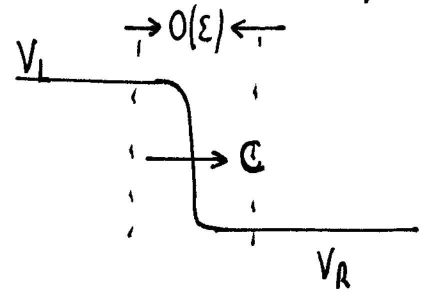
$$w(-\infty) = \alpha, \quad w(\infty) = -\alpha, \quad w(0) = 0.$$

TRY $\quad w = -\alpha \tanh(Bz) \quad \rightarrow \quad -\varepsilon \alpha B \,\mathrm{sech}^2(Bz) = \frac{1}{2}\alpha^2(-\mathrm{sech}^2(Bz)).$

THIS GIVES $\quad B = +\dfrac{\alpha}{2\varepsilon}.$

HENCE $\quad V(z) = \dfrac{(V_R + V_L)}{2} - \dfrac{(V_L - V_R)}{2} \tanh\left(\dfrac{(V_L - V_R)(x - ct)}{4\varepsilon}\right)$

where $\quad C = \frac{1}{2}(V_L + V_R).$



DIFFUSIVE LAYER OF WIDTH $O(\varepsilon)$.

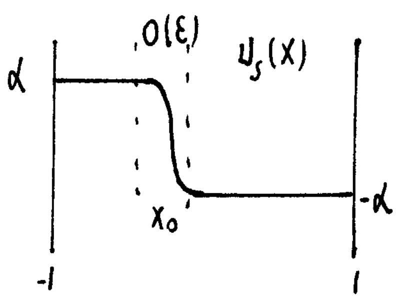
NOTICE THAT IF $V_L = -V_R$ THEN $C = 0$ AND WE HAVE A STATIONARY SOLUTION.

FOR INSTANCE SUPPOSE $\quad U_t + UU_x = \varepsilon U_{xx} \qquad -1 < x < 1, \ t > 0$

$$U(-1,t) = \alpha > 0, \quad U(1,t) = -\alpha < 0, \quad \varepsilon \to 0^+$$

$$U(x,0) = f(x).$$

THEN THERE EXISTS A STEADY-STATE SOLUTION $U_S(x)$ THAT HAS THE FOLLOWING STRUCTURE



- DETERMINING x_0 ANALYTICALLY FOR STEADY STATE PROBLEM REQUIRES MATCHING EXPONENTIALLY SMALL TERMS. BUT $x_0 = 0$ BY SYMMETRY
- FOR THE TIME-DEPENDENT PROBLEM YOU WILL SHOW (HW) $\quad \dfrac{dx_0}{dt} \sim \alpha\left(e^{-\alpha(1+x_0)/\varepsilon} - e^{-\alpha(1-x_0)/\varepsilon}\right).$