DIFFUSIVE WAVES IN A THIN CHANNEL

①

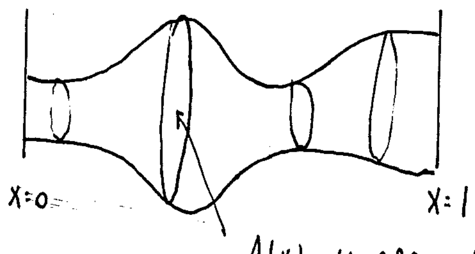
INTERNAL LAYERS AND DIFFUSIVE WAVE PROPAGATION IN A LONG CHANNEL OF VARIABLE CROSS-SECTIONAL AREA $A(x)$. THE MODEL IS A PDE IN 1-D FOR $u(x,t)$ OF THE FORM

$$u_t = \frac{\varepsilon^2}{A(x)} \left[A(x)\, u_x \right]_x + h(u), \quad 0<x<1, \; t>0, \quad \text{WHERE} \quad h(u) = u(1-u)(u-2)$$

WE WILL SPECIFY $u(0,t)=0$, $u(1,t)=2$

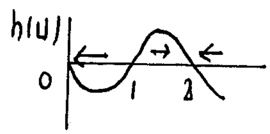
AND THE INITIAL CONDITION $u(x,0) = f(x)$ WITH $f(0)=0$, $f(1)=2$ AND $f'(x)>0$.

THE PLOT OF THE GEOMETRY IS



WHERE $A(x) > 0$

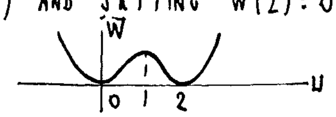
WE WILL ASSUME $\varepsilon \ll 1$. THE NONLINEARITY $h(u)$ IS OF BISTABLE TYPE AS WE HAVE



FOR DYNAMICS OF $du/dt = h(u)$.

<u>REMARK</u> IT IS TYPICAL TO WRITE $\bar{W}'(u) = -h(u)$ AND SETTING $\bar{W}(2)=0$ WE GET WLOG THAT

$$\bar{W}(u) = \frac{1}{4}\left[1-(u-1)^2\right]^2$$



WHICH IS A DOUBLE WELL POTENTIAL WITH MINIMA AT $u=0,2$, LOCAL MAXIMA AT $u=1$.
THEN FOR $\varepsilon=0$ WE HAVE $du/dt = -\bar{W}'(u)$. SO THAT $u \to$ MINIMA OF THE WELL AS t INCREASES.

FOR LARGE TIME, AWAY FROM AN INITIAL TRANSIENT DESCRIBING THE FORMATION OF A THIN LAYER, WE WILL LOOK FOR A SLOWLY EVOLVING THIN DIFFUSIVE LAYER OF THE FORM

$$u(x,t) = \bar{U}\left[(x-m(\tau))\varepsilon^{-1}\right] \quad \text{WITH} \quad \tau = \varepsilon^2 t.$$

THE MOVING INTERNAL OR DIFFUSIVE LAYER HAS COORDINATE $y = \varepsilon^{-1}\left[x - m(\tau)\right]$.
IT EVOLVES OVER THE LONG TIME-SCALE $t = O(1/\varepsilon^2)$.

WE WILL DERIVE AN ODE FOR $m(\tau)$ THAT INVOLVES THE SHAPE OF THE CROSS-SECTION $A(x)$. WE WILL RELATE EQUILIBRIA OF THE ODE FOR $m(\tau)$ TO THE SHAPE $A(x)$. WE WILL SHOW THAT LOCAL MINIMA OF $A(x)$ CORRESPOND TO STABLE EQUILIBRIA, WHEREAS LOCAL MAXIMA OF $A(x)$ ARE UNSTABLE EQUILIBRIA.

WE NOW BEGIN THE INTERNAL LAYER ANALYSIS:

WE WRITE $\dfrac{\varepsilon^2}{A}(A u_x)_x = \varepsilon^2 u_{xx} + \varepsilon^2 \dfrac{A'(x)}{A(x)} u_x.$

THE PDE IS THEN

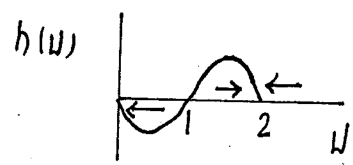
$$u_t = \varepsilon^2 u_{xx} + \varepsilon^2 \frac{A'(x)}{A(x)} u_x + h(u), \quad 0 < x < 1 \qquad h(u) = u(1-u)(u-2)$$

$$u(0,t) = 0, \quad u(1,t) = 2$$

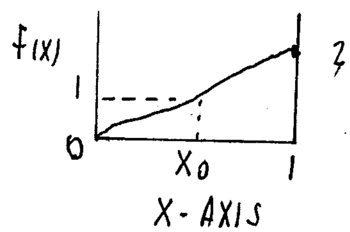
$$u(x,0) = f(x) \quad \text{WITH} \quad f(0) = 0,\ f(1) = 2 \ \text{AND} \ f'(x) > 0.$$

WE FIRST CONSIDER <u>INITIAL TRANSIENT</u> DESCRIBING THE FORMATION OF A FRONT SOLUTION. WE LET $\varepsilon \ll 1$ AND CONSIDER

<u>ODE</u> $\left\{ \begin{array}{l} u_t = h(u) \\ u(x,0) = f(x) \end{array} \right.$



WITH $f(x)$ AS SHOWN BELOW

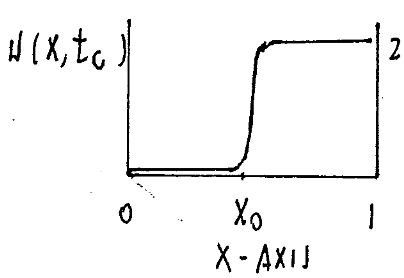


DEFINE x_0 TO BE THE UNIQUE POINT WHERE $f(x_0) = 1$

FROM THE <u>ODE</u>, WE CONCLUDE THAT

$$u \to \left\{ \begin{array}{lll} 0 & \text{IF } 0 < x < x_0 & \text{FOR } t \text{ large} \\ 2 & \text{IF } x_0 < x < 1 & \text{FOR } t \text{ LARGE.} \end{array} \right.$$

THUS, AT THE END OF THE TRANSIENT PERIOD WE HAVE A SOLUTION WITH A "NEAR" DISCONTINUITY AT x_0, WHICH MUST BE SMOOTHED OUT BY $\varepsilon^2 u_{xx} + \varepsilon^2 A' u_x / A$ TERM.



$t_c = O(1)$, x_0 IS DEFINED BY $f(x_0) = 1$.

NEXT, WE LOOK FOR A SLOWLY EVOLVING FRONT SOLUTION OF THE FORM

$$u = U\left[\varepsilon^{-1}(x - m(\tau))\right] \quad \text{WITH} \quad \tau = \varepsilon^2 t, \quad y = \varepsilon^{-1}(x - m(\tau))$$

AND $m(0) = x_0$ IS INITIAL CONDITION FROM END OF TRANSIENT PERIOD

AND $U(\infty) = 2, \quad U(-\infty) = 0, \quad U(0) = 1.$

THE EQUATION IS TRANSFORMED VIA:

$$u_x = \varepsilon^{-1} U_y \qquad u_{xx} = \varepsilon^{-2} U_{yy}$$

$$u_t = -\varepsilon^{-1} \frac{dm}{d\tau}\frac{d\tau}{dt} U_y = -\varepsilon^{-1}\frac{dm}{d\tau}\varepsilon^2 U_y = -\varepsilon \dot{m} U_y \quad \text{WITH} \quad \dot{m} = dm/d\tau$$

ALSO,
$$A'(x)/A(x) = A'(m(\tau)+\varepsilon y)/A(m(\tau)+\varepsilon y) = A'(m)/A(m).$$

THE ORIGINAL PDE TRANSFORMS IN THE INNER REGION TO

$$(*)\begin{cases} -\varepsilon \dot{m} U_y = U_{yy} + \varepsilon \dfrac{A'(m)}{A(m)} U_y + h(U), & -\infty < y < \infty \\ U \to 0 \text{ AS } y \to -\infty \\ U \to 2 \text{ AS } y \to +\infty \\ U(0) = 1 \text{ DEFINES THE FRONT} \end{cases}$$

NOW WE EXPAND $U = U_0 + \varepsilon U_1 + \ldots$, THIS YIELDS THAT,

$$(1)\begin{cases} U_{0yy} + h(U_0) = 0, & -\infty < y < \infty \\ U_0(-\infty) = 0, \quad U_0(\infty) = 2, \quad U_0(0) = 1 \end{cases}$$

$$(2)\begin{cases} U_{1yy} + h'(U_0)U_1 = -\dot{m} U_{0y} - A'(m) U_{0y}/A(m), & -\infty < y < \infty \\ U_1 \to 0 \text{ AS } |y| \to \infty \\ U_1(0) = 0 \end{cases}$$

THE SOLUTION TO THE U_0 PROBLEM IS SIMPLY

$$U_0 = 1 + W$$

WHERE
$$W'' + W - W^3 = 0$$
$$W(\infty) = 1, \quad W(-\infty) = -1, \quad W(0) = 0.$$

PUT $\quad W = \tanh(By) \longrightarrow [B\,\mathrm{sech}^2(By)]' + \tanh(By) - \tanh^3 By = 0$

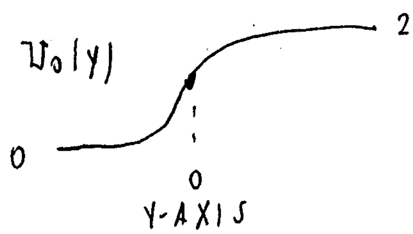
$$-2B\,\mathrm{sech}^2 \tanh + \tanh(1 - \tanh^2) = 0$$
$$-2B^2\,\mathrm{sech}^2 \tanh + \tanh\,\mathrm{sech}^2 = 0 \rightarrow 2B^2 = 1$$

SO $B = 1/\sqrt{2}$.

THEN, $\quad U_0(y) = 1 + \tanh\left(y/\sqrt{2}\right)$

THE FUNCTION U_0 IS THE <u>HETEROCLINIC</u> <u>SOLUTION</u>.

IT LOOKS LIKE



WLOG WE HAVE FIXED THE PHASE INVARIANCE BY SETTING $U_0(0) = 1$.

NOW NOTICE THAT THE U_1 EQUATION CAN BE WRITTEN IN OPERATOR FORM AS

$$(3) \left\{ \begin{array}{l} \mathcal{L} U_1 \equiv U_{1zz} + h'(U_0) U_1 = -U_{0y}\left(\mathring{m} + A'(m)/A(m) \right), \quad -\infty < y < \infty \\ U_1 \rightarrow 0 \text{ AS } |y| \rightarrow \infty. \end{array} \right.$$

NOW DIFFERENTIATE WRT y THE U_0 PROBLEM TO GET

$$(4) \left\{ \begin{array}{l} \mathcal{L} U_{0y} = (U_{0y})_{yy} + h'(U_0) U_{0y} = 0 \\ U_{0y} \rightarrow 0 \text{ AS } |y| \rightarrow \infty. \end{array} \right.$$

THUS, WE REQUIRE THAT THE PROBLEM FOR U_1 MUST SATISFY A SOLVABILITY CONDITION THAT

$$\int^{\infty} -U_{0y}\left(\mathring{m} + A'(m)/A(m) \right) U_{0y}\, dy = 0.$$

THE DERIVATION OF THIS CONDITION IS THAT BY LAGRANGE'S IDENTITY

$$\int_{-\infty}^{\infty} (U_{0y}\, \mathcal{L} U_1 - U_1 \mathcal{L} U_{0y})\, dy = (U_{0y} U_{1y} - U_1 U_{0yy}) \Big|_{-\infty}^{\infty}$$

SINCE $U_1 \to 0$ AS $|y| \to \infty$ AND $U_0 \to$ CONSTANTS AS $|y| \to \infty$, AND $\mathcal{L} U_{0y} = 0$

THIS YIELDS $$\int_{-\infty}^{\infty} U_{0y}\, \mathcal{L} U_1\, dy = \int_{-\infty}^{\infty} U_{0y} \left[- U_{0y} \left(\mathring{m} + A'(m)/A(m) \right) \right] dy = 0.$$

HENCE $$\left[\mathring{m} + A'(m)/A(m) \right] \int_{-\infty}^{\infty} (U_{0y})^2\, dy = 0.$$

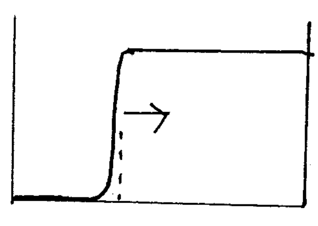
WE CONCLUDE THAT

$$(5) \qquad \frac{dm}{d\tau} = - \frac{A'(m)}{A(m)} \qquad \text{WITH} \quad m(0) = X_0 \quad \text{AND} \quad \tau = \varepsilon^2 t.$$

THE TIME-DEPENDENT SLOW MOVING FRONT SOLUTION SATISFIES

$$(6) \qquad U \sim 1 + \tanh\left[\frac{\varepsilon^{-1}}{\sqrt{2}} [X - m(\tau)] \right] + O(\varepsilon).$$

THE PICTURE IS



(ii) THE EQUILIBRIA OF THE ODE ARE WHERE

$$A'(m_e) = 0$$

LINEARIZING THE ODE NEAR AN EQUILIBRIUM POINT, WE WRITE

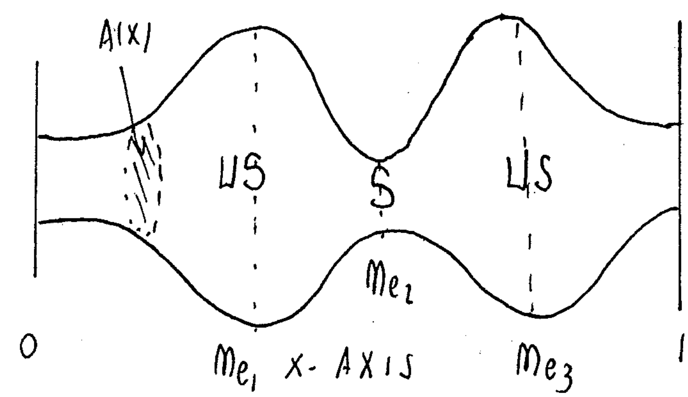
$$m = m_e + \zeta \qquad \zeta \ll 1 \quad \text{TO GET}$$

$$d\zeta/d\tau = - A''(m_e)/A(m_e)\, \zeta.$$ SINCE $A(m_e) > 0$ WE CONCLUDE THAT

m_e is stable when $A''(m_e) > 0 \rightarrow$ LOCAL MIN OF CROSS-SECTIONAL AREA

m_e IS UNSTABLE WHEN $A''(m_e) < 0 \rightarrow$ LOCAL MAX OF CROSS-SECTION.

THUS IF THE CHANNEL CROSS-SECTION (SIDEVIEW) LOOKS LIKE



US = unstable
S: stable.

<u>NOTE:</u>
- EQUILIBRIUM ARE AT DOTTED LINES AT $x = m_{e_1}, m_{e_2}, m_{e_3}$
- m_{e_2} is stable (LOCAL MIN OF $A(x)$), BUT m_{e_1} AND m_{e_3} ARE UNSTABLE

- WE CONCLUDE THAT IF $m(0)$ SATISFIES $m_{e_1} < m(0) < m_{e_2}$ THEN FOR THE ODE $dm/d\tau = -A'(m)/A(m)$ WE HAVE THAT $m \longrightarrow m_{e_2}$ AND THE WAVE IS <u>PINNED</u> AS $t \to \infty$ TO THE LOCAL MINIMUM OF $A(m)$. THE WAVE CANNOT ENTER INTO THE REGION $m > m_{e_2}$.

GENERAL CONSIDERATIONS: INTERNAL LAYER SOLUTIONS

WE CONSIDER

$$u_t = \varepsilon^2 u_{xx} + Q(u, x) \qquad 0 < x < 1, \quad t > 0$$

$$u(0,t) = S_-(0) \qquad u(1,t) = S_+(1)$$

WHERE $Q(u,x)$ HAS THE FOLLOWING PROPERTIES:

$$Q(0,x) = 0 \quad \forall x$$

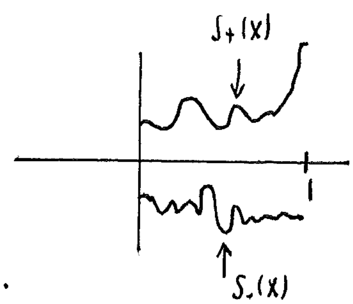
$$Q(S_-, x) = 0 \longrightarrow S_- = S_-(x) \; ; \quad Q(S_+, x) = 0 \longrightarrow S_+ = S_+(x)$$

WE ASSUME THAT

$$S_-(x) < 0 < S_+(x) \qquad \forall x \in [0,1]$$

AND THAT $\quad Q_u(S_\pm, x) < 0 \quad$ WITH $\quad Q_u(0, x) > 0.$

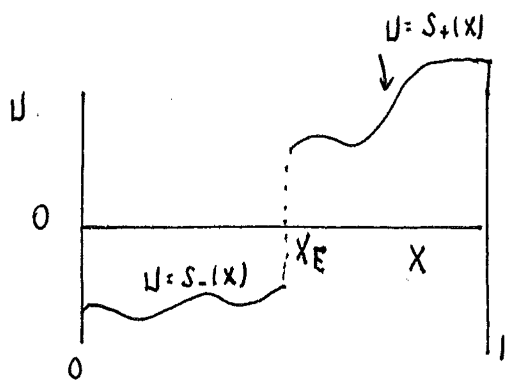
FOR INSTANCE $\quad Q = u\,(S_-(x) - u)(u - S_+(x))$



ASSUME THAT $\exists\, x_E$ SUCH THAT $\displaystyle\int_{S_-(x_E)}^{S_+(x_E)} Q(u, x_E)\, du = 0.$

THEN, WE WILL TRY TO CONSTRUCT A LAYER NEAR $x = x_E$.

THE OUTER SOLUTIONS THAT ARE MATCHABLE ARE $u = S_+(x)$ AND $u = S_-(x)$.



IN THE INNER REGION WE EXPAND $y = \varepsilon^{-1}(x - x_0)$ WITH

$$U = U_0(y) + \varepsilon\, U_1(y) + \dots$$

$$x_0 = x_E + \varepsilon\, x_1 + \dots$$

WE WANT $U(0) = 0$ TO DEFINE LOCATION OF LAYER AS 0 LEVEL SET.

WE OBTAIN

$$U_{0yy} + \varepsilon U_{1yy} + Q(U_0 + \varepsilon U_1, X_0 + \varepsilon y) = 0$$

WITH $X_0 = X_E + \varepsilon X_1$ WE OBTAIN FROM TAYLOR SERIES

$$0 = U_{0yy} + \varepsilon U_{1yy} + Q(U_0, X_E) + \varepsilon U_1 Q_U(U_0, X_E) + \varepsilon(X_1 + y) Q_X(U_0, X_E)$$

COLLECTING POWERS OF ε:

$$U_{0yy} + Q(U_0, X_E) = 0$$

$$U_{1yy} + Q_U(U_0, X_E) U_1 = -(X_1 + y) Q_X(U_0, X_E).$$

NOW THE OUTER SOLUTION HAS

$$S_\pm(X) = S_\pm(X_E + \varepsilon(X_1 + y)) = S_\pm(X_E) + \varepsilon(X_1 + y) S_\pm'(X_E) + \cdots$$

THEREFORE OUR TWO PROBLEMS READ:

$$(1)\quad \begin{cases} U_{0yy} + Q(U_0, X_E) = 0 & -\infty < y < \infty \\ U_0 \to S_+(X_E) & \text{AS } y \to \infty \\ U_0 \to S_-(X_E) & \text{AS } y \to -\infty \\ U_0(0) = 0 \end{cases}$$

$$(2)\quad \begin{cases} LU_1 \equiv U_{1yy} + Q_U(U_0, X_E) U_1 = -(X_1 + y) Q_X(U_0, X_E) \\ U_1 \to (X_1 + y) S_+'(X_E) & \text{AS } y \to \infty \\ U_1 \to (X_1 + y) S_-'(X_E) & \text{AS } y \to -\infty \\ U_1(0) = 0. \end{cases}$$

$\exists$ A UNIQUE SOLUTION TO (1) WITH

$$U_0' \to 0 \quad \text{exponentially as } y \to \pm\infty.$$

NOW $L U_0' = 0$. WE MULTIPLY AND USE LAGRANGE'S IDENTITY

$$\int_{-\infty}^{\infty} U_1 \, L U_0' \, dy - \int_{-\infty}^{\infty} U_0' L U_1 \, dy = \lim_{y \to \infty} \left[U_1 U_0'' - U_0' U_1' \right]$$

NOW $U_1 = O(|y|)$ AS $y \to \pm\infty$, BUT $U_0', U_0'' \to 0$ exponentially.

HENCE $\int_{-\infty}^{\infty} U_0' \, L U_1 \, dy = 0$

$$\longrightarrow \quad \int_{-\infty}^{\infty} (X_1 + y) \, Q_X(U_0, X_E) \, U_0' \, dy = 0.$$

$$\longrightarrow \quad X_1 \int_{-\infty}^{\infty} Q_X(U_0, X_E) \, U_0' \, dy = -\int_{-\infty}^{\infty} y \, Q_X(U_0, X_E) \, U_0' \, dy$$

$$X_1 \int_{S_-(X_E)}^{S_+(X_E)} Q_X(U_0, X_E) \, dU_0 = -\int_{-\infty}^{\infty} y \, Q_X(U_0, X_E) \, U_0' \, dy.$$

NOTICE THAT IF U_0 IS ODD AND $Q_X(U_0; \;) = +Q_X(-U_0; \;)$

THEN $X_1 = 0$.

THIS DETERMINES X_1. TO MAKE THE SOLUTION UNIQUE,

WE WRITE $\quad U_1(y) = U_{1p}(y) + \alpha U_0' \qquad \alpha$ (arbitrary)

where U_{1p} IS A PARTICULAR SOLUTION. SETTING $U_1(0) = 0$ GIVES

$$\alpha = -U_{1p}(0) / U_0'(0).$$

STABILITY OF THIS SOLUTION

WE LET $U_\varepsilon = U_0 + \varepsilon U_1 + \ldots$ AND $y = \dfrac{X - X_0(\varepsilon)}{\varepsilon}$ WITH $X_0(\varepsilon) = X_E + \varepsilon X_1 + \ldots$

WE LINEARIZE TO OBTAIN $U = U_\varepsilon + e^{\lambda t}\phi$ SO THAT

$$L_\varepsilon \phi = \phi_{yy} + Q_U^{\varepsilon}\phi = \lambda\phi$$

$$\phi \longrightarrow 0 \text{ AS } |y| \longrightarrow \infty$$

HERE WE HAVE $Q_U^{\varepsilon} = Q_U\left[U_\varepsilon(y);\, X_0(\varepsilon) + \varepsilon y\right]$.

WE WRITE $Q_U^{0} \equiv Q_U(U_0;\, X_E)$.

THE EIGENVALUE PROBLEM IS

$$L_0\phi = \phi_{yy} + Q_U^{0}\phi = (Q_U^{0} - Q_U^{\varepsilon})\phi + \lambda\phi$$

NOW $L_0 U_0' = 0$ WITH $(U_{0y})_{yy} + Q_U(U_0; X_E)\, U_{0y} = 0$

THEREFORE MULTIPLYING BY U_0' WE GET

$$0 = \int_{-\infty}^{\infty}\left(U_0' L_0\phi - \phi L_0 U_0'\right)dy = \int_{-\infty}^{\infty} U_0'(Q_U^{0} - Q_U^{\varepsilon})\phi + \lambda\int_{-\infty}^{\infty}\phi\, U_0'\, dy.$$

NOW TO LEADING ORDER $\phi \sim U_0' + O(\varepsilon)$ AND WE EXPAND

$$Q_U^{\varepsilon} - Q_U^{0} = \varepsilon Q_{UU}^{0} U_1 + \varepsilon Q_{XU}^{0}(X_1 + y) + \ldots$$

THIS YIELDS THAT $\displaystyle\int_{-\infty}^{\infty}\phi\, U_0'(Q_U^{0} - Q_U^{\varepsilon})\,dy = -\varepsilon\int_{-\infty}\phi\, U_0'\left(Q_{UU}^{0} U_1 + (X_1 + y)\, Q_{UX}^{0}\right)dy$

THIS YIELDS WITH $\phi \sim U_0'$ THAT

$$(*)\qquad \lambda\int_{-\infty}^{\infty}(U_0')^2\, dy \sim \varepsilon\int_{-\infty}^{\infty}(U_0')^2\left[Q_{UU}^{0} U_1 + (X_1 + y)\, Q_{UX}^{0}\right]dy.$$

NOW RECALL FROM THE CONSTRUCTION OF THE EQUILIBRIUM SOLUTION THAT $U_{1yy} + Q_U^{0} U_1 = -(X_1 + y)\, Q_X^{0}$

WE DIFFERENTIATE WRT y TO OBTAIN

$$LU_{1y} \equiv (U_{1y})_{yy} + \overset{\circ}{Q_U}(U_{1y}) = -\left[\overset{\circ}{Q_{UU}}\, U_{0y} U_1 + (X_1+y)\overset{\circ}{Q_{UX}}\, U_{0y}\right] - \overset{\circ}{Q_X}$$

BUT $LU_{0y} = 0$ SO THAT THE SOLVABILITY CONDITION YIELDS:

$$\int_{-\infty}^{\infty} U_0' \, L U_{1y}\, dy = 0 \;\longrightarrow\; \int_{-\infty}^{\infty} \overset{\circ}{Q_X}\, U_{0y}\, dy = -\int_{-\infty}^{\infty} (U_0')^2 \left(\overset{\circ}{Q_{UU}}\, U_1 + (X_1+y)\overset{\circ}{Q_{UX}} \right) dy.$$

THIS YIELDS THE EIGENVALUE FROM (*) ON PREVIOUS PAGE

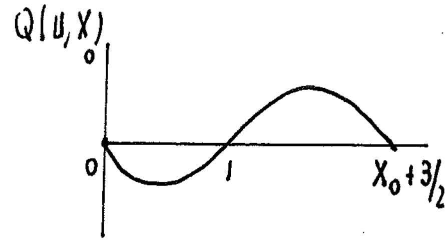
$$\lambda \sim -\varepsilon \frac{\int_{-\infty}^{\infty} \overset{\circ}{Q_X}\, U_{0y}\, dy}{\int_{-\infty}^{\infty} (U_{0y})^2\, dy} \sim -\varepsilon \frac{\int_{S_-(X_E)}^{S_+(X_E)} \overset{\circ}{Q_X}[U_0, X_E]\, dU_0}{\int_{-\infty}^{\infty} (U_{0y})^2\, dy}.$$

WE CONCLUDE THAT $\lambda = O(\varepsilon)$ AND THAT WE HAVE STABILITY IFF $\displaystyle\int_{S_-(X_E)}^{S_+(X_E)} Q_X[U_0, X_E]\, dU_0 > 0.$

EXAMPLE $u_t = \varepsilon^2 u_{xx} + Q(u,x)$

$$u(0,t) = 0, \qquad u(1,t) = 5/2$$

$$Q(u,x) = u(1-u)\left(u - (x + 3/2)\right).$$



EQUILIBRIUM THE OUTER SOLUTION IS

$$u_0 = \begin{cases} 0, & 0 < x < x_E \\ x + 3/2, & x_E < x < 1 \end{cases} \qquad \begin{aligned} S_-(x) &= 0 \\ S_+(x) &= x + 3/2 \end{aligned}$$

IFF $x_E = 1/2$ THEN $\displaystyle\int_{S_-(x_E)}^{S_+(x_E)} u(1-u)(u-2)\,du = \int_0^2 u(1-u)(u-2)\,du = 0.$

THUS $x_E = 1/2$ IS THE EQUILIBRIUM POSITION.

WE THEN CALCULATE

$$Q_x(u,x) = -u(1-u) = u(u-1)$$

WE CALCULATE $S_-(x_E) = 0$, $S_+(x_E) = 2$, $\displaystyle\int_0^2 Q_x(U_0, x_E)\,dU_0 = \int_0^2 u(u-1)\,du = \left(\frac{u^3}{2} - \frac{u^2}{2}\right)\Big|_0^2 = 2 > 0.$

THUS THE SMALL EIGENVALUE $\lambda = O(\varepsilon)$ SATISFIES $\lambda < 0$

AND HENCE THE EQUILIBRIUM SOLUTION IS STABLE.

DYNAMICS WE SUPPOSE THAT WE HAVE A TRANSITION LAYER THAT MOVES IN TIME FOR THE TIME-DEPENDENT PROBLEM:

$$u(x,\tau) = U(z,\tau) \quad \text{WITH} \quad \tau = \varepsilon t, \quad z = \frac{x - m(\tau)}{\varepsilon}$$

NOTICE THE SLOW-TIME SCALE $\tau = O(\varepsilon)$ IS CHOSEN ON THE ORDER OF THE SMALL EIGENVALUE $\lambda = O(\varepsilon)$.

WE CALCULATE $U_t = \varepsilon \bar{U}_\tau - \dot{m}\bar{U}_Z$

THEN $\varepsilon \bar{U}_\tau - \dot{m}\bar{U}_Z = \bar{U}_{ZZ} + Q(\bar{U},m) + Q_x(\bar{U},m)\,\varepsilon Z + \ldots$

THIS LEADS TO THE NONLINEAR EIGENVALUE PROBLEM

$$\begin{cases} \bar{U}_{ZZ} + \dot{m}\,\bar{U}_Z + Q(\bar{U},m) = 0, & -\infty < Z < \infty \\ \bar{U} \to S_+(m) = m + 3/2 & \text{AS } Z \to +\infty \\ \bar{U} \to S_-(m) = 0 & \text{AS } Z \to -\infty \\ \bar{U}(0) = 1 & \text{FIXES THE TRANSLATION INVARIANCE AND "FRONT" LOCATION} \end{cases}$$

RECALL THAT $Q(U) = U(1-U)(U - S_+(m))$ WITH $S_+ > 1$ $\forall m$ IN $0 < m < 1$.

__LEMMA__ CONSIDER

$$(**)\begin{cases} \bar{U}'' - c\bar{U}' + f(\bar{U}) = 0 & \text{WITH } -\infty < Z < \infty \\ \bar{U}(-\infty) = U_1, \quad \bar{U}(\infty) = U_3 & \text{WITH } f(U) = (\bar{U}-U_1)(\bar{U}-U_2)(\bar{U}-U_3) \end{cases}$$

AND $U_1 < U_2 < U_3$. THEN THE SOLUTION WITH $\bar{U}(0) = (U_1 + U_3)/2$ IS

$$\bar{U}(Z) = \frac{U_1 + U_3 \exp(\lambda Z)}{1 + \exp(\lambda Z)} \quad \text{WITH } \lambda = \frac{U_3 - U_1}{\sqrt{2}} \quad \text{AND} \quad C = \sqrt{2}\left[\frac{1}{2}(U_1 + U_3) - U_2\right].$$

__PROOF__ CONSIDER $\bar{U}' = +B(\bar{U} - U_1)(\bar{U} - U_3)$. THEN DIFFERENTIATE

$$\bar{U}'' = +B(2\bar{U} - U_1 - U_3)\bar{U}' = B^2(2\bar{U} - U_1 - U_3)(\bar{U} - U_1)(\bar{U} - U_3).$$

WE CALCULATE

$$\begin{aligned} \mathcal{L}(\bar{U}) \equiv \bar{U}'' - c\bar{U}' + f(\bar{U}) &= B^2(2\bar{U} - U_1 - U_3)(\bar{U} - U_1)(\bar{U} - U_3) - cB(\bar{U} - U_1)(\bar{U} - U_3) \\ &\quad + (\bar{U} - U_1)(\bar{U} - U_3)(\bar{U} - U_2) \\ &= (\bar{U} - U_1)(\bar{U} - U_3)\left[B^2(2\bar{U} - U_1 - U_3) - cB - (\bar{U} - U_2)\right] \end{aligned}$$

NOW CHOOSE B^2 AND C TO ELIMINATE TERM IN SQUARE BRACKET.

CHOOSE $2B^2 = 1$ AND $-B^2(U_1 + U_3) - CB + U_2 = 0$.

THUS

$$(+)\begin{cases} CB = U_2 - B^2(U_1 + U_3). \\ 2B^2 = 1. \end{cases}$$

WE CONCLUDE THAT IF $\bar U$ SATISFIES

$$(++)\quad \Big\}\quad \bar U' = B(\bar U - U_1)(\bar U - U_3)$$

THEN $\bar U$ ALSO SATISFIES $(**)$ PROVIDED THAT

$$cB = U_2 - B^2(U_1 + U_3)\ , \quad 2B^2 = 1.$$

IF WE WANT $\bar U \to U_3$ AS $z \to +\infty$ WE MUST HAVE $B<0$. THUS,

$B = -1/\sqrt{2}$ AND $c = \sqrt{2}\left[\frac{1}{2}(U_1 + U_3) - U_2\right]$ IS THE WAVE SPEED.

WE INTEGRATE $(++)$ TO GET

$$\frac{d\bar U}{(\bar U - U_1)(\bar U - U_3)} = B\,dz \qquad \text{WITH} \quad B = -1/\sqrt{2}.$$

THUS

$$\frac{1}{U_3 - U_1}\left(\frac{1}{\bar U - U_3} - \frac{1}{\bar U - U_1}\right) d\bar U = B\,dz$$

SINCE $U_1 < \bar U < U_3$ THEN

$$\frac{1}{U_3 - U_1}\log\left(\frac{U_3 - \bar U}{\bar U - U_1}\right) = B(z - z_0)$$

SO

$$\log\left(\frac{U_3 - \bar U}{\bar U - U_1}\right) = (U_3 - U_1)B(z - z_0).$$

OR

$$\frac{U_3 - \bar U}{\bar U - U_1} = \exp\Big((U_3 - U_1)B(z - z_0)\Big).$$

WE GET

$$\bar U(z) = \frac{U_3 + \exp\big(\lambda_0(z - z_0)\big)U_1}{1 + \exp\big(\lambda_0(z - z_0)\big)} \qquad \lambda_0 = (U_3 - U_1)B = -\frac{(U_3 - U_1)}{\sqrt{2}}.$$

OR EQUIVALENTLY

$$\bar U(z) = \frac{U_1 + U_3\exp\big(\lambda(z - z_0)\big)}{1 + \exp\big(\lambda(z - z_0)\big)} \qquad \lambda = -\lambda_0 = \frac{U_3 - U_1}{\sqrt{2}},$$

APPLYING THIS RESULT TO OUR PROBLEM WE HAVE

$$C = -\dot{m}, \quad U_1 = 0, \quad U_2 = 1, \quad U_3 = m + 3/2.$$

THUS $\quad C = \sqrt{2}\left[\frac{1}{2}(U_1 + U_3) - U_2\right]$ IS

$$\frac{dm}{d\tau} = -\frac{\sqrt{2}}{2}\left[m - 1/2\right] \quad \text{WITH} \quad m(0) = m_0 \quad \text{INITIAL CONDITION.}$$

WE RECALL THAT WITH $\tau = \varepsilon t$, $\quad m = \frac{1}{2} + \left(m_0 - \frac{1}{2}\right)e^{-\varepsilon t/\sqrt{2}} \to 1/2$ AS $t \to \infty$.

SLOW DRIFT TO EQUILIBRIUM STATE $m_{EQ} = 1/2$ AS $t \to \infty$ WITH $O(\varepsilon)$ speed.

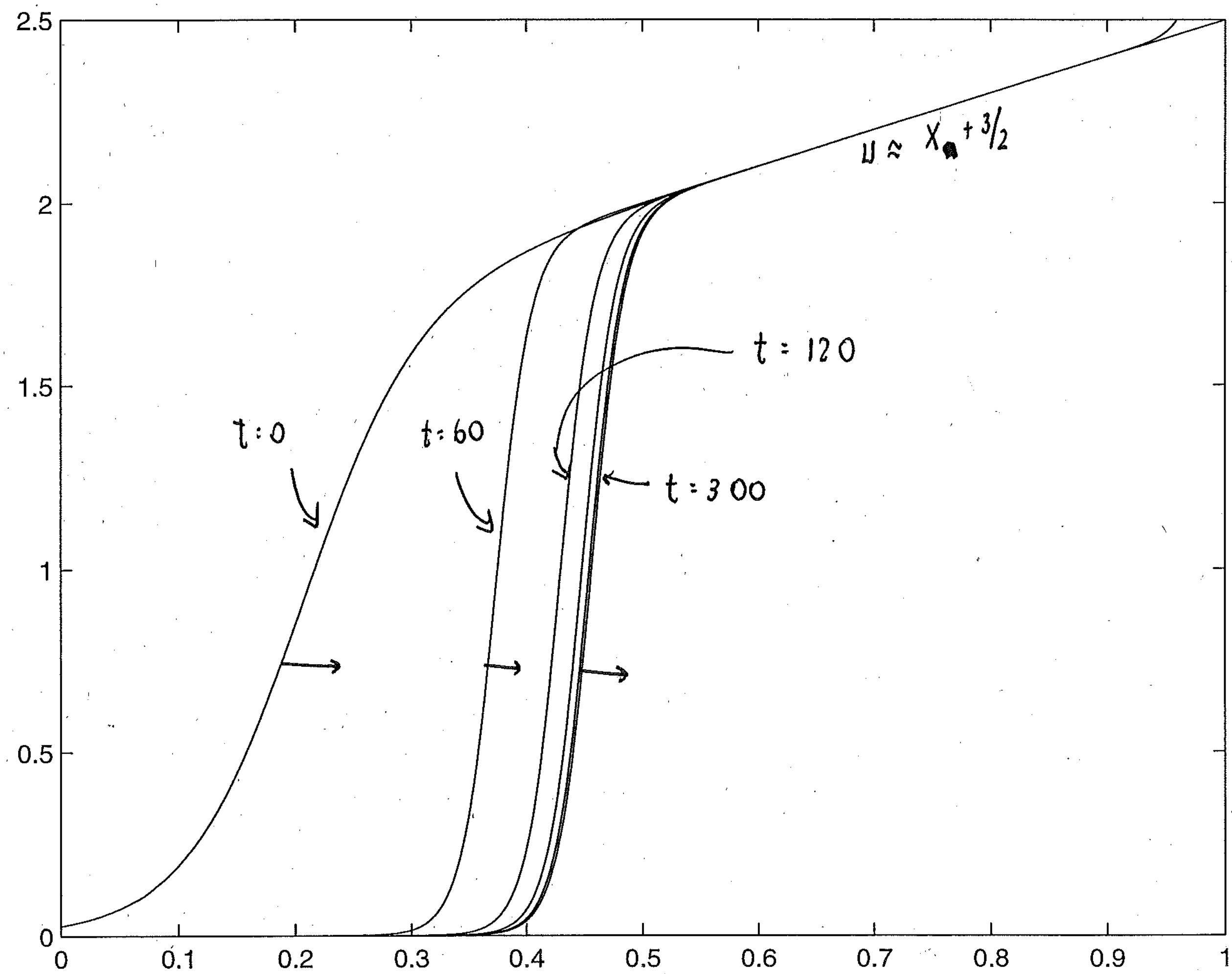
FULL NUMERICAL SOLUTION

$$u_t = \varepsilon^2 u_{xx} + f(u,x) \qquad f(u,x) = u(1-u)\left(u - (x + 3/2)\right).$$

$$u(0,t) = 0 \qquad u(1,t) = 5/2$$

$$u(x,0) = \frac{(x+3/2)}{2}\left[1 + \tanh\left((x-.20)/.10\right)\right]$$

$$\varepsilon = .02$$



DEFINE $x_0(t)$ BY

$$u(x,t) = \frac{1}{2}\left[x_0 + 3/2\right] \to 1/2 \text{ of the jump.}$$

THEN $\quad u[x_0, t] = \frac{1}{2}\left[x_0 + 3/2\right]$

ANALYTICALLY CAN SHOW THAT $\quad \dfrac{dx_0}{d\tau} = -\dfrac{1}{\sqrt{2}}\left[x_0 - 1/2\right] \qquad \tau = \varepsilon t$

SO $\quad x_0 \to 1/2 \quad$ AS $\quad \tau \to \infty$