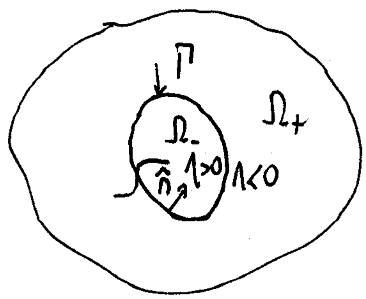
INTERFACE EVOLUTION IN A CHEMOTAXIS MODEL

$$\left.\begin{aligned} U_t &= \nabla\cdot[\,\varepsilon \nabla U - f(U)\nabla V\,] \quad \text{in } \Omega \in \mathbb{R}^2 \\ V_t &= D\,\Delta V + \sigma U - V \quad \text{in } \Omega \end{aligned}\right\} \quad (C1)$$

WITH $\partial_n U = 0$ AND $\partial_n V = 0$ ON $\partial\Omega$. HERE $f(U) = U(1-U)$ WHICH MODELS VOLUME FILLING EFFECT IN AGGREGATION.

WE WILL TRY TO CONSTRUCT A SLOWLY EVOLVING INTERFACE SOLUTION WITH $U \cong 1$ INSIDE Ω_- AND $U \cong 0$ OUTSIDE IN $\Omega_+ = \Omega \setminus \Omega_-$.



NOW

$$U_t = \varepsilon\,\Delta U - f(U)\,\Delta V - f'(U)\,\nabla U\cdot\nabla V.$$

RECALL OUR TANGENTIAL/NORMAL COORDINATE SYSTEM AND THAT $\nabla g = \dfrac{g_s}{1-\kappa\eta}\,\hat{t} + g_\eta\,\hat{n}$

FOR SOME GENERIC FUNCTION g.

IN THIS WAY WE OBTAIN NEAR Γ THAT

$$U_t = \varepsilon\left[U_{\eta\eta} - \frac{\kappa}{1-\kappa\eta}U_\eta + \frac{1}{1-\kappa\eta}\partial_s\left(\frac{U_s}{1-\kappa\eta}\right)\right] - f(U)\left[V_{\eta\eta} - \frac{\kappa}{1-\kappa\eta}V_\eta + \frac{1}{1-\kappa\eta}\partial_s\left(\frac{V_s}{1-\kappa\eta}\right)\right]$$
$$- f'(U)\left[\frac{U_s V_s}{(1-\kappa\eta)^2} + U_\eta V_\eta\right] \qquad (C2)_1$$

$$V_t = D\left[V_{\eta\eta} - \frac{\kappa}{1-\kappa\eta}V_\eta + \frac{1}{1-\kappa\eta}\partial_s\left(\frac{V_s}{1-\kappa\eta}\right)\right] + \sigma U - V. \qquad (C2)_2$$

WE WILL LOOK FOR A SOLUTION THAT DEPENDS ON SLOW TIME AS $\tau = \varepsilon^p t$, SO THAT $U_t = (U_\eta \dot{\eta} + U_s \dot{s})\,\varepsilon^p$, $V_t = \varepsilon^p\,(V_\eta\dot{\eta} + V_s \dot{s})$, WITH $\dot{\eta} = d\eta/d\tau$, $\dot{s} = ds/d\tau$.

NOW LET $\hat{\eta} = \eta/\varepsilon$ AND INTRODUCE INNER LAYER VARIABLES $\bar{U}$, $\bar{V}$.

FROM $\bar{V}$ EQUATION:

$$\varepsilon^{p-1}\,\bar{V}_{\tilde{\lambda}}\,\dot{\lambda} + \varepsilon^{p}\,\bar{V}_{S}\,\dot{S} = D\left[\frac{1}{\varepsilon^{2}}\bar{V}_{\hat{\lambda}\hat{\lambda}} - \frac{\kappa}{\varepsilon}\bar{V}_{\tilde{\lambda}} + O(1)\right] + O(1). \qquad (C3)$$

ASSUMING $p>0$ (SLOW TIME) WE EXPAND

$$\bar{V} = \bar{V}_0 + \varepsilon\bar{V}_1 + \dots$$

TO OBTAIN

$$\bar{V}_{0\hat{\lambda}\hat{\lambda}} = 0 \qquad : O(1/\varepsilon^{2}) \qquad (C4)_1$$

$$\bar{V}_{1\hat{\lambda}\hat{\lambda}} = \kappa\,\bar{V}_{0\tilde{\lambda}} \qquad : O(1/\varepsilon) \qquad (C4)_2$$

NOW SINCE WE MUST MATCH TO AN OUTER SOLUTION THAT IS BOUNDED WE TAKE $\bar{V}_o = A_0(S)$

AND AT NEXT ORDER SINCE $\bar{V}_{0\hat{\lambda}} = 0$ WE OBTAIN

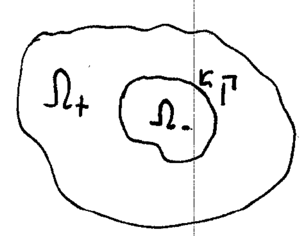
$$\bar{V}_1 = A_1(S)\tilde{\lambda} + B_1(S)$$

AS SUCH A 2-TERM EXPANSION FOR $\bar{V}$ IS SIMPLY

$$\bar{V} \sim A_0(S) + \varepsilon\left[A_1(S)\tilde{\lambda} + B_1(S)\right]. \qquad (C5).$$

NOW TO IDENTIFY $A_0(S)$ AND $A_1(S)$ WE CONSIDER THE OUTER PROBLEM FOR V ON THE TIME-SCALE $\tau = \varepsilon^{p}t$. SINCE $U \doteq 1$ IN Ω_- AND $U \doteq 0$ IN Ω_+ WE HAVE

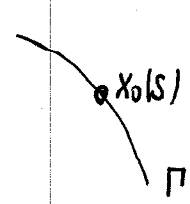
$$(C6)\quad \begin{cases} D\Delta V - V = \begin{cases} -\sigma & \text{IN } \Omega_- \\ 0 & \text{IN } \Omega_+ \end{cases} \\ \partial_n V = 0 \quad \text{ON } \partial\Omega. \end{cases}$$



WE SEEK A $C^1(\Omega)$ SOLUTION.

NOW TO MATCH INNER AND OUTER SOLUTIONS FOR V. AS $\underline{X} \to \underline{X}_0(S) \in \Gamma$ WE HAVE

$$V = V\big|_{\Gamma} + \nabla V\big|_{\Gamma} \cdot (\underline{X} - \underline{X}_0) + \dots$$



BUT $\nabla V = \dfrac{V_S}{1-\kappa\lambda}\,\hat{\underline{t}}(S) + V_\lambda\,\hat{\underline{n}}(S)$ AND $\underline{X} = \underline{X}_0(S) + \lambda\,\hat{\underline{n}}(S)$

SO $\nabla V \cdot (\underline{X} - \underline{X}_0) = \lambda\, V_\lambda .$

WE CONCLUDE THAT AS $\underline{x} \to \Gamma$,

$$V \sim V\big|_{\Gamma} + \eta\, V_{\eta}\big|_{\eta=0} + \cdots$$

NOW FROM (C5) THE FAR-FIELD FORM OF INNER SOLUTION WRITTEN IN TERMS OF $\hat{\eta} = \eta/\varepsilon$ IS $\bar{V} \sim A_0(s) + \varepsilon\left[A_1(s)\frac{\eta}{\varepsilon} + B_1\right] \sim A_0(s) + A_1(s)\,\eta + O(\varepsilon)$.

WE CONCLUDE THAT

$$A_0(s) = V\big|_{\Gamma}, \qquad A_1(s) = V_{\eta}\big|_{\eta=0} = \nabla V \cdot \hat{\underline{n}}\,\big|_{\Gamma} \qquad (C7)$$

WHERE V IS THE SOLUTION TO (C6). THIS IDENTIFIES $A_0(s)$ AND $A_1(s)$.

NOW WE MUST PROCEED TO THE INNER PROBLEM FOR $\bar{U}$ GIVEN BY

$$\varepsilon^{p-1}\bar{U}_{\hat{\eta}}\,\dot{\eta} + \varepsilon^{p}\bar{U}_s\,\dot{s} = \varepsilon\left[\frac{\bar{U}_{\hat{\eta}\hat{\eta}}}{\varepsilon^2} - \frac{\kappa}{1-\kappa\varepsilon\hat{\eta}}\frac{1}{\varepsilon}\bar{U}_{\hat{\eta}} + \frac{1}{1-\kappa\varepsilon\hat{\eta}}\partial_s\left[\frac{\bar{U}_s}{1-\kappa\varepsilon\hat{\eta}}\right]\right]$$
$$- f(\bar{U})\left[\frac{1}{\varepsilon^2}\bar{V}_{\hat{\eta}\hat{\eta}} - \frac{\kappa}{1-\kappa\varepsilon\hat{\eta}}\frac{1}{\varepsilon}\bar{V}_{\hat{\eta}} + \frac{1}{1-\kappa\varepsilon\hat{\eta}}\partial_s\left[\frac{\bar{V}_s}{1-\kappa\varepsilon\hat{\eta}}\right]\right]$$
$$- f'(\bar{U})\left[\frac{1}{\varepsilon^2}\bar{U}_{\hat{\eta}}\bar{V}_{\hat{\eta}} + \frac{\bar{U}_s \bar{V}_s}{1-\kappa\varepsilon\hat{\eta}}\right]$$

NOW WE EXPAND $\bar{V} = \bar{V}_0 + \varepsilon\bar{V}_1 + \varepsilon^2\bar{V}_2$, $\bar{U} = \bar{U}_0 + \varepsilon\bar{U}_1 + \cdots$ WHERE WE RECALL THAT $\bar{V}_{0\hat{\eta}\hat{\eta}} = 0$, $\bar{V}_{1\hat{\eta}\hat{\eta}} = 0$. WE NEED TO COLLECT AND RETAIN TERMS OF ORDER $O(1/\varepsilon)$ AND $O(1)$. WE WILL CHOOSE $p = 1$ TO BALANCE THE TERMS AT $O(1)$. RECALLING $\bar{V}_0 = A_0(s)$, $\bar{V}_1 = A_1(s)\hat{\eta} + B_1(s)$, WE GET

$$\bar{U}_{0\hat{\eta}}\,\dot{\eta} + O(\varepsilon) = \varepsilon\left[\frac{\bar{U}_{0\hat{\eta}\hat{\eta}}}{\varepsilon^2} + \frac{\bar{U}_{1\hat{\eta}\hat{\eta}}}{\varepsilon} - \frac{\kappa}{\varepsilon}\bar{U}_{0\hat{\eta}} + \cdots\right]$$
$$- f(\bar{U}_0)\left[\bar{V}_{2\hat{\eta}\hat{\eta}} - \kappa\bar{V}_{1\hat{\eta}} + \partial_{ss}\bar{V}_0 + \cdots\right]$$
$$- \left[f'(\bar{U}_0) + \varepsilon\bar{U}_1 f''(\bar{U}_0) + \cdots\right]\left[\frac{1}{\varepsilon^2}\left((\bar{U}_{0\hat{\eta}} + \varepsilon\bar{U}_{1\hat{\eta}} + \cdots)(\overset{=0}{\bar{V}_{0\hat{\eta}}} + \varepsilon\bar{V}_{1\hat{\eta}} + \varepsilon^2\bar{V}_{2\hat{\eta}} + \cdots)\right) + \bar{U}_{0s}\bar{V}_{0s} + \cdots\right]$$

WE OBTAIN THAT

$$U_{0\hat{\Lambda}}\,\dot{\Lambda} = \frac{1}{\varepsilon}\bar{U}_{0\hat{\Lambda}\hat{\Lambda}} + \bar{U}_{1\hat{\Lambda}\hat{\Lambda}} - \kappa \bar{U}_{0\hat{\Lambda}} - f(\bar{U}_0)\left[\partial_{\hat{\Lambda}\hat{\Lambda}}\bar{V}_2 - \kappa\bar{V}_{1\hat{\Lambda}} + \partial_{ss}\bar{V}_0\right]$$
$$-\left[f'(\bar{U}_0) + \varepsilon \bar{U}_1 f''(\bar{U}_0) + \dots\right]\left[\frac{1}{\varepsilon}\bar{U}_{0\hat{\Lambda}}\bar{V}_{1\hat{\Lambda}} + \bar{U}_{1\hat{\Lambda}}\bar{V}_{1\hat{\Lambda}} + \bar{U}_{0s}\bar{V}_{0s} + \bar{U}_{0\hat{\Lambda}}\bar{V}_{2\hat{\Lambda}}\right]$$

THIS YIELDS THAT

$$U_{0\hat{\Lambda}}\,\dot{\Lambda} = \frac{1}{\varepsilon}\left[\bar{U}_{0\hat{\Lambda}\hat{\Lambda}} - f'(\bar{U}_0)\bar{V}_{1\hat{\Lambda}}\bar{U}_{0\hat{\Lambda}}\right]$$
$$+\left[\bar{U}_{1\hat{\Lambda}\hat{\Lambda}} - f'(\bar{U}_0)\bar{V}_{1\hat{\Lambda}}\bar{U}_{1\hat{\Lambda}} - \kappa\bar{U}_{0\hat{\Lambda}} - f(\bar{U}_0)\left[\bar{V}_{2\hat{\Lambda}\hat{\Lambda}} - \kappa\bar{V}_{1\hat{\Lambda}} + \bar{V}_{0ss}\right]\right.$$
$$\left. - f'(\bar{U}_0)\bar{U}_{0\hat{\Lambda}}\bar{V}_{2\hat{\Lambda}} - f'(\bar{U}_0)\bar{U}_{0s}\bar{V}_{0s} - f''(\bar{U}_0)\bar{U}_1\bar{U}_{0\hat{\Lambda}}\bar{V}_{1\hat{\Lambda}}\right]$$

THIS GIVES THE LEADING ORDER PROBLEM AT $O\left(\frac{1}{\varepsilon}\right)$ AND NEXT ORDER PROBLEM AT $O(1)$:

$$(C8)_1 \quad \bar{U}_{0\hat{\Lambda}\hat{\Lambda}} - f'(\bar{U}_0)\bar{V}_{1\hat{\Lambda}}\bar{U}_{0\hat{\Lambda}} = 0 \quad \text{IN } -\infty < \hat{\Lambda} < \infty$$

$$(C8)_2 \quad L\bar{U}_1 \equiv \bar{U}_{1\hat{\Lambda}\hat{\Lambda}} - f'(\bar{U}_0)\bar{V}_{1\hat{\Lambda}}\bar{U}_{1\hat{\Lambda}} - f''(\bar{U}_0)\bar{U}_1\bar{U}_{0\hat{\Lambda}}\bar{V}_{1\hat{\Lambda}}$$
$$= U_{0\hat{\Lambda}}\,\dot{\Lambda} + \kappa U_{0\hat{\Lambda}} + f(\bar{U}_0)\left[\bar{V}_{2\hat{\Lambda}\hat{\Lambda}} - \kappa\bar{V}_{1\hat{\Lambda}} + \bar{V}_{0ss}\right]$$
$$f'(\bar{U}_0)\bar{U}_{0\hat{\Lambda}}\bar{V}_{2\hat{\Lambda}} + f'(\bar{U}_0)\bar{U}_{0s}\bar{V}_{0s} \quad \text{ON } -\infty < \hat{\Lambda} < \infty.$$

WE RECALL THAT $f(\bar{U}_0) = \bar{U}_0(1-\bar{U}_0)$ AND THAT

$$\bar{V}_0 = A_0(s), \quad \bar{V}_{1\hat{\Lambda}} = A_1(s)$$

WHERE A_0 AND A_1 ARE DETERMINED BY (C8) FROM OUTER SOLUTION FOR V GIVEN IN (C5). AS SUCH WE CAN WRITE THE PROBLEMS $(C8)_1$ AND $(C8)_2$ USING THAT $\bar{V}_{1\hat{\Lambda}}$ IS INDEPENDENT OF $\hat{\Lambda}$ AS

$$(C9)_1 \qquad \frac{d}{d\hat{\Lambda}}\left[\bar{U}_{0\hat{\Lambda}} - f(\bar{U}_0)A_1(s)\right] = 0$$

$$(C9)_2 \qquad L\bar{U}_1 \equiv \bar{U}_{1\hat{\Lambda}\hat{\Lambda}} - \left[f'(\bar{U}_0)A_1(s)\bar{U}_1\right]_{\hat{\Lambda}}$$
$$= U_{0\hat{\Lambda}}\dot{\Lambda} + \kappa U_{0\hat{\Lambda}} + f(\bar{U}_0)\left[\bar{V}_{2\hat{\Lambda}\hat{\Lambda}} - \kappa A_1(s) + A_{0ss}\right]$$
$$+ f'(\bar{U}_0)\bar{U}_{0s}A_{0s} + f'(\bar{U}_0)\bar{U}_{0\hat{\Lambda}}\bar{V}_{2\hat{\Lambda}}$$

NOW INTEGRATING ONCE AND USING $\bar{U}_0 \to 1$ AS $\hat{\Lambda} \to +\infty$ (Ω_-) AND $\bar{U}_0 \to 0$ AS $\hat{\Lambda} \to -\infty$ (Ω_+).

UNDER THE ASSUMPTION THAT $A_1(s) > 0$ ON Γ WE HAVE

$$\bar{U}_{0\hat{\Lambda}} - f(\bar{U}_0)A_1(s) = 0 \quad (C10)$$

SO THAT $$\bar{U}_0 = \frac{\bar{U}_0(0)}{\bar{U}_0(0) + (1-\bar{U}_0(0))e^{-\hat{\Lambda}A_1(s)}} \quad \text{ON } -\infty < \hat{\Lambda} < \infty$$

WHERE WLOG WE CAN TAKE $\bar{U}_0(0) = 1/2$ WHICH DEFINE Γ BY LEVEL CURVE $\bar{U} = 1/2$.

NOW USING $\bar{U}_{0\hat{\Lambda}} - f(\bar{U}_0)A_1(s) = 0$ IN $(C9)_2$ WE OBTAIN THAT $\kappa(\bar{U}_{0\hat{\Lambda}} - f(\bar{U}_0)A_1)$ VANISHES, SO THAT

$$L\bar{U}_1 \equiv \bar{U}_{1\hat{\Lambda}\hat{\Lambda}} - \left[f'(\bar{U}_0)A_1(s)\bar{U}_1\right]_{\hat{\Lambda}} = \bar{U}_{0\hat{\Lambda}}\dot{\Lambda} + f(\bar{U}_0)\left[\bar{V}_{2\hat{\Lambda}\hat{\Lambda}} + A_{0ss}\right] + f'(\bar{U}_0)\bar{U}_{0\hat{\Lambda}}\hat{\bar{V}}_{2\hat{\Lambda}}$$
$$+ f'(\bar{U}_0)\bar{U}_{0s}A_{0s}.$$

WE CAN WRITE THIS MORE COMPACTLY AS

$$(C11) \quad L\bar{U}_1 \equiv \bar{U}_{1\hat{\Lambda}\hat{\Lambda}} - \left[f'(\bar{U}_0)A_1(s)\bar{U}_1\right]_{\hat{\Lambda}} = \bar{U}_{0\hat{\Lambda}}\dot{\Lambda} + \left(f(\bar{U}_0)\bar{V}_{2\hat{\Lambda}}\right)_{\hat{\Lambda}} + \left[f(\bar{U}_0)A_{0s}\right]_s$$

FINALLY USING ODE FOR $\bar{U}_0$ GIVEN IN (C10) GIVEN BY

$$f(\bar{U}_0) = \frac{\bar{U}_{0\hat{\Lambda}}}{A_1(s)} \quad (C12)$$

SO THAT (C6)

$$\left.\begin{aligned} L\bar{U}_1 \equiv \bar{U}_{1\hat{\eta}\hat{\eta}} - [f'(\bar{U}_0)A_1(s)\bar{U}_1]_{\hat{\eta}} &= \bar{U}_{0\hat{\eta}}\,\dot{\eta} + [f(\bar{U}_0)\bar{V}_{2\hat{\eta}}]_{\hat{\eta}} + \left[\frac{\bar{U}_{0\hat{\eta}}A_{0s}}{A_1(s)}\right]_s \\ \text{ON } -\infty < \hat{\eta} < \infty \quad \text{WITH} \quad \bar{U}_1 &\to 0 \text{ AS } \hat{\eta}\to\pm\infty. \end{aligned}\right\} \quad (C13)$$

OBSERVE THAT L IS NOT SELF-ADJOINT. AS SUCH IN OUR SOLVABILITY CONDITION WE MUST CONSIDER THE ADJOINT OPERATOR L^*.

<u>LEMMA</u> LET $L\phi \equiv \phi'' - [b\phi]'$. THEN WITH $L^*\psi \equiv \psi'' + b\psi'$, WE HAVE BY IBP

$$\int_{-\infty}^{\infty} \psi L\phi \, d\hat{\eta} = \psi[\phi' - (b\phi)]\Big|_{-\infty}^{\infty} - \psi'\phi\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \phi L^*\psi \, d\hat{\eta}.$$

SEEKING A SOLUTION FOR WHICH ψ IS BOUNDED AS $\hat{\eta}\to\pm\infty$ FOR $L^*\psi = 0$ WE OBTAIN $\psi \equiv$ CONSTANT, i.e. $\psi = 1$. IF b IS BOUNDED AS $\hat{\eta}\to\pm\infty$ AND WE IMPOSE $\phi\to 0$ AS $\hat{\eta}\to\pm\infty$ WE CONCLUDE THAT $\int_{-\infty}^{\infty}\psi L\phi\,d\hat{\eta} = \int_{-\infty}^{\infty}\phi L^*\psi\,d\hat{\eta}$ FOR FUNCTIONS ϕ AND ψ THAT SATISFY $\phi\to 0$ AS $\hat{\eta}\to\pm\infty$ AND ψ BOUNDED AS $\hat{\eta}\to\pm\infty$. □

USING THIS RESULT AND NOTING THAT $L^*(1) = 0$ WE OBTAIN THE SOLVABILITY CONDITION FOR (C13) THAT $\int_{-\infty}^{\infty}(1)\,L\bar{U}_1\,d\hat{\eta} = \int_{-\infty}^{\infty}\bar{U}_1\,L^*(1)\,d\hat{\eta} = 0.$

AS SUCH
$$\int_{-\infty}^{\infty}\left[\bar{U}_{0\hat{\eta}}\,\dot{\eta} + (f(\bar{U}_0)\bar{V}_{2\hat{\eta}})_{\hat{\eta}} + \left(\frac{\bar{U}_{0\hat{\eta}}A_{0s}}{A_1(s)}\right)_s\right]d\hat{\eta} = 0.$$

WE OBTAIN
$$(\bar{U}_0(\infty) - \bar{U}_0(-\infty))\,\dot{\eta} + \lim_{\hat{\eta}\to+\infty} f(\bar{U}_0)\bar{V}_{2\hat{\eta}} - \lim_{\hat{\eta}\to-\infty} f(\bar{U}_0)\bar{V}_{2\hat{\eta}}$$
$$+ \left[\frac{A_{0s}}{A_1(s)}\left(\int_{-\infty}^{\infty}\bar{U}_{0\hat{\eta}}\,d\hat{\eta}\right)\right]_s = 0$$

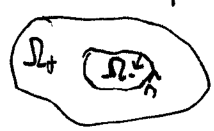
SINCE $f(0) = f(1) = 0$ AND $\bar{U}_0(\infty) = 1$, $\bar{U}_0(-\infty) = 0$ WE OBTAIN THAT LIMIT TERMS CANCEL AND

$$V \equiv \dot{\eta} = -\left[\frac{A_{0s}}{A_1}\right]_s$$

IS THE GEOMETRIC LAW FOR NORMAL VELOCITY V OF Γ. $\Rightarrow$ DIFFUSION ON SURFACE YIELDS V.

RECALLING (C7) WE HAVE $A_0(s) = v\big|_\Gamma$, $A_1(s) = \nabla v\cdot\hat{n}\big|_\Gamma$ WHERE v SATISFIES

$$D\Delta v - v = \begin{cases} -\sigma & \text{IN } \Omega_- \\ 0 & \text{IN } \Omega_+ \end{cases}$$



WITH $\partial_n v = 0$ ON $\partial\Omega$.