MATH 551 HW # 2

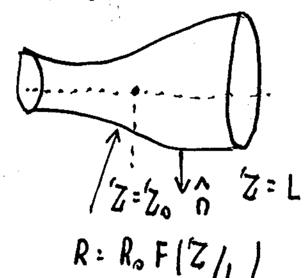
PROBLEM 1 IN A CYLINDER OF REVOLUTION $0 < Z < L$ WITH $0 < R < R_0 F(Z/L)$ CONSIDER THE AXIALLY SYMMETRY GREEN'S FUNCTION SATISFYING

$$G_{RR} + \frac{1}{R} G_R + G_{ZZ} = \frac{\delta(Z - Z_0)\,\delta(R)}{2\pi R} \quad \text{IN } 0 < R < R_0 F(Z/L)$$

$0 < Z < L$. ASSUME $F > 0$ SMOOTH.

ASSUME THAT $\nabla G \cdot \hat{n} = 0$ ON $R = R_0 F(Z/L)$; G BOUNDED AS $R \to 0$ FOR $Z \neq Z_0$

AND $G = 0$ ON $Z = 0, L$.

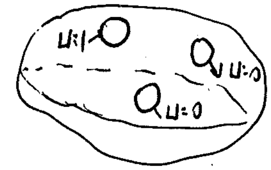


ASSUME THAT Z_0 SATISFIES $0 < Z_0 < L$,

AND FOR CONVENIENCE $F(Z_0/L) = 1$.

FIND AN ASYMPTOTIC APPROXIMATION FOR G IN THE LONG THIN DOMAIN LIMIT $R_0/L \ll 1$. IN YOUR ANALYSIS FIND THE OUTER SOLUTION AWAY FROM THE DIRAC SOURCE AND THE INNER SOLUTION (HINT: BESSEL FUNCTIONS ARE NEEDED HERE FOR INNER SOLUTION). A LEADING-ORDER THEORY IS SUFFICIENT.

PROBLEM 2 IN A 3-D DOMAIN, THE SPLITTING PROBABILITY $u(\underline{x})$ IS DEFINED AS THE PROBABILITY OF REACHING A SPECIFIC TARGET TRAP Ω_{ε_1} FROM AN INITIAL STARTING POINT $\underline{x} \in \Omega \setminus \bigcup_{j=1}^{N} \Omega_{\varepsilon_j}$, BEFORE REACHING ANY OF THE OTHER REMAINING TRAPS Ω_{ε_j} FOR $j = 2, \ldots, N$. IT IS WELL KNOWN THAT $u(\underline{x})$ SATISFIES



$$\Delta u = 0, \quad \text{FOR } \underline{x} \in \Omega \setminus \bigcup_{j=1}^{N} \Omega_{\varepsilon_j} \ ; \quad \partial_n u = 0 \text{ ON } \partial\Omega$$

$$u = 1 \quad \text{FOR } \underline{x} \in \partial\Omega_{\varepsilon_1} \ ; \qquad u = 0 \quad \text{FOR } \underline{x} \in \partial\Omega_{\varepsilon_j},\ j = 2, \ldots, N.$$

ASSUME FOR SIMPLICITY THAT EACH TRAP Ω_{ε_j} IS A SPHERE OF RADIUS εa_j CENTERED AT $\underline{x} = \underline{x}_j$ WITH $a_j > 0$ AND $a_j = O(1)$, $\varepsilon \ll 1$.

i.e. $\Omega_{\varepsilon_j} \equiv \{ |\underline{x}| \ \big|\ |\underline{x} - \underline{x}_j| \leq a_j \varepsilon \}$.

(i) SHOW THAT A TWO-TERM APPROXIMATION FOR $u(\underline{x})$ IN THE OUTER REGION HAS THE FORM FOR $\varepsilon \ll 1$ THAT

$$u(\underline{x}) \sim \frac{C_1}{N\bar{C}} + 4\pi\varepsilon C_1\left[G(\underline{x};\underline{x}_1) - \frac{1}{N\bar{C}}\sum_{j=1}^{N} C_j G(\underline{x};\underline{x}_j)\right] + \varepsilon \chi_1 + O(\varepsilon^2). \quad (*)$$

HERE $\bar{C} = \frac{1}{N}(C_1 + \cdots + C_N)$ WHERE $C_j = a_j$ IS THE CAPACITANCE OF THE j^{th} TRAP, AND $G(\underline{x};\underline{\xi})$ IS THE NEUMANN GREEN'S FUNCTION SATISFYING

$$\Delta G = \frac{1}{|\Omega|} - \delta(\underline{x}-\underline{\xi}), \quad \underline{x} \in \Omega; \quad \partial_n G = 0 \text{ ON } \partial\Omega$$

$$G \sim \frac{1}{4\pi|\underline{x}-\underline{\xi}|} + R(\underline{\xi}) + o(1) \text{ AS } \underline{x} \to \underline{\xi}; \quad \int_\Omega G(\underline{x};\underline{\xi})\, d\underline{x} = 0.$$

HERE $R(\underline{\xi})$ IS THE "<u>REGULAR</u>" PART OF $G(\underline{x};\underline{\xi})$ AT $\underline{x} = \underline{\xi}$.

(ii) DETERMINE EXPLICITLY THE CONSTANT χ_1 IN $(*)$. (<u>HINT</u>: YOU WILL NEED TO USE THE DIVERGENCE THEOREM ON THE PROBLEM AT $O(\varepsilon^2)$).

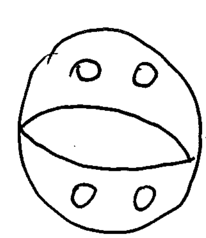
(iii) INTERPRET THE LEADING ORDER TERM IN THE EXPANSION $(*)$. IS IT MORE OR LESS LIKELY TO REACH A SMALL TARGET SPHERE OR A LARGE TARGET SPHERE? EXPLAIN QUALITATIVELY WHAT INFORMATION ON THE SPLITTING PROBABILITY IS CONTAINED IN THE $O(\varepsilon)$ TERM IN $(*)$.

<u>PROBLEM 3</u> LET $\Omega = \{\underline{x} \mid |\underline{x}| \le 1\}$ BE THE UNIT SPHERE IN 3-D AND INTRODUCE THE USUAL SPHERICAL COORDINATES $\underline{x} = (r\cos\phi\sin\varphi, r\sin\phi\sin\varphi, r\cos\varphi)^T$ ON $0 \le r \le 1$ WHERE $0 \le \varphi \le \pi$ IS POLAR ANGLE AND $0 \le \phi \le 2\pi$. SUPPOSE THAT Ω CONTAINS FOUR SMALL SPHERICALLY-SHAPED NON-OVERLAPPING HOLES Ω_{ε_j} OF A COMMON RADIUS $\varepsilon \ll 1$ CENTERED AT SOME $\underline{x}_j$ FOR $j = 1,..,4$. FOR AN ARBITRARY FUNCTION $f(\varphi)$ AND CONSTANTS α_j FOR $j = 1,..,4$ CONSIDER THE PDE

$$\Delta u = 0, \quad \underline{x} \in \Omega \setminus \bigcup_{j=1}^{4} \Omega_{\varepsilon_j}$$

$$u = f(\varphi) \text{ ON } |\underline{x}| = 1$$

$$u = \alpha_j \text{ ON } \partial\Omega_{\varepsilon_j}, \; j = 1,..,4$$



(i) DERIVE A LEADING-ORDER ASYMPTOTIC EXPANSION FOR THE SOLUTION TO THIS PROBLEM FOR $\varepsilon \ll 1$.

(NOTE: YOUR SOLUTION WILL INVOLVE A "PARTICULAR" SOLUTION AND THE GREEN'S FUNCTION FOR THE LAPLACIAN IN A SPHERE).

(ii) LET $f(\varphi) = \sin^2\varphi$ AND $\alpha_j = 1$ FOR $j = 1, \dots 4$.

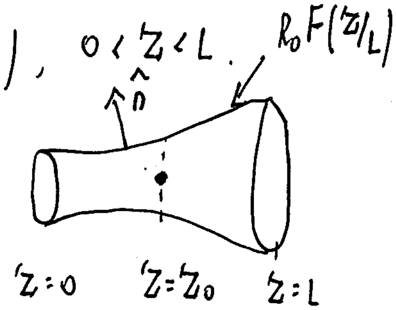
SUPPOSE THAT THE CENTRES OF THE HOLES OF RADIUS ε ARE PLACED SYMMETRICALLY AT $\left(0, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$ IN PLANE $X = 0$.

DERIVE A FORMULA FOR $u(\underline{0})$ VERSUS ε AND PLOT IT NUMERICALLY.

(HINT: THE PARTICULAR SOLUTION IS EASY TO FIND IN TERMS OF LEGENDRE POLYNOMIALS. THE GREEN'S FUNCTION IN PART (i) CAN BE FOUND BY METHOD OF IMAGES).

PROBLEM 1

$$G_{RR} + \frac{1}{R} G_R + G_{ZZ} = \frac{\delta(Z - Z_0)\,\delta(R)}{2\pi R} \quad \text{IN} \quad 0 < R < R_0 F(Z/L), \; 0 < Z < L.$$



WITH $\quad G = 0$ ON $Z = 0, L$; $\quad \nabla G \cdot \hat{n} = 0$ ON $R = R_0 F(Z/L)$

G BOUNDED AS $R \to 0,\; Z \neq Z_0$

ASSUME THAT $0 < Z_0 < L$ AND $F(Z_0/L) = 1$. FIND G WHEN $R_0/L \ll 1$.

SOLUTION WE RESCALE: LET $r = R/R_0$, $z = Z/L$.

THEN USING $\delta(r R_0) = \frac{1}{R_0}\delta(r)$, $\quad \delta(L(z - z_0)) = \frac{1}{L}\delta(z - z_0)$ WITH $z_0 = Z_0/L$,

WE OBTAIN:

$$\frac{1}{R_0^2}\left(G_{rr} + \frac{1}{r} G_r\right) + \frac{1}{L^2} G_{zz} = \frac{\delta(z - z_0)\,\delta(r)}{2\pi (r R_0)(L R_0)}.$$

MULTIPLYING BY R_0^2:

$$G_{rr} + \frac{1}{r} G_r + \frac{R_0^2}{L^2} G_{zz} = \frac{\delta(z - z_0)\,\delta(r)}{2\pi\, r\, L}.$$

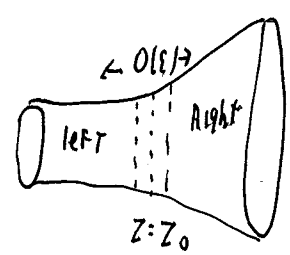
NOW $\quad \nabla G \cdot \hat{n} = \left(\frac{1}{R_0} G_r, \frac{1}{L} G_z\right) \cdot \left(1, -\frac{R_0}{L} F'(z)\right) \Big/ \sqrt{1 + \frac{R_0^2}{L^2} F'^2}$.

SETTING $\nabla G \cdot \hat{n} = 0$ ON $r = F(z)$ WE OBTAIN

$$G_r = \frac{R_0^2}{L^2} F'(z)\, G_z \quad \text{ON} \quad r = F(z).$$

NOW DEFINE $\varepsilon = R_0/L \ll 1$. OUR PROBLEM READS THAT, WITH $0 < z_0 < 1$,

$$(*) \quad \left\{ \begin{array}{ll} G_{rr} + \frac{1}{r} G_r + \varepsilon^2 G_{zz} = \frac{\delta(z - z_0)\,\delta(r)}{2\pi r L} & \text{IN } 0 < z < 1,\; 0 < r < F(z) \\ G_r = \varepsilon^2 F'(z)\, G_z & \text{ON } r = F(z) \\ G = 0 & \text{ON } z = 0, 1. \end{array} \right.$$



WE WILL CONSTRUCT A SOLUTION TO $(*)$ IN 3 REGIONS.

- RIGHT OUTER $\quad z_0 + O(\varepsilon) \ll z \le 1 \longrightarrow G = U$
- LEFT OUTER $\quad 0 \le z \ll z_0 - O(\varepsilon) \longrightarrow G = V$
- INNER REGION $\quad z - z_0 = O(\varepsilon)$.

WE WILL FIRST DO THE <u>INNER REGION</u>:

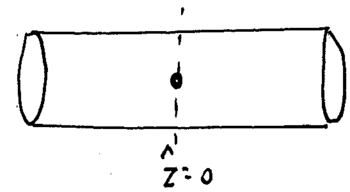
LET $\hat{z} = \varepsilon^{-1}(z - z_0)$. THEN $\delta(z - z_0) = \delta(\varepsilon \hat{z}) = \frac{1}{\varepsilon}\delta(\hat{z})$.

BUT $\varepsilon = R_0/L$ SO THAT $\dfrac{\delta(z-z_0)\,\delta(r)}{2\pi r L} = \dfrac{\delta(\hat{z})\,\delta(r)}{2\pi \varepsilon r L} = \dfrac{\delta(\hat{z})\,\delta(r)}{2\pi r R_0}$.

SINCE $G_z = \varepsilon^{-1} G_{\hat{z}}$ AND $F(z) \cong F(z_0 + \varepsilon \hat{z}) = F(z_0) + O(\varepsilon) = 1 + O(\varepsilon)$,

OUR INNER PROBLEM READS TO LEADING ORDER:

$$(1)\quad \begin{cases} G_{rr} + \frac{1}{r} G_r + G_{\hat{z}\hat{z}} = \dfrac{\delta(\hat{z})\,\delta(r)}{2\pi r R_0} & \text{IN } 0 \le r \le 1,\ -\infty < \hat{z} < \infty \\ G_r = 0 \text{ ON } r = F(0) = 1 & \\ G \text{ BOUNDED AS } r \to 0 \text{ FOR } \hat{z} \neq 0 & \end{cases}$$



NOW IN THE r DIRECTION THE UNDERLYING STURM LIOUVILLE PROBLEM MUST BE FOUND. IGNORE DELTA FUNCTION, SEPARATE VARIABLES $G = \Phi(r) Z(\hat{z})$ AND WE GET

$$\Phi'' + \frac{1}{r}\Phi' + \lambda \Phi = 0 \quad \text{IN } 0 < r < 1$$

$$\Phi'(1) = 0,\quad \Phi(0) \text{ BOUNDED.}$$

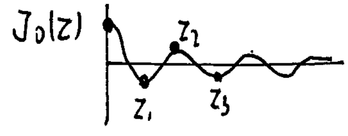
$$\longrightarrow \quad (2)\quad \begin{cases} (r\Phi')' + \underset{\uparrow \text{ weight.}}{r}\lambda \Phi = 0 & \text{IN } 0 < r < 1 \\ \Phi'(1) = 0,\ \Phi(0) \text{ BOUNDED} & \end{cases}$$

THIS IS AN SL EIGENVALUE PROBLEM WITH A WEIGHT FUNCTION $w = r$.

NOTICE THAT (2) IS THE SL PROBLEM. WE GET

$\lambda = 0 \rightarrow \Phi_0 = 1$

$\lambda > 0 \rightarrow \Phi_n(r) = J_0(\sqrt{\lambda_n}\, r)$ WHERE $J_0'(\sqrt{\lambda_n}) = 0$ FOR $n = 1, 2, \ldots$



$J_0'(z_n) = 0,\ n = 1, 2, \ldots$

SO $\lambda_n = z_n^2$.

THIS SUGGESTS THAT WE LOOK FOR A SOLUTION TO (1) IN THE FORM

$$(3)\quad G = A_0(z)\,\Phi_0(r) + \sum_{n=1}^{\infty} A_n(z)\,\Phi_n(r)$$

WHERE $\Phi_0(r) \equiv 1$

$\Phi_n(r) = J_0(\sqrt{\lambda_n}\, r)$

FOR $n = 1, 2, \ldots$

NOW BY ORTHOGONALITY WRT WEIGHT FUNCTION $w = r$ WE HAVE

$$\int_0^1 r\,\Phi_0\,\Phi_n\,dr = \int_0^1 r\,\Phi_n\,dr = 0 \quad \text{FOR} \quad n = 1, 2, \ldots$$

$$\int_0^1 r\,\Phi_n\,\Phi_m\,dr = 0 \quad \text{IF} \quad n \neq m \qquad n, m \in \{1, 2, \ldots$$

IN THIS WAY BY USING ORTHOGONALITY IN (3) WE HAVE

$$\int_0^1 r\,G\,dr = \int_0^1 A_0\,r\,dr \longrightarrow A_0 = 2\int_0^1 r\,G\,dr.$$

$$\int_0^1 r\,G\,\Phi_m\,dr = \sum_{n=1}^{\infty} A_n \int_0^1 r\,\Phi_n\,\Phi_m\,dr \longrightarrow \int_0^1 r\,G\,\Phi_m\,dr = A_m \int_0^1 r\,\Phi_m^2\,dr.$$

HENCE $\quad A_0 = 2\int_0^1 r\,G\,dr, \quad A_m = \dfrac{\int_0^1 r\,G\,\Phi_m\,dr}{\int_0^1 r\,\Phi_m^2\,dr}.$

NOW TO FIND $A_0(z)$ WE WILL INTEGRATE PDE IN (1) OVER THE CROSS-SECTION AFTER MULTIPLYING BY r

$$\int_0^1 r\left[G_{rr} + \frac{1}{r}G_r + G_{\hat{z}\hat{z}}\right]dr = \int_0^1 \frac{\delta(\hat{z})\,\delta(r)}{2\pi R_0}\,dr = \frac{\delta(\hat{z})}{2\pi R_0} \qquad (*)$$

SO $$\int_0^1 r\left[\frac{1}{r}(r\,G_r)\right]dr + \frac{d^2}{dz^2}\int_0^1 r\,G\,dr = \frac{\delta(\hat{z})}{2\pi R_0}$$

$$r\,G_r\Big|_0^1 + \frac{d^2}{dz^2}\left(\frac{A_0}{2}\right) = \frac{\delta(\hat{z})}{2\pi R_0}.$$

THEN THE BC VANISHES AND WE OBTAIN

$$(4) \qquad \frac{d^2 A_0}{dz^2} = \frac{1}{\pi R_0}\,\delta(\hat{z}), \quad -\infty < \hat{z} < \infty \Longrightarrow A_0(z) = a_0 + \begin{cases} b_0\,\hat{z}, & \hat{z} < 0 \\ b_1\,\hat{z}, & \hat{z} > 0 \end{cases}$$

WITH $b_1 - b_0 = 1/\pi R_0$

<u>REMARK</u> IN CALCULATING (*) WE MUST INTERPRET $\int_0^1 \delta(r)\,dr = 1$ SINCE IF WE WERE TO INTEGRATE IN CARTESIAN COORDINATES WE OBTAIN THAT $r = 0$ IS AT CENTER OF A BALL OF RADIUS σ, i.e. $|\underline{x}| \leq \sigma$.

NOW TO CALCULATE $A_n(z)$ WE WRITE:

$$\int_0^1 r\,\Phi_n\left[\frac{1}{r}(r\,G_r)_r + G_{\hat{z}\hat{z}}\right]dr = \int_0^1 r\,\Phi_n(r)\,\frac{\delta(\hat{z})\,\delta(r)}{2\pi r\,R_0}\,dr = \frac{\Phi_n(0)}{2\pi R_0}\,\delta(\hat{z})$$

INTEGRATING BY PARTS TWICE:

$$\frac{d^2}{dz^2}\left(\int_0^1 r\,\Phi_n G\, dr\right) + \underbrace{\left(r G_r \Phi_n - G(r\Phi_n')\right)\Big|_0^1}_{=0} + \int_0^1 G\,(r\Phi_n')'\, dr = \frac{\Phi_n(0)\,\delta(\hat{z})}{2\pi R_0}$$

THUS USING $(r\Phi_n')' = -\lambda_n r \Phi_n$ WE OBTAIN

$$\frac{d^2}{dz^2}\left(\int_0^1 r\,\Phi_n G\, dr\right) - \lambda_n\left(\int_0^1 r\,\Phi_n G\, dr\right) = \frac{\Phi_n(0)\,\delta(\hat{z})}{2\pi R_0}.$$

BUT $\quad A_n = \int_0^1 r\, G\, \Phi_n\, dr \Big/ \int_0^1 r\,\Phi_n^2\, dr$

$$\left.\begin{aligned} &\text{HENCE} \quad A_n'' - \lambda_n A_n = \mathcal{H}_n\, \delta(\hat{z}), \quad \text{ON } -\infty < \hat{z} < \infty \\ &\text{WHERE} \quad \mathcal{H}_n = \frac{\Phi_n(0)}{2\pi R_0 \int_0^1 r\,\Phi_n^2\, dr}. \end{aligned}\right\} \quad (5)$$

NOW $\Phi_n(0) = J_0(0) = 1$ SO $\mathcal{H}_n = \dfrac{1}{2\pi R_0 \int_0^1 r\,\Phi_n^2\, dr}$.

WE OBTAIN THAT $A_n'(0^+) - A_n'(0^-) = \mathcal{H}_n$, AND MAKE A_n CONTINUOUS ACROSS $\hat{z} = 0$ AND DECAY AS $\hat{z} \to \infty$.

WE GET $$A_n(\hat{z}) = -\frac{\mathcal{H}_n e^{-\lambda_n |\hat{z}|}}{2\lambda_n} = -\frac{e^{-\lambda_n|\hat{z}|}}{4\pi R_0 \lambda_n \left(\int_0^1 r\,\Phi_n^2\, dr\right)}. \quad (6)$$

IN SUMMARY, THE INNER SOLUTION IS FROM (3) - (6) THAT

$$G = a_0 + \begin{cases} b_0 \hat{z}, & \text{IF } \hat{z} < 0 \\ b_1 \hat{z}, & \text{IF } \hat{z} > 0 \end{cases} + \sum_{n=1}^{\infty}\left(\frac{-e^{-\lambda_n|\hat{z}|}}{4\pi R_0 \lambda_n \int_0^1 r\,\Phi_n^2\, dr}\right) J_0(\sqrt{\lambda_n}\, r)$$

WHERE $b_1 - b_0 = \dfrac{1}{\pi R_0}$ AND a_0 IS UNKNOWN. FOR $\hat{z} \to \infty$, WE WRITE $\hat{z} = \dfrac{z - z_0}{\varepsilon}$ TO GET

$$G \sim \begin{cases} \dfrac{b_0}{\varepsilon}(z - z_0) + a_0 + \text{EXP. SMALL} & \text{AS } \hat{z} \to -\infty \\ \dfrac{b_1}{\varepsilon}(z - z_0) + a_0 + \text{EXP. SMALL} & \text{AS } \hat{z} \to +\infty. \end{cases} \quad (7)$$

WHICH WE WILL USE TO MATCH TO OUTER SOLUTION. THIS GIVES THE FLUX CONDITIONS $\quad G_z \to \dfrac{b_1}{\varepsilon}$ AS $z \to z_0^+$ AND $G_z \to \dfrac{b_0}{\varepsilon}$ AS $z \to z_0^-$.

NOW IN VIEW OF (X) AND THE MATCHING CONDITION $G_Z \to b_1/\varepsilon$ AS $Z \to Z_0^+$,

WE WRITE $$G = \frac{\bar{U}}{\varepsilon}$$

SO THAT $$\bar{U}_{rr} + \frac{1}{r}\bar{U}_r + \varepsilon^2 \bar{U}_{zz} = 0 \quad \text{IN} \quad 0 < r < F(Z), \; Z_0 < Z < 1$$

$$\text{WITH} \quad \bar{U}_r = \varepsilon^2 F'(Z)\bar{U}_Z \text{ ON } r = F(Z), \quad \bar{U} \text{ BOUNDED AS } r \to 0.$$

$$\bar{U} = 0 \text{ ON } Z = 1, \quad \bar{U}_Z \to b_1 \text{ AS } Z \to Z_0^+.$$

WE EXPAND $\bar{U} = \bar{U}_0 + \varepsilon^2 \bar{U}_1 + \cdots$ AND AS IN OUR PREVIOUS 1-D HEAT CONDUCTION ANALYSIS WE GET $\bar{U}_0 = \bar{U}_0(Z)$ AND

$$\bar{U}_{1rr} + \frac{1}{r}\bar{U}_{1r} = -\bar{U}_{0zz} \longrightarrow (r\bar{U}_{1r})_r = -r\bar{U}_{0zz}, \quad \bar{U}_{1r} = F'(Z)\bar{U}_{0Z} \text{ ON } r = F(Z),$$

WE INTEGRATE TO GET: $$\int_0^{F(Z)} (r\bar{U}_{1r})_r \, dr = -\frac{F^2}{2}\bar{U}_{0zz}.$$

$$\longrightarrow \quad r\bar{U}_{1r}\big|_{r=F} = -\frac{F^2}{2}\bar{U}_{0zz} \longrightarrow FF'\bar{U}_{0Z} = -\frac{F^2}{2}\bar{U}_{0zz}.$$

WE CONCLUDE THAT $$[(F(Z))^2 \bar{U}_{0Z}]_Z = 0 \quad \text{IN} \quad Z_0 < Z < 1.$$

THEN $(F(Z))^2 \bar{U}_{0Z} = B$, $\forall Z$ SO EVALUATING AT $Z = Z_0$ WHERE $F(Z_0) = 1$ YIELDS $$\bar{U}_{0Z}\big|_{Z=Z_0} = B \longrightarrow B = b_1.$$

HENCE $$\bar{U}_{0Z} = \frac{b_1}{[F(Z)]^2}$$

WITH $\bar{U}_0(1) = 0$, WE GET $$\bar{U}_0 = -\int_Z^1 \frac{b_1}{(F(\Lambda))^2}\, d\Lambda \longrightarrow G = -\frac{b_1}{\varepsilon}\int_Z^1 \frac{1}{(F(\Lambda))^2}\, d\Lambda \quad \text{(RIGHT)}$$

NOW IN THE LEFT REGION WE DO AN IDENTICAL ANALYSIS WITH $G_Z \to b_0/\varepsilon$ AS $Z \to Z_0^-$. IN THIS WAY WE OBTAIN

$$G = \begin{cases} \dfrac{b_0}{\varepsilon}\displaystyle\int_0^Z \frac{1}{(F(\Lambda))^2}\, d\Lambda, & \text{FOR } 0 \le Z < Z_0 - O(\varepsilon) \\[2ex] -\dfrac{b_1}{\varepsilon}\displaystyle\int_Z^1 \frac{1}{(F(\Lambda))^2}\, d\Lambda, & \text{FOR } Z_0 + O(\varepsilon) < Z \le 1. \end{cases} \tag{8}$$

WITH THE JUMP CONDITION THAT

$$b_1 - b_0 = \frac{1}{\pi R_0}, \tag{9}$$

AS OBTAINED FROM INNER ANALYSIS.

NOW WE HAVE THE FINAL MATCHING CONDITION THAT

$$G \to q_0 \quad \text{AS} \quad z \to z_0^+ \quad \text{AND AS} \quad z \to z_0^- .$$

IN THIS WAY FROM (8) WE OBTAIN THAT

$$b_0 \int_0^{z_0} \frac{1}{F^2}\, d\lambda = -b_1 \int_{z_0}^{1} \frac{1}{F^2}\, d\lambda$$

$$b_1 - b_0 = \frac{1}{\pi R_0} .$$

THIS YIELDS THAT $$b_0 = \frac{-\int_{z_0}^{1} 1/F^2\, d\lambda}{\pi R_0 \left[\int_0^1 1/F^2\, d\lambda \right]} , \qquad b_1 = \frac{\int_0^{z_0} \frac{1}{F^2}\, d\lambda}{\pi R_0 \left[\int_0^1 1/F^2\, d\lambda \right]}$$

MOREOVER,

$$q_0 = \frac{b_0}{\varepsilon} \int_0^{z_0} \frac{1}{F^2}\, d\lambda = -\frac{1}{\pi R_0 \varepsilon} \frac{\left(\int_0^{z_0} 1/F^2\, d\lambda \right) \left(\int_{z_0}^{1} 1/F^2\, d\lambda \right)}{\int_0^1 1/F^2\, d\lambda} .$$

NOTICE THAT IF $z_0 = 1/2$ AND F IS EVEN ABOUT $z = 1/2$ THEN $b_1 = \frac{1}{2\pi R_0} = -b_0$

AS WE WOULD HAVE EXPECTED.

Math 551: Homework 1 Solutions

Problem 2: In a 3-D domain, the splitting probability $u(\mathbf{x})$ is defined as the probability of reaching a specific target trap Ω_{ε_1} from the initial source point $x \in \Omega \backslash \Omega_p$, before reaching any of the other surrounding traps Ω_{ε_j} for $j = 2, \ldots, N$. Then, it is well-known that $u(\mathbf{x})$ satisfies Laplace's equation

$$\Delta u = 0\,, \quad \mathbf{x} \in \Omega \backslash \Omega_p \equiv \cup_{j=1}^{N} \Omega_{\varepsilon_j}\,; \qquad \partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{0.1 a}$$

$$u = 1\,, \quad \mathbf{x} \in \partial\Omega_{\varepsilon_1}\,; \qquad u = 0\,, \quad \mathbf{x} \in \cup_{j=2}^{N} \partial\Omega_{\varepsilon_j}\,. \tag{0.1 b}$$

Assume for simplicity that each trap Ω_{ε_j} is a sphere of radius εa_j centered at $\mathbf{x}_j \in \Omega$, where $a_j > 0$.

(1) Show that a two-term expansion for $u(\mathbf{x})$ in the outer region has the form

$$u \sim \frac{C_1}{N\bar{C}} + 4\pi\varepsilon C_1 \left[G(\mathbf{x};\mathbf{x}_1) - \frac{1}{N\bar{C}} \sum_{j=1}^{N} C_j G(\mathbf{x};\mathbf{x}_j) \right] + \varepsilon \chi_1 + \mathcal{O}(\varepsilon^2)\,. \tag{0.2}$$

Here $\bar{C} = (C_1 + \ldots + C_N)/N$, where $C_j = a_j$ is the capacitance of the j^{th} trap, and $G(\mathbf{x};\mathbf{x}_0)$ is the Neumann Green's function satisfying

$$\Delta G = \frac{1}{|\Omega|} - \delta(\mathbf{x} - \mathbf{x}_0)\,, \quad \mathbf{x} \in \Omega\,; \qquad \partial_n G = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{0.3 a}$$

$$G(\mathbf{x};\mathbf{x}_0) \sim \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} + R(\mathbf{x}_0;\mathbf{x}_0) + o(1)\,, \quad \mathbf{x} \to \mathbf{x}_0\,; \qquad \int_\Omega G(\mathbf{x};\mathbf{x}_0)\, d\mathbf{x} = 0\,, \tag{0.3 b}$$

where $R(\mathbf{x}_0;\mathbf{x}_0)$ is called the regular part of $G(\mathbf{x};\mathbf{x}_0)$ at $\mathbf{x} = \mathbf{x}_0$.

(2) Determine explicitly the constant χ_1 in (0.2) (Remark: you will need to use the divergence theorem on the problem for the third term of order $O(\varepsilon^2)$ in the outer expansion).

(3) Interpret the leading-order term in the outer expansion. Is it more or less likely to reach a small target sphere or a large target sphere? What information (qualitatively) is contained in the second order term with regards to the probability of reaching the target site?

Solution:

(1) In the outer region, we expand u as

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots\,. \tag{0.4}$$

Here u_0 is an unknown constant, and u_k for $k = 1, 2$ satisfies

$$\Delta u_k = 0\,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}\,; \qquad \partial_n u_k = 0\,, \quad \mathbf{x} \in \partial\Omega\,, \tag{0.5}$$

with certain singularity conditions as $\mathbf{x} \to \mathbf{x}_j$ for $j = 1, \ldots, N$ determined upon matching to the inner solution.

In the inner region near the j^{th} trap, we expand the inner solution $w(\mathbf{y}) \equiv u(\mathbf{x}_j + \varepsilon\mathbf{y})$, with $\mathbf{y} \equiv \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j)$, as

$$w = w_0 + \varepsilon w_1 + \cdots\,. \tag{0.6}$$

Upon substituting (0.6) into (0.1 a) and (0.1 b), we obtain that w_0 and w_1 satisfy

$$\Delta_y w_0 = 0\,, \quad \mathbf{y} \notin \Omega_j\,; \qquad w_0 = \delta_{j1}\,, \quad \mathbf{y} \in \partial\Omega_j\,, \tag{0.7 a}$$

$$\Delta_y w_1 = 0\,, \quad \mathbf{y} \notin \Omega_j\,; \qquad w_1 = 0\,, \quad \mathbf{y} \in \partial\Omega_j\,. \tag{0.7 b}$$

Here $\Omega_j = \varepsilon^{-1}\Omega_{\varepsilon_j}$, and δ_{j1} is Kronecker's symbol. The far-field boundary conditions for w_0 and w_1 are determined by the matching condition as $\mathbf{x} \to \mathbf{x}_j$ between the the inner and outer expansions (0.6) and (0.4), respectively, written as

$$u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \sim w_0 + \varepsilon w_1 + \cdots\,. \tag{0.8}$$

The first matching condition is that $w_0 \sim u_0$ as $|\mathbf{y}| \to \infty$, where u_0 is an unknown constant. Then, the solution for w_0 in the j^{th} inner region is given by

$$w_0 = u_0 + (\delta_{j1} - u_0)\, w_c(\mathbf{y}) , \qquad w_c(\mathbf{y}) = C_j/|\mathbf{y}| , \quad C_j = a_j\, m \tag{0.9}$$

In this way we get

$$w_0 \sim u_0 + (\delta_{j1} - u_0)\, \frac{C_j}{|\mathbf{y}|} , \quad \text{as } \mathbf{y} \to \infty . \tag{0.10}$$

Here $C_j = a_j$ is the capacitance of the sphere Ω_j of radius a_j.

From (0.10) and (0.8), we conclude that u_1 satisfies (0.5) with singular behavior $u_1 \sim (\delta_{j1} - u_0)\, C_j/|\mathbf{x} - \mathbf{x}_j|$ as $\mathbf{x} \to \mathbf{x}_j$ for $j = 1, \ldots, N$. Therefore, in terms of the Dirac distribution, u_1 satisfies

$$\Delta u_1 = -4\pi \sum_{j=1}^{N} (\delta_{j1} - u_0)\, C_j \delta(\mathbf{x} - \mathbf{x}_j) , \quad \mathbf{x} \in \Omega ; \qquad \partial_n u_1 = 0 , \quad \mathbf{x} \in \partial\Omega . \tag{0.11}$$

The solvability condition for u_1, obtained by the divergence theorem, determines the unknown constant u_0 as

$$u_0 = \frac{C_1}{N\bar{C}} , \qquad \bar{C} \equiv \frac{1}{N}\, (C_1 + \cdots C_N) . \tag{0.12}$$

In terms of the Neumann Green's function of (0.3), and an unknown constant χ_1, the solution to (0.11) is

$$u_1 = 4\pi \sum_{i=1}^{N} (\delta_{i1} - u_0)\, C_i G(\mathbf{x}; \mathbf{x}_i) + \chi_1 , \qquad \chi_1 = \frac{1}{|\Omega|} \int_\Omega u_1\, dx . \tag{0.13}$$

At this stage we have the two-terms expansion

$$u \sim \frac{C_1}{N\bar{C}} + 4\pi\varepsilon C_1 \left[G(\mathbf{x}; \mathbf{x}_1) - \frac{1}{N\bar{C}} \sum_{j=1}^{N} C_j G(\mathbf{x}; \mathbf{x}_j) \right] + \varepsilon\chi_1 + \mathcal{O}(\varepsilon^2) , \tag{0.14}$$

where $\bar{C} = (C_1 + \ldots + C_N)/N$, $C_j = a_j$ is the capacitance of the j^{th} trap, and χ_1 is an unknown constant.

(2) Next, we determine χ_1. We expand u_1 as $\mathbf{x} \to \mathbf{x}_j$, and using the local behavior $G(\mathbf{x}; \mathbf{x}_i) \sim 1/(4\pi|\mathbf{x} - \mathbf{x}_i|) + R_{i,i}$ of G as $\mathbf{x} \to \mathbf{x}_i$ from (0.3 b), we obtain that

$$u_1 \sim \begin{cases} \frac{(1-u_0)C_1}{|\mathbf{x}-\mathbf{x}_1|} + A_1 + \chi_1 , & \text{as } \mathbf{x} \to \mathbf{x}_1 , \\ -\frac{u_0 C_j}{|\mathbf{x}-\mathbf{x}_j|} + A_j + \chi_1 , & \text{as } \mathbf{x} \to \mathbf{x}_j , \ j = 2, \ldots, N . \end{cases} \tag{0.15 a}$$

Here, the constants A_j for $j = 1, \ldots, N$ are defined by

$$A_1 = 4\pi C_1 R_{1,1} - 4\pi u_0 \left(C_1 R_{1,1} + \sum_{i=2}^{N} C_i G_{1,i} \right) ; \quad A_j = 4\pi C_1 G_{j,1} - 4\pi u_0 \left(C_j R_{j,j} + \sum_{\substack{j=1 \\ j\neq i}}^{N} C_i G_{j,i} \right) , \quad j = 2, \ldots, N . \tag{0.15 b}$$

Upon substituting (0.15) into the matching condition (0.8), we obtain that the solution w_1 to (0.7 b) must satisfy $w_1 \sim A_j + \chi_1$ as $|\mathbf{y}| \to \infty$. Thus, $w_1 = (A_j + \chi_1)(1 - w_c)$, where $w_c = C_j/|\mathbf{y}|$. Substituting this into the matching condition (0.8), we obtain that u_2 satisfies (0.5) with singularity behavior

$$u_2 \sim -\frac{C_j\, (A_j + \chi_1)}{|\mathbf{x} - \mathbf{x}_j|} , \quad \text{as } \mathbf{x} \to \mathbf{x}_j , \quad j = 1, \ldots, N . \tag{0.16}$$

Therefore, in terms of distributions, u_2 satisfies

$$\Delta u_2 = 4\pi \sum_{j=1}^{N} C_j\, (A_j + \chi_1)\, \delta(\mathbf{x} - \mathbf{x}_j) , \quad \mathbf{x} \in \Omega , \tag{0.17}$$

with $\partial_n u_2 = 0$ on $x \in \partial\Omega$. The solvability condition for u_2, obtained by the divergence theorem, determines χ_1 as

$$\chi_1 = -\frac{1}{N\bar{C}} \sum_{j=1}^{N} A_j C_j \,. \tag{0.18}$$

Finally, we substitute (0.15 b) for A_j into (0.18) and write the resulting expression for χ_1 in matrix form we obtain that

$$\chi_1 = -\frac{4\pi C_1}{N\bar{C}} \left[(\mathcal{G}c)_1 - \frac{1}{N\bar{C}} c^T \mathcal{G} c \right] \,. \tag{0.19}$$

Here $c = (C_1, \ldots, C_N)^T$ is the capacitance vector, and $\mathcal{G}$ is the Green's matrix

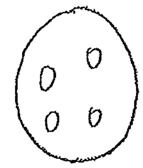
$$\mathcal{G} \equiv \begin{pmatrix} R_{1,1} & G_{1,2} & \cdots & G_{1,N} \\ G_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{N-1,N} \\ G_{N,1} & \cdots & G_{N,N-1} & R_{N,N} \end{pmatrix} \,. \tag{0.20}$$

(3) From (0.2) we observe that $u \sim C_1/(N\bar{C})$, so that there is no leading-order effect on the splitting probability u of either the location of the source, the target, or the surrounding traps. If $C_1 = a_1 < C_j$ for $j = 2, \ldots, N$, then the target sphere is smaller than the other spheres. The leading order term in u predicts that the probability is less than $1/N$, i.e. if all the spheres had the same radius then the leading order probability of hitting the target first is asymptotically $1/N$. However, this probability decreases when the target sphere has smaller radius than the others. The second order term in the expansion of u contains positional information of the initial starting point, the target location, and the location of the other traps. This term can show that the other traps can act as a "shield", making it more difficult to reach the target trap first. For examples of this please see section 3.1 of the paper "Optimizing the Principal Eigenvalue of the Laplacian in a Sphere with Interior Traps", by A. Cheviakov and M.J.Ward on my website.

SOLUTION 3

(i) FIND A 2-TERM ASYMPTOTIC APPROXIMATION TO THE SOLUTION OF

$$\Delta u = 0 \quad \text{IN} \quad \Omega \setminus \bigcup_{j=1}^{4} \Omega_{\varepsilon_j}$$



$$u = f(\varphi) \quad \text{ON} \quad \partial\Omega$$

$$u = \alpha_j \ , \quad x \in \partial\Omega_{\varepsilon_j}, \quad j = 1,..,4.$$

HERE Ω IS UNIT SPHERE AND $\Omega_{\varepsilon_j} = \{ \underline{x} \mid |\underline{x} - \underline{x}_j| \leq \varepsilon \}$, α_j ARE CONSTANTS.

THE LEADING-ORDER SOLUTION IN OUTER REGION IS THE SOLUTION u_0 TO THE UNPERTURBED PROBLEM

$$(1) \begin{cases} \Delta u_0 = 0 & \text{IN} \quad \Omega \\ u_0 = f(\varphi) & \text{ON} \quad \partial\Omega. \\ u_0 \text{ SMOOTH} & \text{IN} \quad \Omega. \end{cases}$$

IN THE OUTER REGION WE EXPAND

$$u = u_0 + \nu u_1 + \cdots \qquad \nu \longrightarrow 0 \text{ AS } \varepsilon \to 0 \text{ TO BE FOUND}$$

WE OBTAIN THAT

$$(2) \begin{cases} \Delta u_1 = 0 & \text{IN} \quad \Omega \setminus \{ \underline{x}_1, .., \underline{x}_4 \} \\ u_1 = 0 & \text{ON} \quad \partial\Omega \\ u_1 \text{ singular as } & x \to x_j, \quad j = 1,..,4 \end{cases}$$

NOW IN THE INNER REGION NEAR x_j WE PUT

$$y = \varepsilon^{-1}(x - x_j), \qquad v(y) = u[x_j + \varepsilon y] = v_0 + \sigma v_1 + \cdots \qquad \sigma \ll 1$$

THEN WE OBTAIN THAT

$$(3) \begin{cases} \Delta_y v_0 = 0 \ , & y \notin \Omega_j \\ v_0 = \alpha_j \ , & y \in \partial\Omega_j, \ j = 1,..,4. \end{cases} \qquad \Omega_j : |y| \leq 1$$

THE MATCHING CONDITION IS

$$U_0 + \nu U_1 + \dots \sim V_0 + \sigma V_1 + \dots$$

WE CONCLUDE THAT

$$(4) \qquad V_0 \to U_0(\underset{\sim}{X}_j) \quad \text{AS} \quad |y| \to \infty.$$

THEN WE WRITE THE EXACT SOLUTION TO (3) AND (4) AS

$$(5) \qquad V_0 = U_0(\underset{\sim}{X}_j) + \left[\alpha_j - U_0(\underset{\sim}{X}_j) \right] / |\underset{\sim}{y}|$$

THIS IS ONLY POSSIBLE SINCE Ω_j IS A SPHERE OF RADIUS 1.

WE WRITE IN OUTER VARIABLES THAT

$$V_0 = U_0(\underset{\sim}{X}_j) + \frac{\varepsilon \left[\alpha_j - U_0(\underset{\sim}{X}_j) \right]}{|\underset{\sim}{X} - \underset{\sim}{X}_j|}$$

THUS FROM THE MATCHING CONDITION WE OBTAIN $\nu(\varepsilon) = \varepsilon$ AND THAT

$$(6) \qquad U_1 \sim \frac{\alpha_j - U_0(\underset{\sim}{X}_j)}{|\underset{\sim}{X} - \underset{\sim}{X}_j|} \quad \text{AS} \quad \underset{\sim}{X} \to \underset{\sim}{X}_j$$

THE PROBLEM (2), (6) CAN BE WRITTEN AS.

$$(7) \quad \left\{ \begin{array}{l} \Delta U_1 = -4\pi \sum_{j=1}^{4} \left(\alpha_j - U_0(\underset{\sim}{X}_j) \right) \delta(\underset{\sim}{X} - \underset{\sim}{X}_j) \\ U_1 = 0 \quad \text{ON} \quad \partial\Omega \end{array} \right.$$

NOW DEFINE THE GREEN'S FUNCTION $G(\underset{\sim}{X}; \underset{\sim}{X}_j)$ BY

$$(8) \quad \begin{array}{l} \Delta G = \delta(\underset{\sim}{X} - \underset{\sim}{X}_j) \quad \text{IN} \quad \Omega \\ G = 0 \quad \text{ON} \quad \partial\Omega \\ G \sim -\dfrac{1}{4\pi |\underset{\sim}{X} - \underset{\sim}{X}_j|} + O(1) \quad \text{AS} \quad \underset{\sim}{X} \to \underset{\sim}{X}_j \end{array}$$

THEN THE SOLUTION TO (7) IS

$$(9) \qquad u_1 = -4\pi \sum_{j=1}^{4} \left(\alpha_j - u_0(\underline{x}_j) \right) G(\underline{x}; \underline{x}_j)$$

IN SUMMARY, WE OBTAIN THE OUTER EXPANSION

$$(9) \qquad u \sim u_0 - 4\pi\varepsilon \sum_{j=1}^{4} \left(\alpha_j - u_0(\underline{x}_j) \right) G(\underline{x}; \underline{x}_j) \qquad \text{FOR} \quad |x - x_j| = O(1) \quad j = 1, \dots, 4$$

AND THE INNER EXPANSION

$$(10) \qquad u \sim u_0(\underline{x}_j) + \varepsilon \left[\alpha_j - u_0(x_j) \right] \Big/ |\underline{x} - \underline{x}_j| \qquad \text{FOR} \quad |x - x_j| = O(\varepsilon)$$

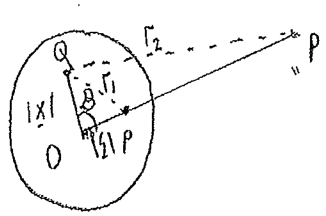
NEXT, WE MUST RECALL HOW TO CALCULATE THE SOLUTION u_0 TO (1) AND WE MUST CALCULATE THE GREEN'S FUNCTION SATISFYING (8) EXPLICITLY.

CALCULATION OF $G(\underline{x}; \underline{\xi})$

LET $G(\underline{x}; \underline{\xi})$ SOLVE

$$(11) \quad \begin{cases} \Delta G = \delta(\underline{x} - \underline{\xi}) & \text{IN } \Omega = \{ x \mid |\underline{x}| \le 1 \} \\ G = 0 & \text{ON } \partial\Omega \\ \text{SO} \quad G \sim -\dfrac{1}{4\pi |\underline{x} - \underline{\xi}|} & \text{AS } \underline{x} \to \underline{\xi} \end{cases}$$

THIS IS A STANDARD METHOD OF IMAGES PROBLEM:



- LET P BE FIXED POINT ξ WHERE DELTA FUNCTION IS LOCATED
  THEN $P = \{ \underline{\xi} \}$ WITH $|\xi| = \text{GIVEN}$ }
- LET Q BE ANY POINT IN THE DISK
  $Q = \{ \underline{x} \}$

O IS CENTER OF UNIT SPHERE

NOW $\quad \text{dist}(O, Q) = |\underset{\sim}{x}| \qquad \text{dist}(P,Q) = r_1 \qquad \text{dist}(Q, P') = r_2.$

$\qquad \text{dist}(O,P) = |\underset{\sim}{\xi}| \qquad \text{dist}(O,P') = R$

WE TAKE OQP TO LIE ON A PLANE IN 3-D WITH β BEING THE ANGLE BETWEEN OQ AND OP. WE CHOOSE POINT P' TO LIE ON THE LINE OP AND TO TAKE $\text{dist}(O,P') = R = 1/|\underset{\sim}{\xi}|$ (i.e. AT "INVERSE" POINT TO SPHERE)

<u>KEY FACT</u> ON $|\underset{\sim}{x}| = 1$, i.e. IF Q IS ON BOUNDARY OF UNIT SPHERE, THEN $\quad r_2 = r_1/|\underset{\sim}{\xi}| \quad$ i.e. $\quad \text{dist}(Q,P') = \dfrac{\text{dist}(P,Q)}{\text{dist}(O,P)}$

<u>PROOF</u> BY LAW OF COSINES:

$$(\text{dist}(QP'))^2 = (\text{dist}(OQ))^2 + (\text{dist}(OP'))^2 - 2\,\text{dist}(OQ)\,\text{dist}(OP')\cos\beta$$

SO $\qquad r_2^2 = |\underset{\sim}{x}|^2 + R^2 - 2\,|\underset{\sim}{x}|\,R\cos\beta.$

NOW $\qquad R = 1/|\underset{\sim}{\xi}| \quad$ SO $\qquad r_2^2 = |\underset{\sim}{x}|^2 + \dfrac{1}{|\underset{\sim}{\xi}|^2} - 2\dfrac{|\underset{\sim}{x}|}{|\underset{\sim}{\xi}|}\cos\beta.$

NOW WHEN Q IS ON $\partial\Omega$, THEN $|\underset{\sim}{x}| = 1$. HENCE

$$r_2^2 = 1 + \frac{1}{|\underset{\sim}{\xi}|^2} - \frac{2}{|\underset{\sim}{\xi}|}\cos\beta = \frac{1}{|\underset{\sim}{\xi}|^2}\left[\,|\underset{\sim}{\xi}|^2 + 1 - 2|\underset{\sim}{\xi}|\cos\beta\,\right]$$

BUT $\quad r_1^2 = |\underset{\sim}{\xi}|^2 + 1 - 2|\underset{\sim}{\xi}|\cos\beta \quad$ FROM LAW OF COSINES. THUS FOR Q ON $\partial\Omega$,

$$r_2^2 = \frac{1}{|\underset{\sim}{\xi}|^2}\,r_1^2 \longrightarrow r_2 = r_1/|\underset{\sim}{\xi}|$$

NOW CONSTRUCT THE GREEN'S FUNCTION. THE FREE SPACE G-FUNCTION IN 3-D IS $G^F = -\frac{1}{4\pi r_1} = -\frac{1}{4\pi|\underline{x}-\underline{\xi}|}$

THIS SATISFIES $\Delta G^F = \delta(\underline{x}-\underline{\xi})$ BUT DOES NOT SATISFY $G=0$ ON $\partial\Omega$.

WE ADD TO G^F A HARMONIC FUNCTION TO MAKE $G=0$ ON $\partial\Omega$.

WE TRY $G = -\frac{1}{4\pi r_1} + \frac{\lambda}{4\pi r_2}$ FOR SOME λ WHERE $r_2 = \mathrm{dist}(Q,P')$

NOW FOR $|\underline{x}|=1$ WE HAVE $r_2 = r_1/|\underline{\xi}|$ AND SO

$$G = -\frac{1}{4\pi r_1} + \frac{\lambda}{4\pi r_1}|\underline{\xi}| \quad \text{ON } \partial D$$

HENCE TAKE $\lambda = 1/|\underline{\xi}|$ TO GET $G=0$ ON $\partial\Omega$.

THUS THE SOLUTION TO (11) IS SIMPLY

$$(12) \qquad G = -\frac{1}{4\pi|\underline{x}-\underline{\xi}|} + \frac{1}{|\underline{\xi}|}\frac{1}{4\pi|\underline{x}-\underline{\xi}'|}$$ WHERE $\underline{\xi}'$ IS LOCATION OF POINT P'.

WITH $\underline{\xi}'$ IS ON THE RAY OP AT A DISTANCE $|\underline{\xi}'| = 1/|\underline{\xi}|$.

COMPUTATION OF SOLUTION u_0

NEXT WE MUST SOLVE (1) FOR u_0. WE LOOK FOR A SOLUTION TO (1) WHICH IS INDEPENDENT OF ϕ (GIVEN THAT BOUNDARY DATA IS INDEPENDENT OF ϕ). THUS IN TERMS OF $u(r,\varphi)$ WE HAVE

$$\frac{1}{r^2}\left(r^2 u_{0r}\right)_r + \frac{1}{r^2 \sin\varphi}\partial_\varphi\left(\sin\varphi \, u_{0\varphi}\right) = 0 \quad \text{IN } 0 \le \varphi \le \pi, \; 0 \le r < 1$$

$$u_0(1,\varphi) = f(\varphi).$$

FROM ONLINE NOTES WE HAVE THAT

$$u_0(r,\varphi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\varphi)$$

WHERE $y = P_n(x)$ IS THE LEGENDRE POLYNOMIAL SATISFYING

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad -1 < x < 1$$

$$y \text{ BOUNDED AS } x \to \pm 1.$$

THE FIRST FEW SUCH POLYNOMIALS NORMALIZED BY $P_n(1) = 1$ ARE

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \quad P_3(x) = \frac{1}{2}(5x^3-3x) \quad \text{etc...}$$

NOW FROM ONLINE NOTES WE HAVE THE ORTHOGONALITY RELATION

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1}\delta_{mn} \qquad \delta_{mn} = \begin{cases} 0 & \text{IF } m \neq n \\ 1 & \text{IF } m = n. \end{cases}$$

THEREFORE, $$\int_0^{\pi} P_n(\cos\varphi) P_m(\cos\varphi) \sin\varphi \, d\varphi = \frac{2}{2n+1}\delta_{mn} \qquad (\text{SET } x = \cos\varphi).$$

THEN WE PUT $$u(1,\varphi) = F(\varphi) = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi)$$

MULTIPLYING BY $P_m(\cos\varphi)\sin\varphi$ ON BOTH SIDES AND INTEGRATING $\int_0^{\pi}$

WE GET $$\int_0^{\pi} F(\varphi) P_n(\cos\varphi) \sin\varphi \, d\varphi = A_n \left[\frac{2}{2n+1}\right]$$

THUS

$$(13) \quad \begin{cases} u_0(r,\varphi) = \displaystyle\sum_{n=0}^{\infty} A_n r^n P_n(\cos\varphi) \\ \text{WITH} \qquad A_n = \displaystyle\frac{2n+1}{2}\int_0^{\pi} F(\varphi) P_n(\cos\varphi)\sin\varphi\, d\varphi. \end{cases}$$

IN SUMMARY, OUR ASYMPTOTIC SOLUTION IS GIVEN BY (9) WHERE THE GREEN'S FUNCTION G AND u_0 ARE GIVEN BY (12) AND (13), RESPECTIVELY.

(ii) NOW SUPPOSE THAT $\alpha_j = 1$ FOR $j = 1, \ldots, 4$ AND $f(\varphi) = \sin^2\varphi$ WITH TRAPS AT $\underline{x}_1 = (0, 1/2, 1/2)$, $\underline{x}_2 = (0, -1/2, 1/2)$, $\underline{x}_3 = (0, 1/2, -1/2)$, $\underline{x}_4 = (0, -1/2, -1/2)$. PLOT $u(\underline{0})$ VERSUS ε FOR $\varepsilon \ll 1$ FROM ASYMPTOTIC SOLUTION.

WE NOW FIND $u_0(r, \varphi)$. WE MUST FIND A_n IN (13), WHICH SATISFY

$$\sin^2\varphi = \sum_{n=0}^{\infty} A_n P_n(\cos\varphi).$$

SO

$$1 - \cos^2\varphi = A_0[1] + A_1[\cos\varphi] + A_2\left[\frac{1}{2}(3\cos^2\varphi - 1)\right] + A_3[\ldots].$$

WE TAKE $A_1 = A_3 = A_4 = A_5 = \ldots = 0$ AND

$$A_0 - \frac{A_2}{2} = 1$$

$$\frac{3}{2}A_2 = -1 \longrightarrow A_2 = -2/3$$

AND $A_0 = 2/3$.

THUS $\sin^2\varphi = 2/3\, P_0(\cos\varphi) - 2/3\, P_2(\cos\varphi)$

THEREFORE WE HAVE

$$(14) \quad u_0(r, \varphi) = \frac{2}{3} P_0(\cos\varphi) - \frac{2}{3} r^2 P_2(\cos\varphi) \qquad \text{WITH } r = |\underline{x}|.$$

AS BEING THE EXACT SOLUTION TO (1).

NOW WE MUST CALCULATE $u_0(\underline{x}_j)$. FOR $\underline{x}_1$ AND $\underline{x}_2$ WE HAVE $\varphi = \pi/4$ WHILE FOR $\underline{x}_3$ AND $\underline{x}_4$ WE HAVE $\varphi = 3\pi/4$. IN BOTH CASES $r_j = \sqrt{(1/2)^2 + (1/2)^2} = 1/\sqrt{2}$

THUS

$$u_0(\underline{x}_1) = u_0(\underline{x}_2) = \frac{2}{3} - \frac{2}{3}\left(\frac{1}{2}\right)\left[P_2(\cos\pi/4)\right]$$

$$u_0(\underline{x}_3) = u_0(\underline{x}_4) = \frac{2}{3} - \frac{2}{3}\left(\frac{1}{2}\right)\left[P_2(\cos 3\pi/4)\right]$$

BUT $P_2(\cos\varphi) = \frac{1}{2}\left[3\cos^2\varphi - 1\right] \rightarrow P_2(\cos\pi/4) = \frac{1}{2}\left(3/2 - 1\right) = 1/2 \; 1/2 = 1/4$

$$P_2(\cos 3\pi/4) = 1/2\left(3/2 - 1\right) = 1/2 \; 1/2 = 1/4$$

THUS WE OBTAIN $u_0(\underset{\sim}{x}_j) = \frac{2}{3} - \frac{1}{3}\left(1/4\right) = 7/12$ FOR $j = 1, 2, 3, 4.$

NOTICE THAT SINCE $\sin^2\varphi$ IS SYMMETRIC ABOUT EQUATOR THEN WE SHOULD HAVE EXPECTED THAT $u_0(\underset{\sim}{x}_j)$ IS INDEPENDENT OF j.

THUS $\quad u_0(\underset{\sim}{x}_j) = 1/2 \quad \forall j \quad$ AND $\quad u_0(\underset{\sim}{0}) = 2/3 \qquad (15)$

NEXT, WE WRITE THAT, FROM (9),

$$u(\underset{\sim}{0}) = u_0(\underset{\sim}{0}) - 4\pi\varepsilon\sum_{j=1}^{4}\left[\alpha_j - u_0(\underset{\sim}{x}_j)\right] G(0; \underset{\sim}{x}_j)$$

WE GET, SINCE $\alpha_j - u_0(\underset{\sim}{x}_j) = (1 - 7/12) = 5/12 \quad \forall j$, THAT

$$(16) \qquad u(\underset{\sim}{0}) = \frac{2}{3} - \frac{5}{3}\pi\varepsilon\sum_{j=1}^{4} G(0; \underset{\sim}{x}_j)$$

IT REMAINS TO CALCULATE $G(0; \underset{\sim}{x}_j)$. FOR THIS WE USE (12) WHICH IS

$$G(\underset{\sim}{x}; \underset{\sim}{\xi}) = -\frac{1}{4\pi|\underset{\sim}{x} - \underset{\sim}{\xi}|} + \frac{1}{|\underset{\sim}{\xi}|}\frac{1}{4\pi|\underset{\sim}{x} - \underset{\sim}{\xi}'|}.$$

NOW $\quad |\underset{\sim}{x}_j| = \sqrt{(1/2)^2 + (1/2)^2} = 1/\sqrt{2} \quad \forall j$. THUS

$$G(0; \underset{\sim}{x}_j) = -\frac{1}{4\pi|\underset{\sim}{x}_j|} + \frac{1}{|\underset{\sim}{x}_j|}\frac{1}{4\pi|-\underset{\sim}{x}_j'|}.$$

BUT $\quad |\underset{\sim}{x}_j'|\,|\underset{\sim}{x}_j| = 1 \quad$ SINCE $\underset{\sim}{x}_j'$ IS INVERSE POINT TO $\underset{\sim}{x}_j$.

THUS $G(0; \underset{\sim}{x}_j) = \dfrac{-1}{4\pi |\underset{\sim}{x}_j|} + \dfrac{1}{4\pi} = \dfrac{1}{4\pi}\left(1 - \sqrt{2}\right)$

THEN (16) GIVES $U(0) = \dfrac{2}{3} - \dfrac{5\pi}{3}\left(\dfrac{1}{4\pi}(1-\sqrt{2})\right)\sum_{j=1}^{4}(1)$

$$U(\underset{\sim}{0}) = \frac{2}{3} - \frac{5}{12}\,\varepsilon\,(1-\sqrt{2})\sum_{j=1}^{4}(1)$$

FINALLY, WE CONCLUDE FOR $\varepsilon \ll 1$ THAT

$$(17)\quad U(\underset{\sim}{0}) = \frac{2}{3} - 2\,\varepsilon\,(1-\sqrt{2}) = \frac{2}{3} + \frac{5}{3}\,\varepsilon\left(\sqrt{2}-1\right).$$

THIS IS A STRAIGHT LINE WITH SLOPE $\dfrac{5}{3}\left(\sqrt{2}-1\right)$.

NOTICE THAT $U(\underset{\sim}{0}) > 2/3$ AS A RESULT OF THE POSITIVE "TEMPERATURE" ON THE OBJECTS.