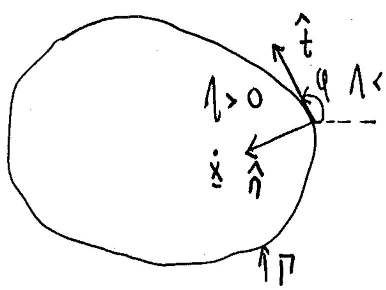
INTERNAL LAYERS AND BOUNDARY FITTED COORDINATES

IN MANY APPLICATIONS WE MUST CONVERT THE LAPLACIAN IN 2-D TO A LOCAL COORDINATE SYSTEM WHERE η IS IN DIRECTION $\perp$ TO SOME INTERFACE Γ AND S IS ARCLENGTH ALONG Γ. THIS WILL GIVE A LOCAL COORDINATE SYSTEM THAT IS ORTHOGONAL.

WE LET η BE SIGNED DISTANCE FROM A POINT $\underline{x}$ TO Γ. OUR CONVENTION IS THAT $\eta > 0$ IF $\underline{x}$ IS INSIDE Γ WHILE $\eta < 0$ IF $\underline{x}$ IS OUTSIDE. WE WILL ASSUME THAT Γ IS SMOOTH SO THAT $\exists$ A UNIT TANGENT $\hat{t}$ AND A UNIT NORMAL $\hat{n}$ AT EACH POINT ON Γ.



LET $(\gamma_1(s), \gamma_2(s))$ PARAMETERIZES Γ WHERE S IS ARCLENGTH ON Γ.

THEN
$$\frac{d\hat{t}}{ds} = \kappa \hat{n}$$

NOTE: $\hat{t} = (\cos\varphi, \sin\varphi)^T$
$d\hat{t}/ds = (-\sin\varphi, \cos\varphi)\underbrace{d\varphi/ds}_{\kappa}$
$= \kappa \hat{n}$.

WHERE κ IS CURVATURE OF Γ AT S WITH $\kappa > 0$ IF Ω IS CONVEX AS SEEN FROM INSIDE.

NOW LET $\underline{x} \equiv (x_1, x_2)$ BE ANY POINT NEAR Γ IN CARTESIAN COORDINATES. THEN WRITE

$$\underline{x} = \underline{\gamma} + \eta \hat{n} \qquad \text{WHERE} \quad \hat{\underline{n}} = (n_1, n_2)^T \quad \text{AND} \quad \underline{\gamma} = (\gamma_1, \gamma_2)^T$$

WHERE BOTH $\gamma_1 = \gamma_1(s)$, $\gamma_2 = \gamma_2(s)$ AND $n_1 = n_1(s)$, $n_2 = n_2(s)$.

THE TANGENT $\hat{t}$ IS $\hat{\underline{t}} = (\gamma_1', \gamma_2')^T$ AND $\hat{\underline{n}} \cdot \hat{\underline{n}} = 1$ (UNIT VECTOR)
$\hat{\underline{t}} \cdot \hat{\underline{t}} = 1$ (UNIT VECTOR)

THEREFORE IN COMPONENT FORM

$$\begin{aligned} x_1 &= \gamma_1(s) + \eta\, n_1(s) \\ x_2 &= \gamma_2(s) + \eta\, n_2(s) \end{aligned} \implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \gamma_1(s) \\ \gamma_2(s) \end{pmatrix} + \eta \begin{pmatrix} n_1(s) \\ n_2(s) \end{pmatrix} \qquad (1)$$

WE NOW DIFFERENTIATE IMPLICITLY WRT x_1 AND x_2. WE GET

$$\left.\begin{aligned} \hat{\underline{e}}_{x_1} &= \underline{\gamma}'(s)\frac{\partial s}{\partial x_1} + \frac{\partial \eta}{\partial x_1}\hat{\underline{n}} + \eta\, \hat{\underline{n}}'(s)\frac{\partial s}{\partial x_1}, \qquad \underline{e}_{x_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \hat{\underline{e}}_{x_2} &= \underline{\gamma}'(s)\frac{\partial s}{\partial x_2} + \frac{\partial \eta}{\partial x_2}\hat{\underline{n}} + \eta\, \hat{\underline{n}}'(s)\frac{\partial s}{\partial x_2}, \qquad \underline{e}_{x_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}\right\} \quad (2)$$

NOW SINCE $\hat{\underline{n}} \cdot \hat{\underline{n}} = 1 \quad \forall s$, WE DIFFERENTIATE TO OBTAIN

$$\frac{d\hat{\underline{n}}}{ds} \cdot \hat{\underline{n}} = 0 \quad \text{AND SO} \quad \frac{d\hat{\underline{n}}}{ds} \text{ IS } \perp \text{ TO } \hat{\underline{n}}.$$

HENCE $\frac{d\hat{\underline{n}}}{ds} = \alpha \hat{\underline{t}}$ FOR SOME α.

<u>CLAIM</u> <u>$\alpha = -\kappa$</u>. <u>PROOF</u> $\frac{d}{ds}\left(\frac{d\hat{\underline{t}}}{ds}\right) = \kappa'(s)\,\hat{\underline{n}} + \kappa \frac{d\hat{\underline{n}}}{ds}$.

NOW TAKE DOT PRODUCT WITH $\hat{t}$: $\longrightarrow \hat{\underline{t}} \cdot \frac{d^2\hat{\underline{t}}}{ds^2} = \kappa\, \hat{\underline{t}} \cdot \frac{d\hat{\underline{n}}}{ds} = \alpha\kappa\, \hat{\underline{t}} \cdot \hat{\underline{t}} = \alpha\kappa$,

WHERE WE USED $\hat{t} \cdot \hat{n} = 0$. NOW SINCE $\hat{t} \cdot \hat{t} = 1$ WE HAVE

$$\frac{d}{ds}\, \hat{t} \cdot \hat{t} = 0. \quad \text{DIFFERENTIATING AGAIN:} \quad \frac{d}{ds}\left(\frac{d\hat{t}}{ds} \cdot \hat{t}\right) = \frac{d\hat{\underline{t}}}{ds} \cdot \frac{d\hat{\underline{t}}}{ds} + \frac{d^2\hat{\underline{t}}}{ds^2} \cdot \hat{\underline{t}} = 0.$$

HENCE USING $d\hat{\underline{t}}/ds = \kappa \hat{\underline{n}}$ AND $\frac{d^2\hat{\underline{t}}}{ds^2} \cdot \hat{\underline{t}} = \alpha\kappa$ WE GET $\kappa^2\, \hat{\underline{n}} \cdot \hat{\underline{n}} + \alpha\kappa = 0 \longrightarrow \alpha = -\kappa$.

IN SUMMARY WE HAVE:

$$\left.\frac{d\hat{\underline{t}}}{ds} = \kappa \hat{\underline{n}} \quad \text{AND} \quad \frac{d\hat{\underline{n}}}{ds} = -\kappa \hat{\underline{t}}. \right\} \qquad (3)$$

NOW IN (2) WE WILL USE (3) AND $\hat{\underline{t}} = \underline{\gamma}'(s)$. WE OBTAIN

$$\left.\begin{aligned} \hat{\underline{e}}_{x_1} &= \hat{\underline{t}}(s)\left[1 - \kappa\eta\right]\frac{\partial s}{\partial x_1} + \hat{\underline{n}}(s)\frac{\partial \eta}{\partial x_1} \\ \hat{\underline{e}}_{x_2} &= \hat{\underline{t}}(s)\left[1 - \kappa\eta\right]\frac{\partial s}{\partial x_2} + \hat{\underline{n}}(s)\frac{\partial \eta}{\partial x_2}. \end{aligned}\right\} \qquad (4)$$

NOW TAKE THE DOT PRODUCT OF (4) WITH $\hat{t} = (t_1, t_2)^T$. FROM EACH OF THE TWO EQUATIONS WE GET

$$t_1 = (1 - \kappa\eta)\frac{\partial s}{\partial x_1} \quad \text{AND} \quad t_2 = (1 - \kappa\eta)\,\partial s/\partial x_2.$$

THIS YIELDS OUR FIRST KEY RESULT:

$$\nabla s \equiv \left(\frac{\partial s}{\partial x_1}, \frac{\partial s}{\partial x_2}\right)^T = \frac{\hat{t}}{1 - \kappa\eta}.$$

NOW IN (4) TAKE THE DOT PRODUCT WITH $\hat{\underline{n}}$. WE GET WITH $\hat{\underline{n}} = (n_1, n_2)$ THAT

$$n_1 = \partial\eta/\partial x_1, \quad n_2 = \partial\eta/\partial x_2,$$

AND THEREFORE $\nabla\eta \equiv \left(\frac{\partial\eta}{\partial x_1}, \frac{\partial\eta}{\partial x_2}\right) = \hat{\underline{n}}$.

IN SUMMARY, WE HAVE THAT PROVIDED THAT $\eta < 1/\kappa \equiv R$ (R = radius of curvature)

$$\nabla\eta = \hat{\underline{n}}, \quad \nabla s = \frac{\hat{t}}{1-\kappa\eta} \qquad (5).$$

THE SECOND STEP OF THIS PROCEDURE IS TO LET $u = u(x_1, x_2)$ AND TO CALCULATE $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)^T$ IN TERMS OF THE (η, s) COORDINATE SYSTEM.

<u>LEMMA</u> FOR $\eta < 1/\kappa$ WE HAVE

$$\nabla u = \frac{u_s}{1-\kappa\eta}\hat{t} + u_\eta \hat{n}; \qquad (6).$$

<u>PROOF</u> WE FIRST WRITE USING (4) THAT

$$\begin{aligned}
\nabla u &= u_{x_1}\hat{\underline{e}}_{x_1} + u_{x_2}\hat{\underline{e}}_{x_2} \\
&= (u_s s_{x_1} + u_\eta \eta_{x_1})\hat{\underline{e}}_{x_1} + (u_s s_{x_2} + u_\eta \eta_{x_2})\hat{e}_{x_2} \\
&= (u_s s_{x_1} + u_\eta \eta_{x_1})\left[(1-\kappa\eta)s_{x_1}\hat{\underline{t}} + \eta_{x_1}\hat{\underline{n}}\right] + (u_s s_{x_2} + u_\eta\eta_{x_2})\left[(1-\kappa\eta)s_{x_2}\hat{\underline{t}} + \eta_{x_2}\hat{\underline{n}}\right] \\
&= u_s(1-\kappa\eta)s_{x_1}^2\hat{\underline{t}} + u_s(1-\kappa\eta)s_{x_2}^2\hat{\underline{t}} + u_\eta(\eta_{x_1}^2 + \eta_{x_2}^2)\hat{\underline{n}} \\
&\quad + u_\eta(1-\kappa\eta)\left[\eta_{x_1}s_{x_1} + \eta_{x_2}s_{x_2}\right]\hat{\underline{t}} + u_s(s_{x_1}\eta_{x_1} + s_{x_2}\eta_{x_2})\hat{\underline{n}}.
\end{aligned}$$

HOWEVER

$$s_{x_1}\eta_{x_1} + s_{x_2}\eta_{x_2} = \nabla s\cdot\nabla\eta = 0 \quad \text{BY (5) SINCE} \quad \hat{\underline{n}}\cdot\hat{t} = 0.$$

THUS $\quad \nabla u = u_s(1-\kappa\eta)\,|\nabla s|^2\,\hat{t} + u_\eta\,\hat{n}\,|\nabla\eta|^2.$

NOW FROM (5) WE HAVE $|\nabla\eta|^2 = \hat{n}\cdot\hat{n} = 1, \quad |\nabla s|^2 = \dfrac{\hat{t}\cdot\hat{t}}{(1-\kappa\eta)^2} = \dfrac{1}{(1-\kappa\eta)^2}.$

WE CONCLUDE THAT

$$\boxed{\nabla u = \frac{u_s}{1-\kappa\eta}\hat{t} + u_\eta\hat{n}} \quad \square$$

NOW THE NEXT STEP IS TO CALCULATE $\Delta\eta$ AND Δs.

<u>LEMMA</u> (7) $\left\{ \Delta\eta = \nabla\cdot(\nabla\eta) = -\dfrac{\kappa}{1-\kappa\eta}, \quad \Delta s = \nabla\cdot(\nabla s) = \dfrac{\kappa'(s)\,\eta}{(1-\kappa\eta)^3} \right.$ FOR $\eta < 1/\kappa$.

<u>PROOF</u> WE USE (5).

$$\Delta\eta \equiv \nabla\cdot(\nabla\eta) = \frac{d\hat{\underline{n}}}{ds}\cdot\nabla s = \frac{d\hat{\underline{n}}}{ds}\cdot\frac{\hat{\underline{t}}}{1-\kappa\eta} = -\frac{\kappa\,\hat{t}\cdot\hat{\underline{t}}}{1-\kappa\eta} = -\frac{\kappa}{1-\kappa\eta} \quad \left(\text{USING } \frac{d\hat{\underline{n}}}{ds} = -\kappa\hat{\underline{t}}\right)$$

NEXT, WE CALCULATE $\Delta S = \nabla\cdot(\nabla S)$.

WE USE $$\nabla\cdot\nabla S = \nabla\cdot\left[\frac{1}{1-\kappa\Lambda}\underline{\hat{t}}\right] = \nabla\left(\frac{1}{1-\kappa\Lambda}\right)\cdot\underline{\hat{t}} + \frac{1}{1-\kappa\Lambda}\nabla\cdot\underline{\hat{t}}$$

$$= \frac{1}{(1-\kappa\Lambda)^2}\nabla(\kappa\Lambda)\cdot\underline{\hat{t}} + \frac{1}{(1-\kappa\Lambda)}\underline{\hat{t}}'\cdot\nabla S$$

$$= \frac{1}{(1-\kappa\Lambda)^2}\left[\kappa\nabla\Lambda + \Lambda\nabla\kappa\right]\cdot\underline{\hat{t}} + \underbrace{\frac{1}{(1-\kappa\Lambda)}\,\underline{\hat{t}}'\cdot\frac{\underline{\hat{t}}}{(1-\kappa\Lambda)}}_{=0}$$

$$\Delta S = \frac{1}{(1-\kappa\Lambda)^2}\left[\kappa\nabla\Lambda + \Lambda\kappa'(s)\nabla S\right]\cdot\underline{\hat{t}}$$

NOW $\nabla\Lambda = \underline{\hat{n}}$ AND $\nabla S = \dfrac{\underline{\hat{t}}}{(1-\kappa\Lambda)}$. THUS $\Delta S = \dfrac{1}{(1-\kappa\Lambda)^2}\left[\kappa\underline{\hat{n}} + \dfrac{\Lambda\kappa'(s)}{(1-\kappa\Lambda)}\underline{\hat{t}}\right]\cdot\underline{\hat{t}}$.

USING $\underline{\hat{t}}\cdot\underline{\hat{t}} = 1$ AND $\underline{\hat{n}}\cdot\underline{\hat{t}} = 0$ WE OBTAIN $$\Delta S = \frac{\kappa'(s)\Lambda}{(1-\kappa\Lambda)^3} \quad \Box.$$

FINALLY, UNDER THE VARIABLE CHANGE $\underline{x}\to(\Lambda,S)$ WE CALCULATE

$$\Delta u = u_{\Lambda\Lambda}|\nabla\Lambda|^2 + u_{SS}|\nabla S|^2 + 2u_{S\Lambda}\nabla S\cdot\nabla\Lambda + u_\Lambda\Delta\Lambda + u_S\Delta S$$

WHERE $\nabla = \left(\partial/\partial x_1, \partial/\partial x_2\right)^T$ AND $\Delta = \partial_{x_1x_1} + \partial_{x_2x_2}$.

NOW USING OUR EARLIER RESULTS THAT

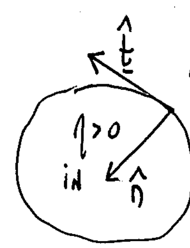
$$\Delta S = \frac{\kappa'\Lambda}{(1-\kappa\Lambda)^3}, \quad \Delta\Lambda = -\frac{\kappa}{1-\kappa\Lambda}, \quad |\nabla S|^2 = \frac{1}{(1-\kappa\Lambda)^2}, \quad |\nabla\Lambda|^2 = 1$$

AND $\nabla S\cdot\nabla\Lambda = 0$ WE OBTAIN

$$\Delta u = u_{\Lambda\Lambda} - \frac{\kappa}{1-\kappa\Lambda}u_\Lambda + \frac{1}{(1-\kappa\Lambda)^2}u_{SS} + \frac{\kappa'(s)\Lambda}{(1-\kappa\Lambda)^3}u_S.$$

THIS IS MOST ELEGANTLY WRITTEN AS

$$\boxed{\Delta u = u_{\Lambda\Lambda} - \frac{\kappa}{1-\kappa\Lambda}u_\Lambda + \frac{1}{(1-\kappa\Lambda)}\frac{\partial}{\partial s}\left(\frac{u_s}{1-\kappa\Lambda}\right)}$$



WHICH IS VALID FOR $0<\Lambda<1/\kappa_{MAX}$ $\quad \kappa_{MAX} = \max_{\Gamma}(\kappa)$.

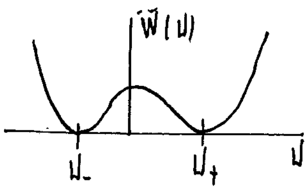
<u>REMARK</u> (LAPLACIAN IN POLAR COORDINATES). CONSIDER Γ TO BE A CIRCLE OF RADIUS R. THEN $S = R\varphi$, $\kappa = 1/R$, $\kappa' = 0$, $\Lambda = R - r$ FOR $0<r<R$. THUS $u_S = Ru_\varphi$, $u_\Lambda = -u_r$ AND $u_{SS} = R^2 u_{\varphi\varphi}$, $u_{\Lambda\Lambda} = u_{rr}$ $\longrightarrow$ $$\Delta u = u_{rr} + \frac{(1/R)u_r}{1-\frac{1}{R}(R-r)} + \frac{R^2}{\left(1-\frac{1}{R}(R-r)\right)^2}u_{\varphi\varphi} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}.$$

EXAMPLE (MOTION BY CURVATURE)

WE WILL DERIVE A GEOMETRIC LAW FOR THE SLOW EVOLUTION OF A THIN DIFFUSIVE LAYER FOR THE ALLEN-CAHN EQUATION

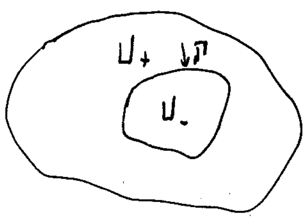
$$(1)\quad \begin{cases} U_t = \varepsilon^2 \Delta U + Q(U), & \underline{x} \text{ in } \Omega \qquad \Omega \in \mathbb{R}^2 \\ \partial_n U = 0 & \text{ON } \partial\Omega \end{cases}$$

HERE $Q(U) = -W'(U)$ WHERE $W(U)$ IS A BALANCED DOUBLE WELL POTENTIAL



THUS $Q(U_\pm) = 0$, $Q'(U_\pm) < 0$ WITH $\int_{U_-}^{U_+} Q(U)\,dU = 0$.

NOW ON AN $O(1)$ TIME-SCALE $U \to U_+$ OR $U \to U_-$ WHICH IS THEN SEPARATED BY A THIN INTERFACE Γ OF WIDTH $O(\varepsilon)$. WE WILL CONSIDER IN 2-D A CLOSED INTERFACE Γ OF WIDTH $O(\varepsilon)$ DEFINED BY $U|_\Gamma = (U_+ + U_-)/2$.

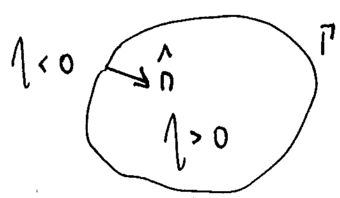


WE WOULD LIKE TO DERIVE A GEOMETRIC LAW FOR THE TIME EVOLUTION OF Γ.

NEAR Γ WE INTRODUCE A LOCAL NORMAL, TANGENTIAL COORDINATE SYSTEM (η, s) TO OBTAIN

$$U_t = \varepsilon^2 \left[U_{\eta\eta} - \frac{\kappa}{1-\kappa\eta} U_\eta + \frac{1}{1-\kappa\eta} \partial_s \left[\frac{1}{1-\kappa\eta} \partial_s U \right] \right] + Q(U) \qquad (2)$$

OUR PLOT IS AS FOLLOWS:



- HERE η IS SIGNED DISTANCE FROM Γ. ($\eta > 0$ IF INSIDE, $\eta < 0$ IF OUTSIDE).
- $\kappa(s)$ IS THE CURVATURE OF Γ AND s IS ARCLENGTH ALONG Γ.

NOW WE LET $\hat{\eta} = \eta/\varepsilon$ AND REPLACE $U \to \bar{U}$ WHERE $\eta = \eta(\tau)$ WITH $\tau = \varepsilon^p t$ WHERE p IS TO BE CHOSEN. THE TIME EVOLUTION WILL CHARACTERIZE THE MOTION OF Γ.

WE SUBSTITUTE INTO (2) TO OBTAIN

$$\varepsilon^{p-1} \bar{U}_{\hat{\eta}} \dot{\eta} + \bar{U}_s \varepsilon^p \dot{s} = \bar{U}_{\hat{\eta}\hat{\eta}} - \kappa\varepsilon \bar{U}_{\hat{\eta}} + O(\varepsilon^2) + Q(\bar{U}) \qquad (3).$$

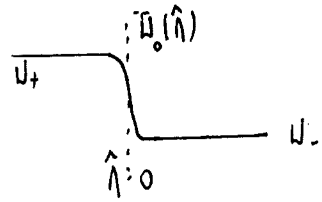
WHERE $\dot{\eta} \equiv d\eta/d\tau$. NOW TO BALANCE TERMS WE CHOOSE $p = 2$ SO THAT THE LONG TIME-SCALE IS $\tau = \varepsilon^2 t$.

WE THEN EXPAND $\bar{U} = \bar{U}_0 + \varepsilon \bar{U}_1 + \ldots$ AND SUBSTITUTE INTO (3).

TO LEADING ORDER WE OBTAIN THAT

$$\left.\begin{aligned} &U_{0\hat{\eta}\hat{\eta}} + Q(U_0) = 0 \quad \text{IN } -\infty < \hat{\eta} < \infty \\ &U_0 \to u_- \quad \text{AS } \hat{\eta} \to +\infty \\ &U_0 \to u_+ \quad \text{AS } \hat{\eta} \to -\infty \end{aligned}\right\} \quad (4) \qquad \begin{aligned} &Q'(u_\pm) < 0 \\ &Q(u_\pm) = 0 \end{aligned}$$

WHERE $U_0(0) = \frac{1}{2}\left[u_- + u_+\right]$ DEFINES Γ AS THE LEVEL CURVE $\hat{\eta} = 0$. $\exists$ A UNIQUE SOLUTION TO THIS PROBLEM



HETEROCLINIC ORBIT.

NOW AT NEXT ORDER WE OBTAIN THAT UPON USING $Q(U_0 + \varepsilon U_1 + \cdots) = Q(U_0) + \varepsilon Q'(U_0)U_1 + \cdots$

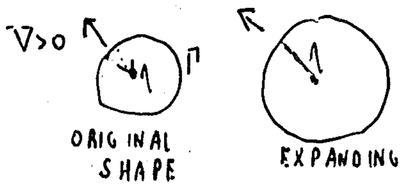
SO THAT

$$\left.\begin{aligned} &L U_1 \equiv U_{1\hat{\eta}\hat{\eta}} + Q'(U_0)U_1 = U_{0\hat{\eta}}\,\dot{\eta} + \kappa U_{0\hat{\eta}} \\ &U_1 \to 0 \quad \text{AS } \hat{\eta} \to \pm\infty. \end{aligned}\right\} \quad (5)$$

WE NOW NOTICE THAT

$$\dot{\eta} = V, \text{ WHICH IS THE NORMAL VELOCITY TO } \Gamma.$$

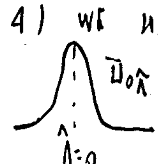
NOTICE THAT $V > 0$, i.e. $\dot{\eta} > 0$ IF Γ IS EXPANDING



NOTICE FROM A SPECIFIC POINT IN Γ WE HAVE $\dot{\eta} > 0$ IF $V > 0$, i.e INTERFACE EXPANDING.

NOW FROM (5) WE CONCLUDE THAT BY DIFFERENTIATING (4) WE HAVE

$$(U_{0\hat{\eta}})_{\hat{\eta}\hat{\eta}} + Q'(U_0)U_{0\hat{\eta}} = 0 \longrightarrow L\,U_{0\hat{\eta}} = 0$$



BY USING LAGRANGE'S IDENTITY WE HAVE TO IMPOSE A FREDHOLM ALTERNATIVE CONDITION FOR (5):

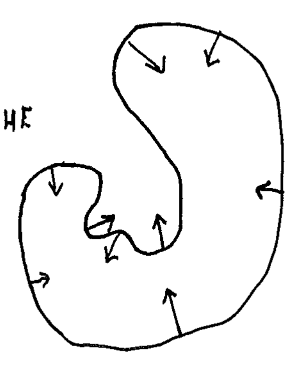
$$\int_{-\infty}^{\infty} \big(U_{0\hat{\eta}}\, L U_1 - U_1 \underbrace{L U_{0\hat{\eta}}}_{=0}\big)\, d\hat{\eta} = 0 \longrightarrow \text{WE NEED} \quad \int_{-\infty}^{\infty} U_{0\hat{\eta}}\left(U_{0\hat{\eta}}\,\dot{\eta} + \kappa U_{0\hat{\eta}}\right) d\hat{\eta} = 0.$$

THUS WE HAVE THAT

$$\dot{\eta}\int_{-\infty}^{\infty} (U_{0\hat{\eta}})^2\, d\hat{\eta} = -\kappa \int_{-\infty}^{\infty} (U_{0\hat{\eta}})^2\, d\hat{\eta}.$$

WE CONCLUDE THAT Γ EVOLVES BY ITS OWN CURVATURE, i.e.

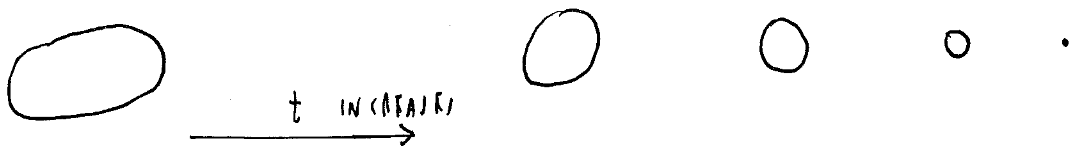
$$\dot{\eta} \equiv V = -\kappa. \qquad \dot{\eta} \equiv d\eta/d\tau$$



NOW IN ORIGINAL TIME VARIABLES $\tau = \varepsilon^2 t$ WE HAVE THAT THE NORMAL VELOCITY V TO Γ EVOLVES AS

$$V = -\varepsilon^2 \kappa, \quad \text{WHERE } \kappa \text{ IS CURVATURE OF } \Gamma.$$

NOW SUPPOSE THAT Γ IS CONVEX AS SEEN FROM INSIDE Γ. THEN $\kappa > 0$ AND $V < 0$ SO MOTION IS INWARDS. A FAMOUS PDE RESULT OF GAGE (1988) SAYS THAT ALL SUCH CONVEX INTERFACES WILL TEND TO CIRCULARIZE AND THEN SHRINK TO A POINT.



IN PARTICULAR SUPPOSE THAT AT $t = 0$, Γ IS A DISK OF RADIUS a. WE WILL LOOK FOR A SOLUTION TO $\dot{\eta} = -\kappa$ WHERE Γ IS CIRCULAR $\forall\, t \geq 0$ (UNTIL EXTINCTION). IF AT TIME t, Γ IS A CIRCLE OF RADIUS r THEN $V = d\eta/d\tau = dr/d\tau$.

THUS $dr/d\tau = -1/r$, $r(0) = a$.

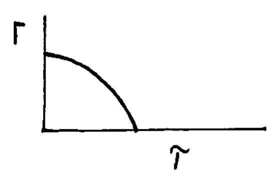
WE SOLVE THIS ODE

$$r\,dr = -\,d\tau \longrightarrow \frac{r^2}{2} - \frac{a^2}{2} = -\,\tau .$$

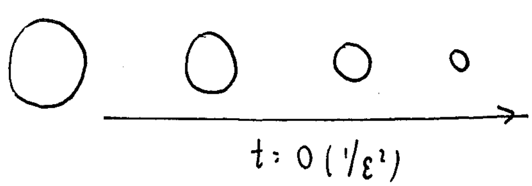
THUS $r = \sqrt{a^2 - 2\tau}$

WHICH IN ORIGINAL VARIABLES WITH $\tau = \varepsilon^2 t$ IS

$$r = \sqrt{a^2 - 2\varepsilon^2 t}.$$

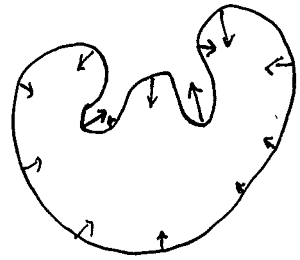


WE CONCLUDE THAT THE CIRCULAR SHAPED INTERFACE SHRINKS TO A POINT WHEN $t = a^2/2\varepsilon^2$.

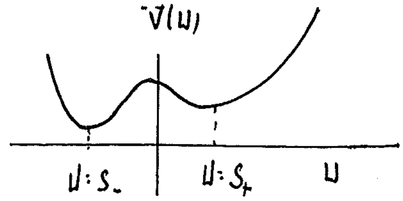


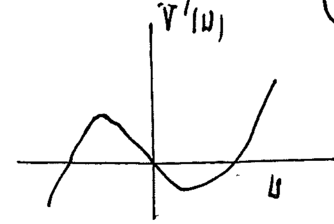
IN APPENDIX A WE DERIVE SOME KEY PROPERTIES FOR THE GEOMETRIC EVOLUTION LAW $V = -\kappa$.

THE ARROWS AND THEIR LENGTHS INDICATE DEFORMATION OF CURVE (SOLID) AFTER Δt TIME UNITS



CAHN-HILLIARD TO MULLINS-SEKERKA





WE BEGIN WITH THE CAHN-HILLIARD MODEL

$$u_t = \Delta\left[-\varepsilon^2 \Delta u + V'(u)\right] \quad \text{in } \Omega$$

$$\partial_n u = 0\,, \quad \partial_n[\Delta u] = 0 \quad \text{on } \partial\Omega$$

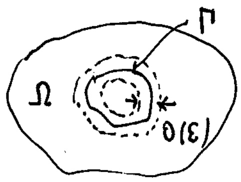
WE ASSUME THAT $V(u)$ IS A DOUBLE-WELL POTENTIAL AND ALLOW THE WELLS OF UNEQUAL DEPTH.

WE WILL INTRODUCE A SLOW TIME $\tau = \varepsilon t$ AND w BY

$$\left.\begin{aligned} &\varepsilon u_\tau = \Delta w\,, && w_n = 0 \text{ on } \partial\Omega \\ &-\varepsilon^2 \Delta u + V'(u) = +w\,, && u_n = 0 \text{ on } \partial\Omega. \end{aligned}\right\} \quad (1)$$

<u>REMARK</u> IF $\partial_n u = 0$, $\partial_n[\Delta u] = 0 \longrightarrow \partial_n[\varepsilon^2 \Delta u - V'(u)] = \varepsilon^2 \partial_n[\Delta u] - V''(u)\partial_n u = 0. \to \partial_n w = 0.$

NOW WE CONSIDER LIMIT $\varepsilon \to 0$ WHERE THERE IS A LAYERED SOLUTION SHOWN AS



IN THE LIMIT $\varepsilon \to 0$ THE "CENTER" OF THE LAYER IS A CURVE Γ. INSIDE Γ, u IS A CONSTANT, WHILE OUTSIDE Γ, u IS ANOTHER CONSTANT. THE THIN LAYER ALLOWS FOR A FAST TRANSITION BETWEEN THE TWO STATES.

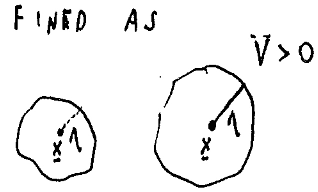
WE WILL NOW CONVERT (1) TO OUR NORMAL-TANGENTIAL COORDINATE SYSTEM:

WE OBTAIN FOR (1) THAT

$$\left.\begin{aligned} &\varepsilon\left(u_\Lambda \Lambda' + u_s s'\right) = w_{\Lambda\Lambda} - \frac{\kappa}{1-\kappa\Lambda} w_\Lambda + \frac{1}{1-\kappa\Lambda}\partial_s\left(\frac{w}{1-\kappa\Lambda}\right) \\ &\varepsilon^2\left(u_{\Lambda\Lambda} - \frac{\kappa}{1-\kappa\Lambda} u_\Lambda + \frac{1}{1-\kappa\Lambda}\partial_s\left(\frac{u}{1-\kappa\Lambda}\right)\right) - V'(u) = -w \end{aligned}\right\} \quad (2)$$

NOW THE NORMAL VELOCITY TO THE INTERFACE V IS DEFINED AS

$$V = d\Lambda/d\tau\,.$$



NOTICE THAT IF $d\Lambda/d\tau > 0$ THEN $V > 0$, i.e. Γ IS EXPANDING.

ALSO $\kappa > 0$ IF Γ IS CONVEX AS SEEN FROM THE INSIDE.

<u>OUTER EXPANSION</u> IN THE OUTER REGION AWAY FROM AN $O(\varepsilon)$ REGION NEAR Γ WE EXPAND

$$w = w_0 + \varepsilon w_1 + \cdots\,, \quad u = u_0 + \varepsilon u_1 + \cdots\cdots.$$

FROM (1) WE OBTAIN <u>AWAY FROM</u> Γ THAT FROM (1)

$$\left.\begin{aligned} &O(1): && \Delta w_0 = 0\,, \quad V'(u_0) = w_0\,, \quad \partial_n w_0 = 0 \text{ on } \partial\Omega \\ &O(\varepsilon): && \Delta w_1 = u_{0\tau}\,, \quad V''(u_0)\,u_1 = w_1\,, \quad \partial_n w_1 = 0 \text{ on } \partial\Omega \end{aligned}\right\} \quad (3)$$

INNER EXPANSION NEAR Γ WE INTRODUCE LOCAL NORMAL COORDINATE $\hat{\eta} = \eta/\varepsilon$ AND TANGENTIAL COORDINATE S. WE REPLACE $u \to \bar{U}$, $w \to \bar{W}$. THEN FROM (2)

$$\left.\begin{aligned} &\frac{1}{\varepsilon^2}\bar{W}_{\hat{\eta}\hat{\eta}} - \frac{\kappa}{\varepsilon(1-\kappa\varepsilon\hat{\eta})}\bar{W}_{\hat{\eta}} + \frac{1}{(1-\kappa\varepsilon\hat{\eta})}\partial_s\left(\frac{\bar{W}}{(1-\kappa\varepsilon\hat{\eta})}\right) = \varepsilon\left(\frac{1}{\varepsilon}\bar{U}_{\hat{\eta}}V + \bar{U}_s\right) \\ &\varepsilon^2\left(\frac{1}{\varepsilon^2}\bar{U}_{\hat{\eta}\hat{\eta}} - \frac{\kappa}{\varepsilon(1-\kappa\varepsilon\hat{\eta})}\bar{U}_{\hat{\eta}} + \frac{1}{(1-\kappa\varepsilon\hat{\eta})}\partial_s\left(\frac{\bar{U}}{(1-\kappa\varepsilon\hat{\eta})}\right)\right) - V'(\bar{U}) = -\bar{W}. \end{aligned}\right\} \quad (4)$$

WE THEN EXPAND

$$\bar{W} = \bar{W}_0 + \varepsilon\bar{W}_1 + \varepsilon^2\bar{W}_2 + \dots \qquad \bar{U} = \bar{U}_0 + \varepsilon\bar{U}_1 + \dots$$

FROM THE FIRST EQUATION OF (4) WE HAVE

$$\left.\begin{aligned} &O(1/\varepsilon^2): \quad \bar{W}_{0\hat{\eta}\hat{\eta}} = 0 && (5a) \\ &O(1/\varepsilon): \quad \bar{W}_{1\hat{\eta}\hat{\eta}} - \kappa\bar{W}_{0\hat{\eta}} = 0 && (5b) \\ &O(1): \quad \bar{W}_{2\hat{\eta}\hat{\eta}} - \kappa\bar{W}_{1\hat{\eta}} - \kappa^2\hat{\eta}\bar{W}_{0\hat{\eta}} + \bar{W}_{0ss} = \bar{U}_{0\hat{\eta}}V && (5c) \end{aligned}\right\}$$

<u>REMARK</u> IN DERIVING (5c) WE USED BINOMIAL APPROXIMATION $\frac{\kappa}{\varepsilon}(1-\varepsilon\kappa\hat{\eta})^{-1} = \frac{\kappa}{\varepsilon} + \kappa^2\hat{\eta}$.

NOW FROM THE $\bar{U}$ EQUATION IN (4) WE OBTAIN THAT

$$\left.\begin{aligned} &O(1): \quad \bar{U}_{0\hat{\eta}\hat{\eta}} - V'(\bar{U}_0) = -\bar{W}_0 && (6a) \\ &O(\varepsilon): \quad \bar{U}_{1\hat{\eta}\hat{\eta}} - V''(\bar{U}_0)\bar{U}_1 = \kappa\bar{U}_{0\hat{\eta}} - \bar{W}_1. && (6b) \end{aligned}\right\}$$

NOW THE SOLUTION TO (5a) IS LINEAR IN $\hat{\eta}$. HOWEVER, IF WE WRITE $W_0 = a_0 + b_0\hat{\eta}$ THEN WE MUST CHOOSE $b_0 = 0$ OTHERWISE $W_0 = b_0\eta/\varepsilon + a_0 = O(1/\varepsilon)$ IN THE FAR-FIELD WHICH IS INCONSISTENT WITH THE OUTER EXPANSION.

HENCE, WE MUST HAVE $\bar{W}_0$ = CONSTANT.

NOW IN THE OUTER REGION, WE OBTAIN FROM (3) THAT $\Delta W_0 = 0$ IN Ω WITH $\partial_n W_0 = 0$ ON $\partial\Omega$, GIVES W_0 = CONSTANT, IN OUTER REGION. $\longrightarrow W_0 = \bar{W}_0$.

AND IN OUTER REGION FROM (3), $V'(U_0) = W_0$.

THEN (6a) YIELDS THE LAYER PROBLEM:

$$\bar{U}_{0\hat{\eta}\hat{\eta}} - V'(\bar{U}_0) = -\bar{W}_0 \qquad (7).$$

TO MATCH TO THE OUTER SOLUTION WHERE $\bar{V}'(\bar{U}_0) = W_0$

WE MUST FIND CONSTANTS $\bar{U}_{0-}, \bar{U}_{0+}$ AND $\bar{W}_{0m}$ THAT SATISFY

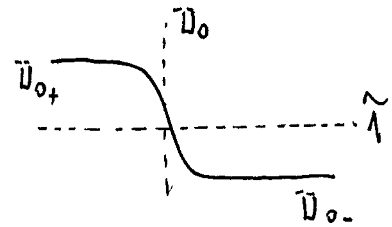
$$\bar{V}''(\bar{U}_{0\pm}) > 0\,, \quad \bar{V}'(\bar{U}_{0+}) = \bar{W}_{0m}\,, \quad \bar{V}'(\bar{U}_{0-}) = \bar{W}_{0m} \qquad (8a)$$

AND THE EQUAL AREA CONDITION

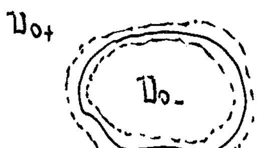
$$\int_{\bar{U}_{0-}}^{\bar{U}_{0+}} \left[-\bar{V}'(\bar{U}_0) + \bar{W}_{0m} \right] d\bar{U}_0 = 0 \qquad (8b)$$

WHICH ENSURES THAT (7) HAS A HETEROCLINIC CONNECTION SATISFYING

$$(9)\;\left\{\begin{array}{l} \bar{U}_{0\hat\lambda\hat\lambda} - \bar{V}'(\bar{U}_0) + \bar{W}_{0m} = 0\,, \quad -\infty < \hat\lambda < \infty \\ \bar{U}_0 \to \bar{U}_{0+} \quad \text{AS} \quad \hat\lambda \to -\infty \\ \bar{U}_0 \longrightarrow \bar{U}_{0-} \quad \text{AS} \quad \hat\lambda \to +\infty \end{array}\right.$$



THE PLOT IS SIMPLY AS SHOWN.



WE CAN DEFINE Γ AS THE LEVEL CURVE $\hat\lambda = 0$ FOR WHICH $\bar{U}_0(0) = (\bar{U}_{0-} + \bar{U}_{0+})/2$,
THE CONDITION $\bar{V}''(\bar{U}_{0\pm}) > 0$ MEANS $\bar{U}_0$ DECAYS ONTO $\bar{U}_{0\pm}$ AS $\hat\lambda \to \pm\infty$.

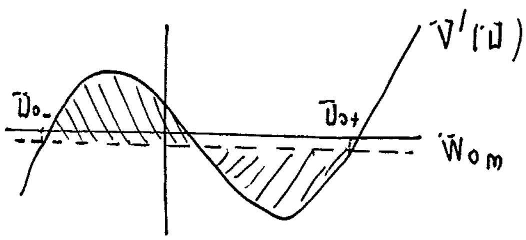
<u>REMARK</u> (i) FROM BASIC HETEROCLINIC THEORY WE RECALL THAT $\exists$ A HETEROCLINIC SOLUTION $\phi(y)$ TO

$$\left.\begin{array}{l} \phi'' + Q(\phi) = 0 \quad \text{IN} \quad -\infty < y < \infty \\ \text{WITH} \quad \phi \to \phi_- \ \text{AS} \ y \to +\infty\,, \quad \phi \to \phi_+ \ \text{AS} \ y \to -\infty \\ \text{WHERE } Q(\phi_\pm) = 0,\ Q'(\phi_\pm) = 0 \text{ AND THE EQUAL AREA RULE} \end{array}\right\} \quad (10)$$

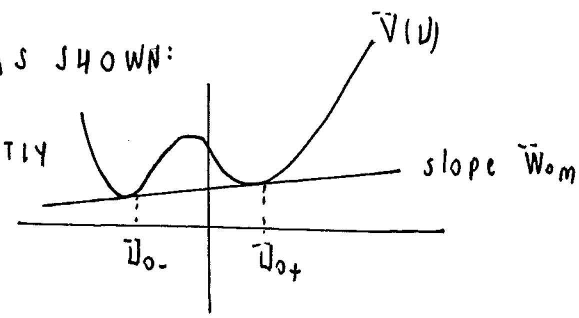
$\int_{\phi_-}^{\phi_+} Q(\phi)\, d\phi = 0$ HOLDS.

NOTICE THAT IN (9), WE HAVE $Q(\bar{U}_0) = -\bar{V}'(\bar{U}_0) + \bar{W}_{0m}$, AND EQUATIONS (8a), (8b) ARE PRECISELY THE CONDITIONS IN (10).

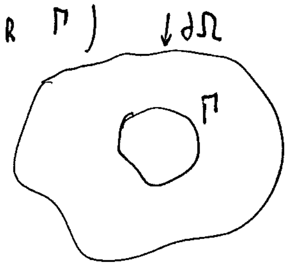
IN EQUATION (8b) THE EQUAL AREA RULE IS AS SHOWN:



OR EQUIVALENTLY



IN SUMMARY, TO LEADING ORDER WE HAVE:



$$
(11)\left\{\begin{array}{l}
W \sim \bar{W}_{0m} \quad \text{EVERYWHERE (IN OUTER REGION } \underline{\text{AND}} \text{ NEAR } \Gamma) \\
U \sim \bar{U}_{0-} \quad (\text{INSIDE } \Gamma)\ ; \quad U \sim \bar{U}_{0+} \ (\text{OUTSIDE } \Gamma) \\
U \sim \bar{U}_0(\hat{\eta}) \quad \text{IN } O(\varepsilon) \text{ REGION NEAR } \Gamma.
\end{array}\right.
$$

WE WILL NOW PROCEED TO ONE HIGHER ORDER. FROM THE $O(\varepsilon)$ TERM IN (3) WE HAVE SINCE $\bar{U}_{0\pm}$ ARE INDEPENDENT OF τ,

$$\Delta W_1 = 0 \quad \text{AWAY FROM } \Gamma\ ; \quad \partial_n W_1 = 0 \quad \text{ON} \quad \partial\Omega. \quad \Big\} \quad (12)$$

OUR GOAL IS TO DERIVE TRANSMISSION (OR JUMP) CONDITIONS FOR W_1 AND $W_{1\eta}$ ACROSS Γ. NOW FROM (5b), SINCE $\bar{W}_0 = \bar{W}_{0m}$ WE HAVE $\bar{W}_{1\hat{\eta}\hat{\eta}} = 0$ AND SO

$$\bar{W}_1 = A\hat{\eta} + B.$$

THE MATCHING CONDITION WILL BE THAT

$$\bar{W}_{0m} + \varepsilon W_1 + \cdots \sim \bar{W}_{0m} + \varepsilon \bar{W}_1 + \cdots \qquad (12)$$
$$\text{AS } \underset{\sim}{x} \to \Gamma \qquad\qquad \hat{\eta} \to \pm\infty$$

AS SUCH WE $\underline{\text{CANNOT}}$ ALLOW LINEAR GROWTH IN $\hat{\eta}$ SINCE IT WOULD GIVE $\bar{W}_1 = A\eta/\varepsilon + B$ AND SO $\varepsilon \bar{W}_1 = A\eta + \varepsilon B$ WHICH CAN'T BE MATCHED BY OUTER EXPANSION. AS SUCH WE MUST SET $\bar{W}_1 = B$.

THEN FROM THE $O(\varepsilon)$ PROBLEM IN (6b) WE HAVE

$$L\bar{U}_1 \equiv \bar{U}_{1\hat{\eta}\hat{\eta}} - V''(\bar{U}_0)\bar{U}_1 = \mu \bar{U}_{0\hat{\eta}} - B \Big\} \qquad (13)$$

SINCE $L(\bar{U}_{0\hat{\eta}}) = 0$, AS SEEN BY DIFFERENTIATING (6a) WRT $\hat{\eta}$, THE SOLVABILITY CONDITION FOR (13) IS THAT

$$\int_{-\infty}^{\infty} (\mu \bar{U}_{0\hat{\eta}} - B)\bar{U}_{0\hat{\eta}}\, d\hat{\eta} = 0 \longrightarrow \mu\int_{-\infty}^{\infty} (\bar{U}_{0\hat{\eta}})^2\, d\hat{\eta} = B\int_{-\infty}^{\infty} \bar{U}_{0\hat{\eta}}\, d\hat{\eta} = B\left(\bar{U}_{0-} - \bar{U}_{0+}\right)$$

WE CONCLUDE THAT IN THE LAYER

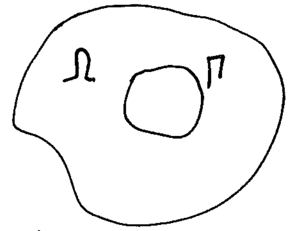
$$\bar{W}_1 = B = \frac{\mu \displaystyle\int_{-\infty}^{\infty} (\bar{U}_{0\hat{\eta}})^2\, d\hat{\eta}}{\bar{U}_{0-} - \bar{U}_{0+}}$$

NOW INVOKING THE MATCHING CONDITION WE CONCLUDE THAT, FROM (12),

$$W_1 \longrightarrow \bar{W}_1 \quad \text{AS} \quad \underset{\sim}{x} \longrightarrow \Gamma.$$

THEREFORE, AT THIS STAGE WE CONCLUDE THAT W_1 SATISFIES

$$(14)\quad \begin{cases} \Delta W_1 = 0 & \text{INSIDE AND OUTSIDE } \Gamma, \text{ i.e. IN } \Omega \\ \partial_n W_1 = 0 & \text{ON } \partial\Omega \\ W_1 = \gamma \kappa & \text{ON } \Gamma \end{cases}$$



WHERE $\kappa = \kappa(s)$ IS THE CURVATURE OF Γ AND $\gamma \equiv \int_{-\infty}^{\infty} (\bar{U}_{0\hat{\Lambda}})^2 \, d\hat{\Lambda} \Big/ (\bar{U}_{0-} - \bar{U}_{0+})$.

THIS IS KNOWN AS THE <u>MULLINS-SEKERKA</u> FLOW. THE FINAL STEP IS TO DETERMINE THE NORMAL VELOCITY V AND JUMP CONDITION FOR W_{1n} ACROSS Γ.

SINCE $\bar{W}_0 = \bar{W}_{0m}$, $\bar{W}_{1\hat{\Lambda}} = 0$ WE OBTAIN FROM (5c) THAT

$$\bar{W}_{2\hat{\Lambda}\hat{\Lambda}} = \bar{V}\, \bar{U}_{0\hat{\Lambda}}$$

WHERE $\bar{U}_0(\hat{\Lambda})$ IS THE HETEROCLINIC CONNECTION SATISFYING (9).

NOW INTEGRATING ONCE WE OBTAIN

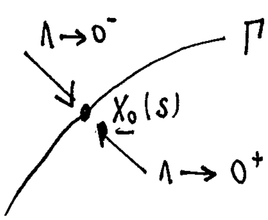
$$\left.\bar{W}_{2\hat{\Lambda}}\right|_{-\infty}^{\infty} = \bar{V} \int_{-\infty}^{\infty} \bar{U}_{0\hat{\Lambda}} \, d\hat{\Lambda} \longrightarrow \bar{W}_{2\hat{\Lambda}}(\infty) - \bar{W}_{2\hat{\Lambda}}(-\infty) = \bar{V}\left[\bar{U}_{0-} - \bar{U}_{0+}\right] \Big\} \quad (15)$$

NOW (15) IS A JUMP CONDITION FOR THE DERIVATIVE OF $\bar{W}_2$. WE NOW EXAMINE WHAT THIS IMPLIES FOR OUR MATCHING CONDITION:

$$\left.\begin{matrix} \overset{\text{(OUTER)}}{\bar{W}_{0m} + \varepsilon W_1 + \varepsilon^2 W_2 + \cdots} & \sim & \overset{\text{(INNER)}}{\bar{W}_{0m} + \varepsilon \bar{W}_1 + \varepsilon^2 \bar{W}_2 + \cdots} \\ \Lambda \to 0 \text{ (OUTER LIMIT)} & & \text{AS } \hat{\Lambda} = \Lambda/\varepsilon \to \pm\infty \end{matrix}\right\} \quad (16)$$

WE WILL NOW SHOW THAT THE JUMP IN THE DERIVATIVE OF W_1 ACROSS Γ, i.e. ITS NORMAL DERIVATIVE, IS RELATED TO THE JUMP IN (15) FOR $\bar{W}_{2\hat{\Lambda}}$.

WE NOW DO THIS MATCHING PROPERLY. WE WILL EXPAND THE LEFT SIDE OF (16) AS $\underline{X} \to \underline{X}_0(S)$ WHERE $\underline{X}_0(S)$ IS A POINT ON Γ.



NOW AS WE APPROACH Γ FROM $\pm$ DIRECTION WE GET BY TAYLOR EXPANSION THAT

$$W_1 \sim \left.W_1\right|_{\Lambda=0} + \left.\nabla W_1\right|_{\Lambda=0^{\pm}} \cdot (\underline{X} - \underline{X}_0) + \cdots$$

NOW WE RECALL FROM EQUATION (6) ON PAGE 3 THAT

$$\left.\nabla W_1\right|_{\Lambda=0^{\pm}} = \left.\frac{W_S}{1-\kappa\Lambda}\right|_{\Lambda=0^{\pm}} \underline{\hat{t}} + \left.W_{1\Lambda}\right|_{\Lambda=0^{\pm}} \underline{\hat{n}} = \left.W_{1S}\right|_{\Lambda=0^{\pm}} \underline{\hat{t}} + \left.W_{1\Lambda}\right|_{\Lambda=0^{\pm}} \underline{\hat{n}}.$$

NOW $\underline{x} - \underline{x}_0(s) = \eta \, \hat{n}$.

THUS, $\nabla w_1 \cdot (\underline{x} - \underline{x}_0) = (w_{1s}\,\hat{\underline{t}} + w_{1\eta}\,\hat{\underline{n}}) \cdot \eta\,\hat{\underline{n}} = \eta \left(w_{1\eta}\big|_{\eta = 0^\pm} \right).$

SINCE $\hat{\underline{t}} \cdot \hat{\underline{n}} = 0$ AND $\hat{\underline{n}} \cdot \hat{\underline{n}} = 1$.

$$\text{HENCE} \quad \nabla w_1\big|_\Gamma \cdot (\underline{x} - \underline{x}_0) = \eta \left(w_{1\eta}\big|_{\eta = 0^\pm} \right). \qquad (17)$$

THEREFORE, WE CONCLUDE THAT AS $\eta \to 0^\pm$ THE LEFT SIDE OF (16) AS WE APPROACH THE INTERFACE Γ IS

$$\bar{W}_{0m} + \varepsilon \left[w_1\big|_{\eta=0} + w_{1\eta}\big|_{\eta=0^+} \eta \right] + \varepsilon^2 w_2\big|_{\eta=0} + \cdots \quad \text{AS } \eta \to 0^+ \text{ FROM INSIDE}$$

$$\bar{W}_{0m} + \varepsilon \left[w_1\big|_{\eta=0} + w_{1\eta}\big|_{\eta=0^-} \eta \right] + \varepsilon^2 w_2\big|_{\eta=0} + \cdots \quad \text{AS } \eta \to 0^- \text{ FROM OUTSIDE}$$

NOW THIS IS WRITTEN IN TERMS OF $\hat{\eta}$ GIVEN BY $\eta = \varepsilon \hat{\eta}$. IN TERMS OF THE INNER VARIABLE $\hat{\eta}$ THIS IMPLIES THAT THE LHS OF (16) IS

$$(18) \quad \bar{W}_{0m} + \varepsilon w_1\big|_{\eta=0} + \varepsilon^2 \left[w_{1\eta}\big|_{\eta=0^\pm} \hat{\eta} + w_2\big|_{\eta=0} \right] + \cdots \quad \text{AS } \eta \to 0^\pm \quad \begin{array}{l} + \text{ INSIDE} \\ - \text{ OUTSIDE} \end{array}$$

NOW (18) MUST MATCH WITH THE RHS OF (16) GIVEN BY

$$\bar{W}_{0m} + \varepsilon \bar{W}_1 + \varepsilon^2 \bar{W}_2 \quad \text{AS } \hat{\eta} \to \pm\infty.$$

WE CONCLUDE THAT $\quad \bar{W}_2 \sim w_{1\eta}\big|_{\eta=0^+} \hat{\eta} \quad$ AS $\hat{\eta} \to +\infty$

$$\bar{W}_2 \sim w_{1\eta}\big|_{\eta=0^-} \hat{\eta} \quad \text{AS } \hat{\eta} \to -\infty.$$

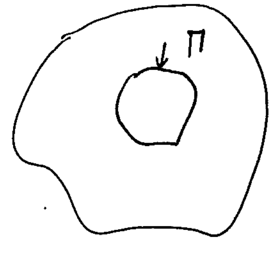
DIFFERENTIATING AND SUBTRACTING GIVES $\quad \bar{W}_{2\hat{\eta}}(\infty) - \bar{W}_{2\hat{\eta}}(-\infty) = w_{1\eta}\big|_{\eta=0^+} - w_{1\eta}\big|_{\eta=0^-}$.

BY (15) WE CONCLUDE THAT

$$[w_{1\eta}]\big|_{\eta=0} \equiv \underbrace{w_{1\eta}\big|_{\eta=0^+}}_{\text{ONSIDE}} - \underbrace{w_{1\eta}\big|_{\eta=0^-}}_{\text{OUTSIDE}} = \bar{V}\left(\bar{U}_{0-} - \bar{U}_{0+} \right).$$

WE CONCLUDE THAT WE HAVE THE <u>MULLINS-SEKERKA FLOW</u>:

$$(19) \quad \left\{ \begin{array}{l} \Delta w_1 = 0 \quad \text{IN } \Omega \text{ BUT AWAY FROM } \Gamma \\ \partial_n w_1 = 0 \quad \text{ON } \partial\Omega \\ w_1 = \gamma \kappa \quad \text{ON } \Gamma \quad \text{WHERE} \quad \gamma = \dfrac{\int_{-\infty}^{\infty} (\bar{U}_{0\hat{\eta}})^2 \, d\hat{\eta}}{\bar{U}_{0-} - \bar{U}_{0+}} \\ [w_{1\eta}]\big|_\Gamma = \bar{V}\left(\bar{U}_{0-} - \bar{U}_{0+} \right) \end{array} \right.$$

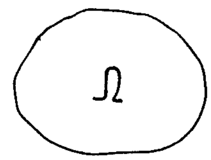


BY A RESCALING OF w_1 AND $\bar{V}$ WE CAN WLOG SET $\gamma = 1$, $\bar{U}_{0-} - \bar{U}_{0+} = 1$

FINALLY, WE USE OUR (η, s) COORDINATE SYSTEM TO EXAMINE SHORT-TIME BEHAVIOR OF A HEAT EQUATION IN $\Omega \subset \mathbb{R}^2$.

<u>EXAMPLE</u> SUPPOSE $u(\underline{x}, t)$ SATISFIES

$$(1)\left\{\begin{array}{l} u_t = D\Delta u \quad \text{FOR } \Omega \subset \mathbb{R}^2,\ \Omega \text{ BOUNDED WITH SMOOTH BOUNDARY} \\ u = u_b \quad \text{ON } \partial\Omega \quad \forall\, t \geq 0 \\ u(\underline{x}, 0) = 0 \quad \text{IN } \Omega. \end{array}\right.$$



FIND THE ASYMPTOTIC BEHAVIOR OF THE SOLUTION AS $t \to 0$. HERE u_b IS TIME-INDEPENDENT.

<u>SOLUTION</u> TO LEADING ORDER WE SHOULD EXPECT AN ERFC PROFILE IN DIRECTION $\perp$ TO $\partial\Omega$; HOWEVER, AT HIGHER ORDER THERE SHOULD BE A CORRECTION DUE TO CURVATURE.

WE TAKE LAPLACE TRANSFORMS:

$$\bar{U}(\underline{x}, \sigma) = \int_0^\infty e^{-\sigma t}\, u(\underline{x}, t)\, d\underline{x}.$$

WE OBTAIN THAT $D\Delta\bar{U} = \sigma\bar{U}$. NOW SHORT TIME $t \to 0^+$ CORRESPONDS TO $\sigma \to \infty$.

WE THEN WRITE

$$\frac{D}{\sigma}\Delta\bar{U} - \bar{U} = 0 \quad \text{WITH} \quad \bar{U} = u_b/\sigma \quad \text{ON } \partial\Omega.$$

WE WRITE $$\frac{D}{\sigma}\Delta\bar{U} - \bar{U} = 0 \quad \text{WITH} \quad \bar{U} = \frac{D}{\sigma}\,\frac{u_b}{D} \quad \text{ON } \partial\Omega.$$

DEFINE $\varepsilon \equiv (\sigma/D)^{-1}$. WE MUST SOLVE

$$\varepsilon\Delta\bar{U} - \bar{U} = 0 \quad \text{ON } \partial\Omega \quad \text{WITH} \quad \bar{U} = \frac{u_b}{\sigma} \quad \text{ON } \partial\Omega.$$

IT IS CONVENIENT TO DEFINE $\bar{V}$ BY

$$\bar{U} = \frac{1}{\sigma}\bar{V}$$

SO THAT
$$\left.\begin{array}{l} \varepsilon\Delta\bar{V} - \bar{V} = 0 \quad \text{IN } \Omega \\ \bar{V} = u_b \quad \text{ON } \partial\Omega \end{array}\right\} \quad (2)$$

WHERE $\varepsilon = D/\sigma \ll 1$. WE WILL NOW SOLVE THIS BY BOUNDARY LAYER ANALYSIS USING OUR FITTED COORDINATES (η, s). LET $\eta > 0$ BE DISTANCE FROM $\underline{x} \in \Omega$ TO $\partial\Omega$. WE HAVE THAT (2) BECOMES

$$(3)\left\{\ \varepsilon\left(\bar{V}_{\eta\eta} - \frac{\kappa}{1-\kappa\eta}\bar{V}_\eta + \frac{1}{1-\kappa\eta}\frac{\partial}{\partial s}\left(\frac{\bar{V}_s}{1-\kappa\eta}\right)\right) - \bar{V} = 0 \quad \text{NEAR } \partial\Omega;\ \bar{V} = u_b \text{ ON } \eta = 0.\right.$$

HERE κ IS CURVATURE OF $\partial\Omega$, i.e. $\kappa = \kappa(s)$, WITH $\kappa > 0$ IF Ω CONVEX AS SEEN FROM INSIDE. WE THEN INTRODUCE THE BOUNDARY LAYER SCALING

$$\left.\tilde{\eta} = \eta/\varepsilon^{1/2}\right\} \quad (4)$$

AND THEN OBTAIN FROM (3) THAT

$$\bar{V}_{\hat{\eta}\hat{\eta}} - \kappa\varepsilon^{1/2}\bar{V}_{\hat{\eta}} + O(\varepsilon) - \bar{V} = 0$$

WE THEN EXPAND $\bar{V} = \bar{V}_0 + \varepsilon^{1/2}\bar{V}_1 + \dots$

AND FROM (3) OBTAIN

$$\left.\begin{aligned} &\bar{V}_{0\hat{\eta}\hat{\eta}} - \bar{V}_0 = 0 \quad \text{IN } \hat{\eta} \geq 0 \\ &\bar{V}_0 = \bar{u}_b \quad \text{ON } \hat{\eta} = 0; \quad \bar{V}_0 \to 0 \quad \text{AS } \hat{\eta} \to \infty \end{aligned}\right\} \quad (5a)$$

$$\left.\begin{aligned} &\bar{V}_{1\hat{\eta}\hat{\eta}} - \bar{V}_1 = \kappa\bar{V}_{0\hat{\eta}} \quad \text{IN } \hat{\eta} \geq 0 \\ &\bar{V}_1 = 0 \quad \text{ON } \hat{\eta} = 0; \quad \bar{V}_1 \to 0 \quad \text{AS } \hat{\eta} \to \infty \end{aligned}\right\} \quad (5b)$$

HERE $\bar{u}_b = \bar{u}_b(s)$ AND $\kappa = \kappa(s)$ WHERE s IS ARCLENGTH ALONG $\partial\Omega$.

THE SOLUTION TO (5a) IS

$$\bar{V}_0 = \bar{u}_b e^{-\tilde{\eta}}$$

AND THEN WE HAVE THAT (5b) BECOMES

$$\bar{V}_{1\hat{\eta}\hat{\eta}} - \bar{V}_1 = -\bar{u}_b\kappa e^{-\tilde{\eta}} \longrightarrow V_{1p} = A\eta e^{-\tilde{\eta}} \Rightarrow \bar{V}_{1p} = \frac{\bar{u}_b\kappa}{2}\hat{\eta}e^{-\hat{\eta}}.$$

WE CONCLUDE THAT $\bar{V}_1 = \bar{V}_{1p}$ SATISFIES $\bar{V}_1 = 0$ ON $\hat{\eta} = 0$. THUS,

$$\bar{V}_1 = \frac{\bar{u}_b\kappa}{2}\hat{\eta}e^{-\hat{\eta}}.$$

THE INNER SOLUTION TO TWO TERMS IS

$$\bar{V} \sim \bar{V}_0 + \varepsilon^{1/2}\bar{V}_1 + \dots = \bar{u}_b e^{-\hat{\eta}} + \frac{\varepsilon^{1/2}\bar{u}_b\kappa}{2}\hat{\eta}e^{-\tilde{\eta}} + \dots$$

WE HAVE THAT THE OUTER SOLUTION IS $\bar{V} \doteq 0$ (EXPONENTIALLY SMALL).

OUR LAST STEP IS TO PUT $\hat{\eta} = \eta/\varepsilon^{1/2}$ AND $\varepsilon = \frac{D}{\sigma}$ AND INVERT LAPLACE TRANSFORM.

FROM (2), WE HAVE

$$\tilde{U} \doteq \frac{1}{\sigma}\tilde{V} = \frac{1}{\sigma}\left[U_b e^{-\Lambda/\varepsilon^{1/2}} + \frac{\varepsilon^{1/2} U_b K}{2}\left(\frac{\Lambda}{\varepsilon^{1/2}}\right) e^{-\Lambda/\varepsilon^{1/2}}\right] \quad \text{VALID FOR } \Lambda = O(\varepsilon^{1/2}).$$

NOW $\varepsilon = D/\sigma$ SO THAT

$$\tilde{U} \cong \frac{1}{\sigma} U_b e^{-\Lambda\sqrt{\sigma}/\sqrt{D}} + \frac{K U_b}{2\sigma}\Lambda e^{-\Lambda\sqrt{\sigma}/\sqrt{D}} \quad \text{FOR } \Lambda = O(\varepsilon^{1/2}).$$

NOW RECALL THAT

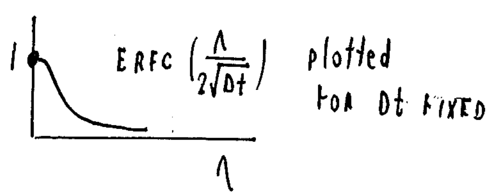
$$\mathcal{L}^{-1}\left[\frac{e^{-\lambda\sqrt{\sigma}}}{\sigma}\right] = \text{ERFC}\left(\frac{\lambda}{2\sqrt{t}}\right).$$

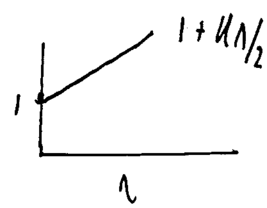
SO THAT
$$\tilde{U} \cong \frac{U_b}{\sigma}\left(1 + \frac{K\Lambda}{2}\right) e^{-\Lambda\sqrt{\sigma}/\sqrt{D}} \quad \text{FOR } \Lambda = O(1/\sigma^{1/2}).$$

WE OBTAIN WITH $\lambda = \Lambda/\sqrt{D}$ THAT

$$u(x,t) \cong U_b\left(1 + K\Lambda/2\right)\text{erfc}\left(\Lambda/2\sqrt{Dt}\right). \quad \text{FOR } \Lambda \ll 1.$$

WE CONCLUDE THAT WE HAVE A $O(K\Lambda)$ CORRECTION TO THE ERFC PROFILE. FOR $\Lambda > 0$ SMALL BUT FIXED, THE TEMPERATURE DEVIATES LESS FROM U_b IN REGIONS OF HIGH CURVATURE. NOTICE $1 + K\Lambda/2 \uparrow$ AS $\Lambda\uparrow$ AND $\text{ERFC}\left(\frac{\Lambda}{2\sqrt{Dt}}\right)\downarrow$ AS $\Lambda\uparrow$.



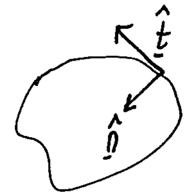


THE PRODUCT INDICATES THAT FOR Dt FIXED, u REMAINS CLOSER TO U_b AT A FIXED DISTANCE FROM THE BOUNDARY IN REGIONS OF LARGE CURVATURE.

APPENDIX (CURVE PROPERTIES AND CURVE-SHORTENING FLOW) (A1)

CONSIDER A CLOSED CURVE $\underline{X}(s)$ WITH $s \in [0, a]$ FOR WHICH

$$\underline{X}(0) = \underline{X}(a) \quad \text{AND} \quad \underline{X}'(0) = \underline{X}'(a), \text{ AND } \underline{X} \text{ IS SMOOTH.}$$



THE UNIT TANGENT IS

$$\hat{\underline{t}} = \frac{\underline{X}'}{|\underline{X}'|} \qquad (1)$$

THE LENGTH L OF THE CURVE, IN TERMS OF THE PARAMETRIZATION s IS

$$L = \int_0^a |\underline{X}'(s)|\, ds .$$

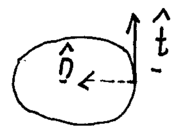
NOW THE ARCLENGTH DIFFERENTIAL dS IS DEFINED BY $dS = |\underline{X}'(s)|\, ds$ SO THAT

$$\frac{ds}{dS} = \frac{1}{|\underline{X}'(s)|} . \qquad (2)$$

NOW THE UNIT NORMAL $\hat{\underline{n}}$ IS DEFINED BY

$$\hat{\underline{n}} = R\,\hat{\underline{t}} \quad \text{WHERE} \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ IS } 90^\circ \text{ ROTATION MATRIX.}$$

NOTE: IF $\hat{\underline{t}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ THEN $\hat{\underline{n}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ AS SHOWN



THUS IF s IS COUNTERCLOCKWISE PARAMETRIZATION, $\hat{\underline{n}}$ POINTS ORTHOGONALLY INSIDE THE CURVE.

NOW IF $s = S$ IS ARCLENGTH PARAMETRIZATION, THEN $|\underline{X}'(S)| = 1$.

IN THIS CASE, WE HAVE THAT

$$\frac{d\hat{\underline{t}}}{dS} = \kappa\,\hat{\underline{n}} \quad \text{AND} \quad \frac{d\hat{\underline{n}}}{dS} = -\kappa\,\hat{\underline{t}} \qquad (3)$$

WHERE κ IS THE CURVATURE OF $\underline{X}(S)$.

NOW TO COMPUTE THE CURVATURE WHEN WE DO NOT PARAMETRIZE THE CURVE BY ARCLENGTH, BUT INSTEAD USE $\underline{X}(s)$ WE USE (1) AND (2), COMBINED WITH FIRST OF (3), TO GET

$$\frac{d\hat{\underline{t}}}{dS} = \frac{d\hat{\underline{t}}}{ds}\frac{ds}{dS} = \frac{d}{ds}\left(\frac{\underline{X}'}{|\underline{X}'|}\right)\frac{1}{|\underline{X}'|} = \kappa\,\hat{\underline{n}} .$$

SINCE $|\hat{\underline{n}}| = 1$ AS $\hat{\underline{n}}$ UNIT VECTOR, WE CONCLUDE THAT

$$\kappa = \frac{\left|\left(\underline{X}'/|\underline{X}'|\right)'\right|}{|\underline{X}'|} \qquad (4).$$

FROM THIS EXPRESSION (4) WE OBSERVE THAT κ IS AN INTRINSIC PROPERTY OF THE CURVE AND IS INDEPENDENT OF THE SPECIFIC PARAMETRIZATION USED. i.e. IF $\tilde{s} = f(s)$ IS ANOTHER PARAMETRIZATION THEN CAN USE (4) WITH $'$ DENOTING DERIVATIVES IN $\hat{s}$.

NOW IN (4) WE CALCULATE

$$\left.\frac{d}{ds}\frac{\underline{X}'}{|\underline{X}'|} = \frac{\underline{X}''}{|\underline{X}'|} - \frac{\underline{X}'}{|\underline{X}'|^2}\frac{d}{ds}|\underline{X}'|\right\} (5)$$

BUT $\frac{d}{ds}|\underline{X}'|^2 = \frac{d}{ds}(\underline{X}'\cdot\underline{X}') = 2\,\underline{X}'\cdot\underline{X}''$

$$\rightarrow \quad 2\,|\underline{X}'|\frac{d}{ds}|\underline{X}'| = 2\,\underline{X}'\cdot\underline{X}''.$$

WE CONCLUDE THAT

$$\frac{d}{ds}|\underline{X}'| = \frac{\underline{X}'\cdot\underline{X}''}{|\underline{X}'|}. \qquad (6)$$

SUBSTITUTING INTO (5) WE OBTAIN, USING (4) THAT κ IS GIVEN BY

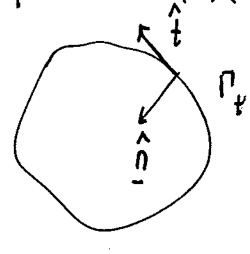
$$\kappa = \frac{1}{|\underline{X}'|}\left|\frac{\underline{X}''}{|\underline{X}'|} - \frac{\underline{X}'(\underline{X}'\cdot\underline{X}'')}{|\underline{X}'|^3}\right|.$$

WITH THIS BACKGROUND WE NOW DISCUSS CURVE-SHORTENING FLOW.

WE DERIVED EARLIER THAT $\dot{n} = -\kappa$

IN THIS WAY IF WE WRITE THE CURVE Γ_t AS A CURVE $\underline{X}(s,\tau)$ WITH $s \in S^1$ BEING A PARAMETRIZATION OF Γ_t (i.e. $0 \le s \le 2\pi$, s periodic) DEFINED SO THAT

$$\frac{d\underline{X}}{d\tau} = -\dot{n}\,\hat{\underline{n}} \qquad (7),$$



THE CURVE WILL DEFORM AS τ INCREASES.

IF $\dot{n} > 0$ (EXPANDING CURVE), $d\underline{X}/d\tau$ IS OPPOSITE TO NORMAL. CURVE SHORTENING FLOW THEN CORRESPONDS TO

$$\frac{d\underline{X}}{d\tau} = -V\hat{\underline{n}} \qquad \text{WITH} \quad V = -\kappa. \qquad (7)_2$$

THE UNIT TANGENT IS $\hat{\underline{t}} = \dfrac{\underline{X}_s}{|\underline{X}_s|}$ AND $\hat{\underline{n}} = R\,\hat{\underline{t}}$, WHERE $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

IS ROTATION MATRIX.

THE LENGTH OF Γ_t IS

$$L = \int_{S^1} |\underline{X}_s|\, ds.$$ WE WANT TO CALCULATE HOW L CHANGES WRT τ.

THEN
$$\frac{dL}{d\tau} = \int_{S_1} \frac{d}{d\tau}|\underline{X}_s|\, ds = \int_{S_1} \frac{\underline{X}_s \cdot \underline{X}_{s\tau}}{|X_s|}\, ds \qquad \text{(SEE EQUATION (6))}$$

$$= \int_{S_1} \hat{\underline{t}} \cdot (\underline{X}_\tau)_s\, ds = \int_{S_1} T \cdot (\underline{X}_\tau)_s\, |\underline{X}_s|\, ds$$

$$= -\int_{S_1} \hat{\underline{t}} \cdot [V\, \hat{\underline{n}}]_s\, |\underline{X}_s|\, ds \qquad \left(\text{USING } \underline{X}_\tau = -V\hat{\underline{n}}\right)$$

$$= -\int_{S_1} \hat{\underline{t}} \cdot [V_s\, \hat{\underline{n}} + V\, \hat{\underline{n}}_s]\, |\underline{X}_s|\, ds$$

$$= -\int_{S_1} (\hat{\underline{t}} \cdot \underline{n}_s)\, V\, |\underline{X}_s|\, ds \qquad \text{SINCE } \hat{\underline{n}} \cdot \hat{\underline{t}} = 0.$$

$$= -\int_{S_1} V(-\kappa)\,(\hat{\underline{t}} \cdot \hat{\underline{t}})\, |\underline{X}_s|\, ds \qquad \text{SINCE } \hat{\underline{n}}_s = -\kappa\, \hat{\underline{t}}.$$

$$\frac{dL}{d\tau} = \int_{S_1} \kappa V\, |\underline{X}_s|\, ds \qquad \text{SINCE } \hat{\underline{t}} \cdot \hat{\underline{t}} = 1$$

WE CONCLUDE THAT IF $\underline{X}_\tau = -V\hat{\underline{n}}$ IS GEOMETRICAL LAW OF CURVE EVOLUTION, THEN

$$dL/d\tau = \int_{S_1} \kappa V\, |\underline{X}_s|\, ds. \qquad (8)_1$$

FINALLY SINCE $dS = |\underline{X}_s|\, ds$, THEN SINCE $V = -\kappa$, WE HAVE

$$dL/d\tau = -\int_{\Gamma_t} \kappa^2\, dS < 0 \qquad (8)_2$$

$\longrightarrow$ CURVE SHORTENING FLOW. $L \downarrow$ AS $\tau \uparrow$.

NOW WITH REGARDS TO THE AREA WE HAVE (SEE HW) THAT

$$\frac{dA}{d\tau} = \int_{\Gamma_t} V\, dS. \qquad (9)$$

REMARK WE CAN WRITE

$$A = \frac{1}{2}\oint_{\Gamma_t}(x\,dy - y\,dx) = \frac{1}{2}\int_{S_1}(x y_\xi - y x_\xi)\,d\xi = \frac{1}{2}\int_{S_1} R\underline{X}\cdot\underline{X}_\xi\,d\xi$$

USING CHAIN RULE WE GET

$$dA/d\tau = \frac{1}{2}\int_{S_1}\left(R\,\underline{X}_\tau\cdot\underline{X}_\xi + R\,\underline{X}\cdot\underline{X}_{\xi\tau}\right)d\xi$$

AND WE USE $\underline{X}_\xi = \hat{\underline{t}}\,|\underline{X}_\xi|$, AND USE IBP ON SECOND TERM.

CONTINUING ON IF $\underline{X}_\tau = -V\hat{\underline{n}}$ WE OBTAIN, AFTER SOME MORE CALCULATION THAT

$$dA/d\tau = \int_{\Gamma_t} V\,ds\,.$$

FOR CURVE SHORTENING FLOW $V = -\kappa$ AND SO

$$dA/d\tau = \int_{\Gamma_t}(-\kappa)\,ds = -2\pi.$$

WE CONCLUDE THAT $A = A(0) - 2\pi\tau$

AND SO AREA INSIDE Γ_t IS EXTINGUISHED AT TIME $\tau = \dfrac{A(0)}{2\pi}$.

NEXT WE DERIVATIVE A FEW FURTHER PROPERTIES OF CURVE SHORTENING FLOW.

LEMMA LET Γ_t BE A C^2 SMOOTH FAMILY OF CURVES SATISFYING THE MEAN CURVATURE FLOW $\dfrac{d\underline{X}}{d\tau} = \kappa\,\hat{\underline{n}}$ WHERE $\underline{X} = \underline{X}(\xi,\tau)$ CHARACTERIZES FAMILY OF CURVES WITH $\xi \in S^1$. THEN, IF s IS ARCLENGTH ON Γ_t, WE HAVE

(i) $\underline{X}_\tau = \underline{X}_{ss}$ WHERE s IS ARCLENGTH ALONG Γ_t.

(ii) IF $g(s,\tau)$ IS ANY C^2 FUNCTION, THEN $g_{s\tau} = g_{\tau s} + \kappa^2 g_s$.

(iii) $\hat{\underline{t}}_\tau = \kappa_s\,\hat{\underline{n}}$, $\hat{\underline{n}}_\tau = -\kappa_s\,\hat{\underline{t}}$.

(iv) $\kappa_\tau = \kappa_{ss} + \kappa^3$.

PROOF

(i) WE RECALL THAT $\frac{d\hat{\underline{t}}}{ds} = \kappa\hat{\underline{n}}$ AND $\frac{d\hat{\underline{n}}}{ds} = -\kappa\hat{\underline{t}}$.

BUT IF s IS ARCLENGTH $\hat{\underline{t}} = \underline{X}_s$ SINCE $|\underline{X}_s| = 1$ MUST HOLD. SINCE $\kappa\hat{\underline{n}} = \underline{X}_\tau$, WE OBTAIN THAT $\frac{d\hat{\underline{t}}}{ds} = \kappa\hat{\underline{n}} \longrightarrow \underline{X}_{ss} = \underline{X}_\tau$.

ALTHOUGH $\underline{X}$ APPEARS TO SATISFY A DIFFUSION EQUATION THAT IS LINEAR, THIS IS MISLEADING SINCE s DEPENDS ON τ, OWING TO THE FACT THAT Γ_t CHANGES AS t INCREASES

(ii) IF $g(\xi,\tau)$ IS C^2 SMOOTH IN ξ AND τ, WE HAVE $g_{\xi\tau} = g_{\tau\xi}$.

HOWEVER, IN TERMS OF ARCLENGTH WE NOW SHOW THAT $g_{s\tau} \neq g_{\tau s}$. SINCE $\frac{d\xi}{ds} = \frac{1}{|\underline{X}_\xi|}$

$$g_{s\tau} = \left(\frac{g_\xi}{|\underline{X}_\xi|}\right)_\tau = \frac{g_{\xi\tau}}{|\underline{X}_\xi|} - g_\xi \frac{\underline{X}_\xi\cdot\underline{X}_{\xi\tau}}{|\underline{X}_\xi|^3}$$

SINCE $\underline{X}_{\xi\tau} = \underline{X}_{\tau\xi}$ AND $g_{\xi\tau} = g_{\tau\xi}$ WE GET

$$g_{s\tau} = \frac{g_{\tau\xi}}{|\underline{X}_\xi|} - g_s \frac{\underline{X}_\xi}{|\underline{X}_\xi|}\cdot\frac{(\underline{X}_\tau)_\xi}{|\underline{X}_\xi|^2} = g_{\tau s} - g_s\,\hat{\underline{t}}\cdot(\underline{X}_\tau)_s$$

BUT $\underline{X}_\tau = \kappa\hat{\underline{n}}$ SO THAT

$$g_{s\tau} = g_{\tau s} - g_s\,\hat{\underline{t}}\cdot[\kappa\hat{\underline{n}}]_s = g_{\tau s} - g_s\,\hat{\underline{t}}\cdot[\kappa_s\hat{\underline{n}} + \kappa\hat{\underline{n}}_s].$$

SINCE $\hat{\underline{t}}\cdot\hat{\underline{n}} = 0$ AND $\hat{\underline{n}}_s = -\kappa\hat{\underline{t}}$ WITH $\hat{\underline{t}}\cdot\hat{\underline{t}} = 1$ WE GET

$$g_{s\tau} = g_{\tau s} - g_s\,\hat{\underline{t}}\cdot(-\kappa^2\hat{\underline{t}}) = g_{\tau s} + \kappa^2 g_s.$$

THUS $g_{s\tau} = g_{\tau s} + \kappa^2 g_s$. (*)

(iii) USE $\underline{X}_\tau = \kappa\hat{\underline{n}}$ AND $\hat{\underline{t}} = \underline{X}_s$. LET $g_s = \underline{X}_s$.

WE CALCULATE USING (*) THAT

$$\begin{aligned}\hat{\underline{t}}_\tau = (\underline{X}_s)_\tau &= (\underline{X}_\tau)_s + \kappa^2\underline{X}_s \\ &= (\kappa\hat{\underline{n}})_s + \kappa^2\hat{\underline{t}} \\ &= \kappa_s\hat{\underline{n}} + \kappa\hat{\underline{n}}_s + \kappa^2\hat{\underline{t}} = \kappa_s\hat{n} + \kappa\cancel{(-\kappa\hat{\underline{t}})} + \cancel{\kappa^2\hat{t}}\end{aligned}$$

THUS $\hat{\underline{t}}_\tau = \kappa_s\hat{\underline{n}}$.

NOW SINCE THE ROTATION MATRIX R IS INDEPENDENT OF τ,

$$\hat{\underline{n}}_\tau = R\,\hat{\underline{t}}_\tau = \kappa_s R\,\hat{\underline{n}} \qquad \text{BUT} \qquad R\,\hat{\underline{n}} = R^2\hat{\underline{t}} = -\hat{\underline{t}}$$

HENCE $\hat{\underline{n}}_\tau = -\kappa_s\,\hat{\underline{t}}$.

(iv) TO DERIVE THE PDE FOR THE CURVATURE, WE DIFFERENTIATE $\hat{\underline{t}}_s = \kappa\,\hat{\underline{n}}$ IN τ AND USE (iii) AND (i)

NOW
$$(\hat{\underline{t}}_s)_\tau = (\hat{\underline{t}}_\tau)_s + \kappa^2\,\hat{\underline{t}}_s$$
$$= (\kappa_s\hat{\underline{n}})_s + \kappa^2\,\hat{\underline{t}}_s$$
$$(\hat{\underline{t}}_s)_\tau = \kappa_{ss}\,\hat{\underline{n}} + \kappa_s\,\hat{\underline{n}}_s + \kappa^2\,\hat{\underline{t}}_s.$$

NOW $\hat{\underline{n}}_s = -\kappa\,\hat{\underline{t}}$ AND SO WE GET USING $\hat{\underline{t}}_s = \kappa\,\hat{\underline{n}}$ THAT

$$(\hat{\underline{t}}_s)_\tau = (\kappa_{ss} + \kappa^3)\,\hat{\underline{n}} - \kappa\,\kappa_s\,\hat{\underline{t}}. \qquad (*)$$

NOW WE CALCULATE THE LEFT SIDE DIRECTLY USING $\hat{\underline{t}}_s = \kappa\,\hat{\underline{n}}$.

$$(\hat{\underline{t}}_s)_\tau = [\kappa\,\hat{\underline{n}}]_\tau = \kappa_\tau\,\hat{\underline{n}} + \kappa\,\hat{\underline{n}}_\tau = \kappa_\tau\,\hat{\underline{n}} - \kappa\,\kappa_s\,\hat{\underline{t}}. \quad (+)$$

COMPARING $(*)$ AND $(+)$ WE OBTAIN THAT THE TANGENTIAL COMPONENTS AGREE, AND THE NORMAL COMPONENTS WILL AGREE WHEN

$$\kappa_\tau = \kappa_{ss} + \kappa^3. \qquad (**)$$

THIS IS A NONLINEAR PDE FOR THE CURVATURE OF Γ_t, BUT s STILL DEPENDS ON τ. IT CAN BE USED TO ESTABLISH A CONVEXITY RESULT.

<u>THEOREM</u> LET Γ_t BE A C^2 FAMILY OF CURVES SATISFYING THE CURVE SHORTENING FLOW $\underline{X}_\tau = \kappa\,\hat{\underline{n}}$. SUPPOSE THAT Γ_0 (i.e. at $\tau = 0$) IS CONVEX. THEN Γ_t IS CONVEX FOR ALL $\tau \in [0,T]$.

<u>PROOF</u> WE HAVE CONVEXITY IFF $\kappa \geq 0$ $\forall s$ AT EACH FIXED τ. WE NEED ONLY APPLY MAXIMUM PRINCIPLE TO $(**)$. SUPPOSE $\exists\, \tau_0 \in (0,T)$ WHICH IS THE FIRST TIME κ VANISHES AT SOME $s = s^*$, SO THAT

$$\kappa(\tau_0, s^*) = 0, \quad \kappa(\tau_0, s) \geq 0 \quad \forall s.$$

THUS $\kappa_{ss}(\tau_0, s^*) \geq 0$, BUT FROM PDE $(**)$ $\kappa_\tau(\tau_0, s^*) = (\kappa_{ss} + \kappa^3)\big|_{s^*,\tau_0} \geq 0$.

THUS $\kappa_\tau \geq 0$ AND CANNOT DECREASE BELOW ZERO. □

THE FOLLOWING CELEBRATED THEOREM OF GAGE-HAMILTON (1986) AND GRAYSON (1987) SUMMARIZES THE MAIN FACTS FOR CURVE SHORTENING FLOW.

<u>THEOREM</u> SUPPOSE THAT AT $\tau = 0$, Γ_0 IS A C^2 EMBEDDED CURVE IN $\mathbb{R}^2$ WITH BOUNDED CURVATURE AND ENCLOSING AN AREA A_0. THEN $\exists$ A SMOOTH FAMILY OF EMBEDDED CURVES, INDEPENDENT OF THE SPECIFIC PARAMETRIZATION OF Γ_t, THAT SATISFIES $\underline{X}_\tau = \kappa \hat{\underline{D}}$, OR EQUIVALENTLY $\underline{X}_\tau = \underline{X}_{ss}$, ON $0 \leq \tau < \tau_\infty \equiv A_0/2\pi$. THE FOLLOWING CHARACTERISTICS HOLD.

(i) IF Γ_0 IS NONCONVEX, $\exists\, \tau_1$ IN $(0, \tau_\infty)$ WHERE Γ_{τ_1} IS CONVEX. IT REMAINS CONVEX $\forall\, \tau > \tau_1$.

(ii) FLOW SHRINKS TO A ROUND POINT IN THE SENSE THAT

$$\frac{\operatorname{MIN}_{\Gamma_t} \kappa}{\operatorname{MAX}_{\Gamma_t} \kappa} \rightarrow 1 \text{ AS } \tau \rightarrow \tau_\infty \quad \text{AND} \quad \underline{X} \rightarrow \underline{X}_0 \text{ AS } \tau \rightarrow \tau_\infty.$$