Strong Localized Perturbations and the Biharmonic Problem

(*Received 15 January 2013*)

1 Biharmonic Equation with Hole

Problem 1: *Consider the Biharmonic equation in the two-dimensional concentric annulus, formulated as*

$$\triangle^2 u = 0 \,, \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon \,, \tag{1.1 a}$$

$$u = 1 \,, \quad u_r = 0 \,, \quad \text{on} \quad r = 1 \,, \tag{1.1 b}$$

$$u = u_r = 0 \,, \quad r = \varepsilon \,. \tag{1.1 c}$$

Here Ω is the unit disk centered at the origin, containing a small hole of radius ε centered at $\mathbf{x} = 0$, i.e. $\Omega_\varepsilon = \{\mathbf{x} \mid |\mathbf{x}| \le \varepsilon\}$. Calculate the exact solution, and from it determine an approximation to the solution in the outer region $|\mathbf{x}| \gg \mathcal{O}(\varepsilon)$. Can you re-derive this result from singular perturbation theory in the limit $\varepsilon \to 0$? (Hint: the leading-order outer problem for Case I is different from what you might expect).

Solution

We first find the exact solution of (1.1) and then expand it for $\varepsilon \to 0$. Since the radially symmetric solutions to (1.1 a) are linear combinations of $\{r^2, r^2 \log r, \log r, 1\}$, we can write the solution to (1.1 a), which satisfies (1.1 c), as

$$u = A\left(r^2 - 1\right) + B r^2 \log r - (2A + B) \log r + 1 \,, \tag{1.2}$$

for any constants A and B. Then, imposing that $u = u_r = 0$ on $r = \varepsilon$, we get two equations for A and B:

$$2A\left(1 - \varepsilon^2\right) A + B\left(1 - \varepsilon^2 - 2\varepsilon^2 \log \varepsilon\right) = 0 \,, \tag{1.3 a}$$

$$A\left(1 + 2\log\varepsilon - \varepsilon^2\right) + B\left(1 - \varepsilon^2\right) \log \varepsilon = 1 \,. \tag{1.3 b}$$

Equation (1.3 a) gives

$$A = -\frac{B}{2}\left(1 - \frac{2\varepsilon^2 \log \varepsilon}{1 - \varepsilon^2}\right) . \tag{1.4}$$

Upon substituting this into (1.3 b), we obtain that B satisfies

$$-\frac{B}{2}\left(1-\frac{2\varepsilon^2\log\varepsilon}{1-\varepsilon^2}\right)\left(1+\frac{2\log\varepsilon}{1-\varepsilon^2}\right)+B\log\varepsilon=\frac{1}{1-\varepsilon^2} \tag{1.5 a}$$

$$-\frac{B}{2}-\frac{B\log\varepsilon}{1-\varepsilon^2}+\frac{B\varepsilon^2\log\varepsilon}{1-\varepsilon^2}+\frac{2\varepsilon^2 B(\log\varepsilon)^2}{(1-\varepsilon^2)^2}+B\log\varepsilon=\frac{1}{1-\varepsilon^2} \tag{1.5 b}$$

$$-\frac{B}{2}-B(1+\varepsilon^2)\log\varepsilon+B\varepsilon^2(1+\varepsilon^2)\log\varepsilon+2\varepsilon^2(\log\varepsilon)^2 B+B\log\varepsilon\sim 1+\varepsilon^2 \tag{1.5 c}$$

$$-\frac{B}{2}+2\varepsilon^2(\log\varepsilon)^2 B\sim 1+\mathcal{O}(\varepsilon^2)\,. \tag{1.5 d}$$

The last line of (1.5) determines B, while (1.4) determines A. In this way, we get

$$B\sim -2-8\varepsilon^2\left(\log\varepsilon\right)^2\,,\qquad A\sim 1+4\varepsilon^2\left(\log\varepsilon\right)^2\,. \tag{1.6}$$

Upon substituting (1.6) into (1.2), we obtain the following two-term expansion in the outer region $r\gg\mathcal{O}(\varepsilon)$:

$$u\sim u_0(r)+\varepsilon^2\left(\log\varepsilon\right)^2 u_1(r)+\cdots\,, \tag{1.7}$$

where $u_0(r)$ and $u_1(r)$ are defined by

$$u_0(r)=r^2-2r^2\log r\,,\qquad u_1=4\left(r^2-1\right)-8r^2\log r\,. \tag{1.8}$$

It is interesting to note that the leading-order outer solution $u_0(r)$ is not a C^2 smooth function, but that it does satisfy the point constraint $u_0(0)=0$. Hence, in the limit of small hole radius the ε-dependent solution does not tend to the unperturbed solution in the absence of the hole. This unperturbed solution would have $B=0$ and $A=0$ in (1.2), and consequently $u=1$ in the outer region.

Next, we show how to recover (1.7) from a matched asymptotic expansion analysis. In the outer region we expand the solution as

$$u\sim w_0+\sigma w_1+\cdots\,, \tag{1.9}$$

where $\sigma\ll 1$ is an unknown gauge function, and where w_0 satisfies the following problem with a point constraint:

$$\triangle^2 w_0=0\,,\quad 0<r<1\,;\quad w_0(1)=1\,,\quad w_{0r}(1)=0\,,\quad w_0(0)=0\,. \tag{1.10}$$

The solution is readily calculated as

$$w_0=r^2-2r^2\log r\,. \tag{1.11}$$

The problem for w_1 is

$$\triangle^2 w_1=0\,,\quad 0<r<1\,;\quad w_1(1)=w_{1r}(1)=0\,. \tag{1.12}$$

The solution to (1.12) is given in terms of unknown coefficients α_1 and β_1 as

$$w_1=\alpha_1\left(r^2-1\right)+\beta_1 r^2\log r-(2\alpha_1+\beta_1)\log r\,. \tag{1.13}$$

The behavior of w_1 as $r\to 0$, as found below by matching to the inner solution, will determine α_1 and β_1.

In the inner region we set $r=\varepsilon\rho$ and obtain from (1.11) that the terms of order $\mathcal{O}(\varepsilon^2\log\varepsilon)$ and $\mathcal{O}(\varepsilon^2)$ will be generated in the inner region. Therefore, this suggests that in the inner region we expand the solution as

$$v(\rho) = u(\varepsilon\rho) = \left(\varepsilon^2 \log \varepsilon\right) v_0(\rho) + \varepsilon^2 v_1(\rho) + \cdots . \tag{1.14}$$

The functions v_0 and v_1 must satisfy $v_j(1) = v_{j\rho}(1) = 0$. Therefore, we obtain for $j = 0, 1$ that

$$v_j = A_j \left(\rho^2 - 1\right) + B_j \rho^2 \log \rho - (2A_j + B_j) \log \rho . \tag{1.15}$$

We substitute (1.15) into (1.14), and write the resulting expression in terms of the outer variable $r = \varepsilon\rho$. A short calculation gives that the far-field behavior of (1.14) is

$$v \sim -(\log \varepsilon)^2 B_0 r^2 + (\log \varepsilon) \left[(A_0 - B_1) r^2 + B_0 r^2 \log r\right] + A_1 r^2 + B_1 r^2 \log r + 2A_0 \varepsilon^2 (\log \varepsilon)^2 + \mathcal{O}(\varepsilon^2 \log \varepsilon) . \tag{1.16}$$

In contrast, the two-term outer solution from (1.9), (1.11), and (1.13), is

$$u \sim r^2 - 2r^2 \log r + \sigma \left[\alpha_1 \left(r^2 - 1\right) + \beta_1 r^2 \log r - (2\alpha_1 + \beta_1) \log r\right] + \cdots . \tag{1.17}$$

Upon comparing (1.17) with (1.16), we conclude that

$$B_0 = 0 , \quad B_1 = A_0 , \quad A_1 = 1 , \quad B_1 = -2 , \quad \sigma = \varepsilon^2 (\log \varepsilon)^2 . \tag{1.18}$$

This leaves the unmatched constant term $-4\varepsilon^2 (\log \varepsilon)^2$ on the right-hand side of (1.16). Consequently, it follows that the outer correction w_1 is bounded as $r \to 0$ and has the point value $w_1(0) = -4$. Consequently, $2\alpha_1 + \beta_1 = 0$ and $\alpha_1 = 4$ in (1.17). This gives $\beta_1 = -8$, and specifies the second-order term as

$$w_1 = 4 \left(r^2 - 1\right) - 8r^2 \log r . \tag{1.19}$$

This expression reproduces that obtained in (1.8) from the perturbation of the exact solution.

In Problem 9 below we elaborate on why it is impossible to match to an outer solution u_0 that does not satisfy $u_0(0) = 0$. In addition, we further remark that point constraints are possible with the Biharmomic operator, since the free-space Green's function has singularity $\mathcal{O}\left(|\mathbf{x} - \mathbf{x}_0|^2 \log |\mathbf{x} - \mathbf{x}_0|\right)$ as $\mathbf{x} \to \mathbf{x}_0$. However, with a point constraint we will not have C^2 smoothness.

2 A Biharmonic Eigenvalue Problem

Problem 2: *Consider the following Biharmonic eigenvalue problem in a two-dimensional bounded domain Ω containing a small circular hole Ω_ε of radius ε centered at $\mathbf{x}_0 \in \Omega$,*

$$\triangle^2 u - \lambda u = 0 , \quad \mathbf{x} \in \Omega \backslash \Omega_\varepsilon ; \quad u = \partial_n u = 0 , \quad \mathbf{x} \in \partial\Omega ; \quad u = \partial_n u = 0 , \quad \mathbf{x} \in \partial\Omega_\varepsilon . \tag{2.1}$$

Let $\lambda_{0\varepsilon}$ denote the first positive eigenvalue of this problem. Let λ_0 be the first eigenvalue of the unperturbed problem with no hole, with corresponding eigenfunction $u_0(\mathbf{x})$. Assume that $u_0(\mathbf{x}_0) \neq 0$. By using a matched asymptotic expansion argument, show that $\lambda_{0\varepsilon}$ does not approach λ_0 as $\varepsilon \to 0$, in contrast to that for Laplacian eigenvalue problems in perforated domains. Instead, show that $\lambda_{0\varepsilon} \to \lambda_0^\star$ as $\varepsilon \to 0$, where $\lambda_0^\star$ is the first eigenvalue of the following problem with a point constraint:

$$\triangle^2 u^\star - \lambda^\star u^\star = 0 , \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_0\} ; \quad u^\star = \partial_n u^\star = 0 , \quad \mathbf{x} \in \partial\Omega ; \quad u^\star(\mathbf{x}_0) = 0 . \tag{2.2}$$

Finally, calculate the asymptotic behavior of the difference $\lambda_{0\varepsilon} - \lambda_0^\star$ *as* $\varepsilon \to 0$ *using a matched asymptotic analysis.*

Solution: Let $\lambda_{0\varepsilon}$ and $u_{0\varepsilon}(\mathbf{x})$ be the principal eigenvalue of the Biharmonic eigenvalue problem with a hole, given by (2.1) with normalization condition $\int_{\Omega\setminus\Omega_\varepsilon} u_{0\varepsilon}^2 \, d\mathbf{x} = 1$. Next, let λ_0 and $u_0(\mathbf{x})$ be the first eigenpair of the unperturbed problem with no hole

$$\triangle^2 u - \lambda u = 0\,, \quad \mathbf{x} \in \Omega\,; \quad u = \partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{2.3}$$

with normalization condition $\int_\Omega u^2 \, d\mathbf{x} = 1$.

We now show that $\lambda_{0\varepsilon}$ does not tend to λ_0 as $\varepsilon \to 0$. To show this, suppose to the contrary that for some $\sigma \ll 1$ we have

$$\lambda_{0\varepsilon} = \lambda_0 + \sigma\lambda_1 + \cdots\,. \tag{2.4}$$

In the outer region we expand the outer eigenfunction as

$$u_\varepsilon(\mathbf{x}) = u_0(\mathbf{x}) + \sigma u_1(\mathbf{x}) + \cdots\,. \tag{2.5}$$

Now at $\mathbf{x} = \mathbf{x}_0$, we assume that $u_0(\mathbf{x}_0) \neq 0$.

In the inner region we introduce the new variables $\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_0)$ and $v_\varepsilon(\mathbf{y}) = u_\varepsilon(\mathbf{x}_0 + \varepsilon\mathbf{y})$. Then, for some gauge function μ, we put

$$v_\varepsilon(\mathbf{y}) = \mu v_0(\rho)\,, \qquad \rho \equiv |\mathbf{y}|\,. \tag{2.6}$$

Upon substituting (2.6) into (2.1), we obtain that v_0 satisfies

$$\triangle^2 v_0 = 0\,, \quad \rho = |\mathbf{y}| \geq 1\,; \qquad v_0(1) = v_{0\rho}(1) = 0\,. \tag{2.7}$$

The general solution of this problem has the form

$$v_0 = a\rho^2 + b\rho^2 \log\rho + c\log\rho + d\,, \quad \rho \geq 1\,. \tag{2.8}$$

The matching condition is that the outer solution as $\mathbf{x} \to \mathbf{x}_0$ must agree with the inner expansion as $\rho = |\mathbf{y}| \to \infty$. Therefore,

$$u_0(\mathbf{x}_0) + \cdots + \sigma u_1 \sim \mu v_0(\rho) + \cdots\,. \tag{2.9}$$

The only possibility for matching is that $a = b = 0$, and that $c = u_0(\mathbf{x}_0)$ with $\mu = -1/\log\varepsilon$. However, this choice leaves only one free parameter d to satisfy the two boundary conditions $v_0(1) = v_{0\rho}(1) = 0$, which is impossible.

Therefore, we conclude that if we assume that the perturbed eigenfunction is close to the unperturbed eigenfunction with no hole in the outer region, then asymptotic matching is impossible. This suggests that this assumption must be modified, and that the limiting problem as $\varepsilon \to 0$ is not the problem with no hole.

Instead, we let $\lambda_0^\star$ and $u_0^\star(\mathbf{x})$ be the principal eigenpair of the Biharmonic eigenvalue problem with a point constraint, given by (2.3). In other words, we claim that the limiting problem $\varepsilon \to 0$ corresponds to the eigenvalue problem (2.2) **with point constraint**. Point constraints are compatible with Biharmonic problems, but not with Laplace's equation.

We then look for an eigenvalue of (2.1) close to $\lambda_0^\star$. For some gauge function $\sigma \ll 1$, we expand

$$\lambda_{0\varepsilon} = \lambda_0^\star + \sigma\lambda_1 + \cdots \,. \tag{2.10}$$

In the outer region, we expand the eigenfunction as

$$u_{0\varepsilon}(\mathbf{x}) = u_0^\star(\mathbf{x}) + \sigma u_1(\mathbf{x}) + \cdots \,. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.1), we obtain that λ_1 and $u_1(\mathbf{x})$ satisfy

$$\triangle^2 u_1 - \lambda_0^\star u_1 = \lambda_1 u_0^\star \,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_0\} \,, \tag{2.12 a}$$

$$u_1 = \partial_n u_1 = 0 \,, \quad \mathbf{x} \in \partial\Omega \,; \quad \int_\Omega u_0^\star u_1 \, d\mathbf{x} = 0 \,, \tag{2.12 b}$$

$$u_1 \quad \text{singular as } \mathbf{x} \to \mathbf{x}_0 \,. \tag{2.12 c}$$

Next, we must derive a singularity condition for u_1 as $\mathbf{x} \to \mathbf{x}_0$.

In the inner region, we introduce the new variables

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_0) \,, \qquad v(\mathbf{y}) = u(\mathbf{x}_0 + \varepsilon\mathbf{y}) \,. \tag{2.13}$$

In terms of the gauge function $\mu \ll 1$, we then expand

$$v_\varepsilon(\mathbf{y}) = \mu v_0(\mathbf{y}) \,, \quad \rho = |\mathbf{y}| \,. \tag{2.14}$$

Since $u_0(\mathbf{x}_0) = 0$, the matching condition is that the outer expansion of the eigenfunction as $\mathbf{x} \to \mathbf{x}_0$ must agree with the far-field form of the inner expansion as $\mathbf{y} \to \infty$,

$$\nabla u_0^\star \cdot (\mathbf{x} - \mathbf{x}_0) + \cdots + \sigma u_1 \sim \mu v_0(\mathbf{y}) + \cdots \,. \tag{2.15}$$

Here we have defined

$$\nabla u_0^\star \equiv \nabla u_0^\star(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_0} \,. \tag{2.16}$$

The problem for v_0 is

$$\triangle^2 v_0 = 0 \,, \quad \rho = |\mathbf{y}| \geq 1 \,; \qquad v_0(1) = v_{0\rho}(1) = 0 \,. \tag{2.17}$$

For any vector $\mathbf{a}$, there is a solution to (2.17) of the form

$$v_0 = \mathbf{A} \cdot \mathbf{e}_\theta v_c(\rho) \,, \tag{2.18 a}$$

where $\mathbf{e}_\theta \equiv (\cos\theta, \sin\theta)$ and $v_c(\rho)$ is given by

$$v_c = \rho \log \rho - \rho \log \left[e^{1/2} \right] + \frac{1}{2\rho} \,. \tag{2.18 b}$$

Notice that this is the Stokes solution given in equation (4.21) of the workshop notes.

We then write the far-field expansion of the inner solution in terms of the outer variables as

$$\mu v_0(\mathbf{y}) \sim \varepsilon^{-1} \mu \mathbf{A} \cdot \mathbf{e}_\theta |\mathbf{x} - \mathbf{x}_0| \left[\log |\mathbf{x} - \mathbf{x}_0| - \log \left(\varepsilon e^{1/2} \right) \right] \,. \tag{2.19}$$

This far-field expression suggests that we define μ and ν by

$$\nu = -\frac{1}{\log\left[\varepsilon e^{1/2}\right]}\,, \qquad \mu = \varepsilon\nu \tag{2.20}$$

Then, the matching condition (2.15) becomes

$$\nabla u_0^\star \cdot (\mathbf{x}-\mathbf{x}_0) + \cdots + \sigma u_1 \sim \mathbf{A}\cdot\mathbf{e}_\theta|\mathbf{x}-\mathbf{x}_0| + \mathbf{A}\cdot\mathbf{e}_\theta \nu|\mathbf{x}-\mathbf{x}_0|\log|\mathbf{x}-\mathbf{x}_0| + \cdots\,. \tag{2.21}$$

Therefore, we conclude that

$$\mathbf{A} = \nabla u_0^\star\,, \qquad \sigma = \nu \tag{2.22}$$

The matching condition (2.21) shows that the solution u_1 to (2.12) must have the singularity behavior

$$u_1 \sim \nabla u_0^\star \cdot (\mathbf{x}-\mathbf{x}_0)\log|\mathbf{x}-\mathbf{x}_0|\,, \quad \text{as} \quad \mathbf{x}\to\mathbf{x}_0\,. \tag{2.23}$$

Finally, we apply the divergence theorem to (2.12) over Ω_0, where $\Omega_0 \equiv \Omega\backslash\Omega_\gamma$, and Ω_γ is a small disk of radius $\gamma \ll 1$, centered at $\mathbf{x}_0$. In this way, we get

$$\lambda_1 = 4\pi|\nabla u_0^\star|^2\,, \qquad \sigma = \nu\,. \tag{2.24}$$

In summary, the principal eigenvalue of (2.1) has the two-term asymptotic expansion

$$\lambda_{0\varepsilon} \sim \lambda_0^\star + 4\pi\nu|\nabla u_0^\star|^2 + \cdots\,, \qquad \nu = -\frac{1}{\log\left[\varepsilon e^{1/2}\right]}\,. \tag{2.25}$$

Here $u_0^\star$ and $\lambda_0^\star$ are the principal eigenpair of the problem (2.2) with point constraint.