MATH 551 HW 5

PROBLEM 1 CONSIDER THE TIME-DEPENDENT PROBLEM

$$(*)\quad \begin{cases} u_t = \dfrac{\partial}{\partial x}\left[D(x, x/\varepsilon) \dfrac{\partial u}{\partial x} \right] - f(x, x/\varepsilon) & 0 < x < 1,\ t > 0 \\ u(0,t) = 0, \quad u(1,t) = 1, \quad u(x,0) = g(x, x/\varepsilon) \end{cases}$$

WITH $g(0,0) = 0,\ g(1, 1/\varepsilon) = 1$.

ASSUME THAT $f(x,y)$, $g(x,y)$, $D(x,y)$ ARE ALL PERIODIC IN THE y-VARIABLE WITH PERIOD 2π.

(i) CALCULATE THE EFFECTIVE STEADY-STATE EQUATION FROM HOMOGENIZATION THEORY.

(ii) PLOT THE EXACT STEADY-STATE SOLUTION AND THE HOMOGENIZED STEADY-STATE SOLUTION FOR THE SPECIAL CASE

$$D(x, x/\varepsilon) = \frac{1}{1 + \beta \cos(x/\varepsilon)} \qquad f(x, x/\varepsilon) = 1 + \cos(x/\varepsilon)$$

WITH $\beta = .95$ AND $\varepsilon = 0.01$.

WHAT HAPPENS IF ONE MISTAKINGLY USES THE AVERAGE

$$D_{avg}(x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{dy}{1 + \beta \cos y}$$

IN HOMOGENIZED EQUATION RATHER THAN THE HARMONIC MEAN?

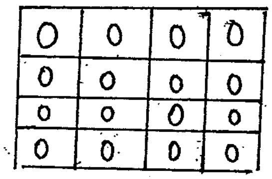
(WITH $\beta = .95$)

(iii) FOR THE TIME-DEPENDENT PROBLEM WHAT IS THE HOMOGENIZED EQUATION ON THE x, t SCALES?

(iv) DISCUSS ANY TRANSIENT-IN-TIME DYNAMICS FOR $(*)$. IN OTHER WORDS, WHAT IS THE PROBLEM TO BE SOLVED ON SUITABLY SHORT TIME SCALES? DERIVE FROM IT THE INITIAL CONDITION TO BE USED FOR YOUR HOMOGENIZED PROBLEM IN (iii).

PROBLEM 2 (THE CLAUSIUS-MOSSOTTI FORMULA)

WE WANT TO FIND THE EFFECTIVE CONDUCTIVITY OF A PERIODIC ARRAY OF SPHERES IN THE LOW-VOLUME-FRACTION LIMIT. THE PERTURBED PROBLEM IS



$$(1) \qquad \nabla \cdot \left[D(x/\varepsilon) \nabla u \right] = f(x) \quad \text{IN } \mathbb{R}^N$$

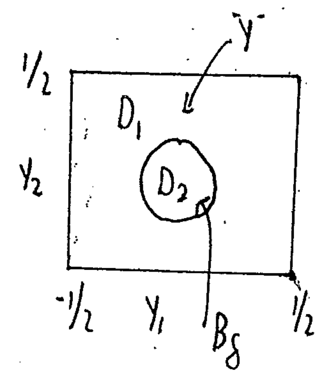
WHERE $D(y)$ IS 1-PERIODIC RESPECT TO THE UNIT CELL Y

$$Y = \{ (y_1, .., y_3) \mid -1/2 < y_j < 1/2 , \ j = 1, .. 3 \}$$

IN THE UNIT CELL Y WE ASSUME THAT

$$D = \begin{cases} D_1 & \text{IN } Y \setminus B_\delta \\ D_2 & \text{IN } B_\delta \end{cases}$$

WHERE $B_\delta = \{ y \mid |y| \le \delta \text{ WITH } 0 < \delta < 1 \}$



SIDE-VIEW PICTURE

HERE $D_1 > 0$, $D_2 > 0$ ARE CONSTANTS. ACROSS EACH SPHERE WE HAVE

u IS CONTINUOUS AND $D_1 \nabla u \cdot \hat{n} \big|_{OUT} = D_2 \nabla u \cdot \hat{n} \big|_{IN}$

(i) FOR $\varepsilon \to 0$ DERIVE THE LEADING ORDER PROBLEM FOR u TO BE SOLVED ON THE MACROSCALE x. DETERMINE THE "CELL PROBLEM" THAT NEEDS TO BE SOLVED.

(ii) IN THE LOW-VOLUME-FRACTION LIMIT $\delta \ll 1$ SOLVE THE UNIT CELL PROBLEM ASYMPTOTICALLY USING TECHNIQUES FROM STRONG LOCALIZED PERTURBATION THEORY. IN PARTICULAR IF $f = 4\pi\delta^3/3 \ll 1$ IS THE VOLUME FRACTION SHOW THAT

$$D_{eff} = D_1 + O(f) + ...$$

AND CALCULATE THE COEFFICIENT OF THE $O(f)$ **TERM** EXPLICITLY IN TERMS OF D_1 AND D_2.

(HINT: PLEASE READ AND STUDY NOTES FOR 2-D PROBLEM ONLINE)