Math 551: Two Strong Localized Perturbation Problems in 2-D

**Worked Example 1:** Consider the following eigenvalue problem in a **2-D domain** with $K$ small holes:

$$\triangle u + \lambda u = 0\,, \quad \mathbf{x} \in \Omega \backslash \Omega_p\,; \qquad \Omega_p \equiv \cup_{j=1}^{K} \Omega_{\varepsilon_j}\,, \tag{0.1 a}$$

$$\partial_n u = 0\,, \quad \mathbf{x} \in \partial\Omega\,; \qquad \int_{\Omega\backslash\Omega_p} u^2\, d\mathbf{x} = 1 \tag{0.1 b}$$

$$u = 0\,, \quad \mathbf{x} \in \partial\Omega_{\varepsilon_j}\,, \quad j = 1, \ldots, K\,. \tag{0.1 c}$$

We assume that each hole $\Omega_{\varepsilon_j}$ is centered at $\mathbf{x}_j \in \Omega$. Assume further that the holes have a common logarithmic capacitance $d \equiv d_1 = \ldots = d_K$.

(1) Derive a two-term expansion for the lowest eigenvalue $\lambda_0$ of this problem in the form

$$\lambda_0 \sim \lambda_{00}\nu + \lambda_{01}\nu^2 + O(\nu^3)\,, \tag{0.2}$$

where $\lambda_{00}$ and $\lambda_{01}$ are to be found, and $\nu \equiv -1/\log(\varepsilon d)$. (Hint: the result for $\lambda_{01}$ will involve a sum of the entries of a certain $K \times K$ Green's function matrix).

(2) For the case of a concentric annular domain $\varepsilon < r < 1$ with $r = |\mathbf{x}|$, show that general your two-term asymptotic result above reduces to $\lambda_0 \sim 2\nu + 3\nu^2/2$ where $\nu = -1/\log\varepsilon$. Verify that this two-term result agrees with the result obtained from an asymptotic approximation of the transcendental equation for the exact lowest eigenvalue. The exact transcendental equation for $\lambda$ is obtained by making $u(r) = J_0(\sqrt{\lambda} r) + aY_0(\sqrt{\lambda} r)$ satisfy $u'(1) = u(\varepsilon) = 0$. (Hint: in approximating the solution to the transcendental equation you will need the following behavior for $J_0(z)$ and $Y_0(z)$ for $z \to 0$:)

$$J_0(z) \sim 1 - z^2/4 + z^4/64 + \cdots\,; \qquad Y_0(z) \sim \frac{2}{\pi}\left[ (\log(z/2) + \gamma_e)\left(1 - \frac{z^2}{4}\right) + \frac{z^2}{4}\right] + \cdots\,,$$

where $\gamma_e$ is Euler's constant.

(3) Recall that the first passage time $w(\mathbf{x})$ for Brownian motion in a 2-D domain starting a point $\mathbf{x} \in \Omega$ in a domain with $K$ traps, and with diffusivity D, satisfies

$$\triangle w = -\frac{1}{D}\,, \quad \mathbf{x} \in \Omega\backslash\Omega_p\,; \qquad \Omega_p \equiv \cup_{j=1}^{K} \Omega_{\varepsilon_j}\,, \tag{0.3 a}$$

$$\partial_n w = 0\,, \quad \mathbf{x} \in \partial\Omega\,; \qquad w = 0\,, \quad \mathbf{x} \in \partial\Omega_{\varepsilon_j}\,, \quad j = 1, \ldots, K\,. \tag{0.3 b}$$

From your answer for $\lambda_0$ above, calculate a two-term asymptotic expansion for the average mean first passage time, defined by $\bar{w} = |\Omega\backslash\Omega_p|^{-1} \int_{\Omega\backslash\Omega_p} w\, d\mathbf{x}$. Here $|\Omega\backslash\Omega_p| \sim |\Omega| + O(\varepsilon^2)$ denotes the area of the domain with the holes removed.

(4) Show how your result for $\lambda_0$ above immediately applies to determining a critical value of the diffusivity $D$ for the extinction threshold of a population satisfying the diffuse logistic model $U_t = D\triangle U + \mu U\,(1 - U/\beta)$ in a 2-D domain with reflecting outer boundary, and with localized regions where the population is extinct. Here $\mu$ and $\beta$ are positive constants. (I am looking for a simple explanation here)

**Solution:**

(1) We look for a two-term expansion for the principal eigenvalue $\lambda_0(\varepsilon)$ as

$$\lambda_0(\varepsilon) = \lambda_1\nu + \lambda_2\nu^2 + \cdots\,, \qquad \nu = -1/\log(\varepsilon d)\,. \tag{0.4}$$

In the outer region, away from $\mathcal{O}(\varepsilon)$ neighbourhoods of the holes, we expand the outer solution for $u$ as

$$u = u_0 + \nu u_1 + \nu^2 u_2 + \cdots\,. \tag{0.5}$$

The leading-order term is

$$u_0 = |\Omega|^{-1/2}\,, \tag{0.6}$$

where $|\Omega|$ is the area of $\Omega$. Upon substituting (0.4) and (0.5) into (0.1 $a$) and (0.1 $b$), and collecting powers of $\nu$, we obtain that $u_1$ satisfies

$$\triangle u_1 = -\lambda_1 u_0 \,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_1, \ldots, \mathbf{x}_K\} \,; \qquad \int_\Omega u_1 \, d\mathbf{x} = 0 \,, \tag{0.7 a}$$

$$\partial_n u_1 = 0 \,, \quad \mathbf{x} \in \partial\Omega \,; \quad u_1 \text{ singular as } \mathbf{x} \to \mathbf{x}_j \,, \quad j = 1, \ldots, K \,, \tag{0.7 b}$$

while $u_2$ satisfies

$$\triangle u_2 = -\lambda_2 u_0 - \lambda_1 u_1 \,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_1, \ldots, \mathbf{x}_K\} \,; \qquad \int_\Omega \left( u_1^2 + 2 u_0 u_2 \right) \, d\mathbf{x} = 0 \,, \tag{0.8 a}$$

$$\partial_n u_2 = 0 \,, \quad \mathbf{x} \in \partial\Omega \,; \quad u_2 \text{ singular as } \mathbf{x} \to \mathbf{x}_j \,, \quad j = 1, \ldots, K \,. \tag{0.8 b}$$

Now in the $j^{\text{th}}$ inner region we introduce the new variables by

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j) \,, \qquad v(\mathbf{y}) = u(\mathbf{x}_j + \varepsilon \mathbf{y}) \,. \tag{0.9}$$

We then expand the inner solution as

$$v(\mathbf{y}) = \nu A_{0j} v_{cj}(\mathbf{y}) + \nu^2 A_{1j} v_{cj}(\mathbf{y}) + \cdots \,. \tag{0.10}$$

Upon substituting (0.9) and (0.10) into (0.1 $a$) and (0.1 $c$), we obtain that $v_{cj}$ satisfies

$$\triangle_{\mathbf{y}} v_{cj} = 0 \,, \quad \mathbf{y} \notin \Omega_j \,; \quad v_{cj} = 0 \,, \quad \mathbf{y} \in \partial\Omega_j \,, \tag{0.11 a}$$

$$v_{cj}(\mathbf{y}) \sim \log|\mathbf{y}| - \log d + o(1) \,, \quad \text{as} \quad |\mathbf{y}| \to \infty \,. \tag{0.11 b}$$

Here $\triangle_{\mathbf{y}}$ is the Laplacian in the $\mathbf{y}$ variable, and $\Omega_j \equiv \varepsilon^{-1}\Omega_{\varepsilon_j}$. We consider the special case where $d$ is independent of $j$.

Upon using the far-field form (0.11 $b$) in (0.10), and writing the resulting expression in outer variables, we get

$$v = A_{0j} + \nu \left[ A_{0j} \log|\mathbf{x} - \mathbf{x}_j| + A_{1j} \right] + \nu^2 \left[ A_{1j} \log|\mathbf{x} - \mathbf{x}_j| + A_{2j} \right] + \cdots \,. \tag{0.12}$$

The far-field behavior (0.12) must agree with the local behavior of the outer expansion (0.5). Therefore, we obtain that

$$A_{0j} = u_0 = |\Omega|^{-1/2} \,, \quad j = 1, \ldots K \,, \tag{0.13 a}$$

$$u_1 \sim u_0 \log|\mathbf{x} - \mathbf{x}_j| + A_{1j} \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j \,, \quad j = 1, \ldots, K \,, \tag{0.13 b}$$

$$u_2 \sim A_{1j} \log|\mathbf{x} - \mathbf{x}_j| + A_{2j} \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j \,, \quad j = 1, \ldots, K \,. \tag{0.13 c}$$

Equations (0.13 $b$) and (0.13 $c$) give the required singularity structure for $u_1$ and $u_2$ in (0.7) and (0.8), respectively.

The problem for $u_1$ with singular behavior (0.13 $b$) can be written in terms of the delta function as

$$\triangle u_1 = -\lambda_1 u_0 + 2\pi A_0 \sum_{j=1}^{K} \delta(\mathbf{x} - \mathbf{x}_j) \,, \quad \mathbf{x} \in \Omega \,; \qquad \int_\Omega u_1 \, d\mathbf{x} = 0 \,, \tag{0.14 a}$$

$$\partial_n u_1 = 0 \,, \quad \mathbf{x} \in \partial\Omega \,. \tag{0.14 b}$$

Upon using the divergence theorem we obtain that $-\lambda_1 u_0 \int_\Omega 1 d\mathbf{x} + 2\pi A_0 K = 0$, so that with $u_0 = A_0$ from (0.13 $a$), we get

$$\lambda_1 = \frac{2\pi K}{|\Omega|} \,. \tag{0.15}$$

The solution to (0.14) can be written in terms of the Neumann Green's function as

$$u_1 = -2\pi u_0 \sum_{i=1}^{K} G_N(\mathbf{x}; \mathbf{x}_i) \,, \tag{0.16}$$

where the Neumann Green's function $G_N(\mathbf{x};\boldsymbol{\xi})$ satisfies

$$\triangle G_N = \frac{1}{|\Omega|} - \delta(\mathbf{x}-\boldsymbol{\xi}) \,, \quad \mathbf{x}\in\Omega \,; \quad \partial_n G_N = 0 \,, \quad \mathbf{x}\in\partial\Omega \,, \tag{0.17 a}$$

$$G_N(\mathbf{x};\boldsymbol{\xi}) \sim -\frac{1}{2\pi}\log|\mathbf{x}-\boldsymbol{\xi}| + R_N(\boldsymbol{\xi};\boldsymbol{\xi}) + o(1) \,, \quad \text{as} \quad \mathbf{x}\to\boldsymbol{\xi} \,; \quad \int_\Omega G_N(\mathbf{x};\boldsymbol{\xi})\, d\mathbf{x} = 0 \,. \tag{0.17 b}$$

The constant $R_N(\boldsymbol{\xi};\boldsymbol{\xi})$ is the regular part of $G_N$ at the singularity. Since $G_N$ has a zero spatial average, it follows from (0.16) that $\int_\Omega u_1\, d\mathbf{x} = 0$, as required in (0.14 *a*).

Next, we expand $u_1$ as $\mathbf{x}\to\mathbf{x}_j$. We use the local behavior for $G_N$, given in (0.17 *b*), to obtain from (0.16) that

$$u_1 \sim u_0 \log|\mathbf{x}-\mathbf{x}_j| - 2\pi u_0 \left[ R_{Njj} + \sum_{\substack{i=1\\ i\neq j}}^{K} G_{Nij} \right] \,, \quad \mathbf{x}\to\mathbf{x}_j \,, \tag{0.18}$$

where $G_{Nji} = G_N(\mathbf{x}_j;\mathbf{x}_i)$ and $R_{Njj} = R_N(\mathbf{x}_j;\mathbf{x}_j)$. Comparing (0.18) and the required singularity behavior (0.13 *b*), we obtain that

$$A_{1j} = -2\pi u_0 \left[ R_{Njj} + \sum_{\substack{i=1\\ i\neq j}}^{K} G_{Nij} \right] \,, \qquad j = 1,\ldots,N \,. \tag{0.19}$$

Next, we write the problem (0.8) in $\Omega$ as

$$\triangle u_2 = -\lambda_2 u_0 - \lambda_1 u_1 + 2\pi \sum_{j=1}^{K} A_{1j}\delta(\mathbf{x}-\mathbf{x}_j) \,, \quad \mathbf{x}\in\Omega \,; \quad \partial_n u_2 = 0 \,, \quad \mathbf{x}\in\partial\Omega \,. \tag{0.20}$$

Since $\int_\Omega u_1\, d\mathbf{x} = 0$ and $u_0 = |\Omega|^{-1/2}$, the divergence theorem applied to (0.20) determines $\lambda_2$ as $\lambda_2 u_0|\Omega| = 2\pi\sum_{j=1} A_{1j}$. Finally, we use (0.19) for $A_{1j}$, we get

$$\lambda_2 = -\frac{4\pi^2}{|\Omega|} p(\mathbf{x}_1,\ldots,\mathbf{x}_K) \,, \qquad p(\mathbf{x}_1,\ldots,\mathbf{x}_K) \equiv \sum_{j=1}^{N} \left( R_{Njj} + \sum_{\substack{i=1\\ i\neq j}}^{K} G_{Nji} \right) \,. \tag{0.21}$$

Combining (0.4) with (0.15) and (0.21) we get the two-term expansion given in equations (5.27) and (5.28) of the Corollary in §5 of the workshop notes given by

$$\lambda_0(\varepsilon) \sim \frac{2\pi\nu K}{|\Omega|} - \frac{4\pi^2\nu^2}{|\Omega|} p(\mathbf{x}_1,\ldots,\mathbf{x}_K) + \cdots \,, \qquad \nu = -1/\log(\varepsilon d) \,. \tag{0.22}$$

(2) For the case of one circular hole of radius $\varepsilon$ (for which $d = 1$) in a circle of area $|\Omega| = \pi$, the result above reduces to

$$\lambda \sim 2\nu - 4\pi\nu^2 R_{N11} \,, \qquad \nu \equiv -1/\log\varepsilon \,. \tag{0.23}$$

Here $R_{N11}$ is the regular part of the Neumann Green's function at the center of the hole. For the unit disk and a source point at the origin, so that $\mathbf{x}_1 = 0$, the Neumann Green's function $G_N(r;0)$, satisfying (0.17), is radially symmetric and has the form

$$G_N(r;0) = \frac{r^2}{4\pi} - \frac{1}{2\pi}\log r + A \sim -\frac{1}{2\pi}\log r + A + o(1) \,, \quad r\to 0 \,, \tag{0.24}$$

where $A \equiv R_{N11}$ is a constant to be found from the constraint $\int_\Omega G_N\, dx = 0$ . Notice that $\partial_r G_N = 0$ on $r = 1$.

The integral constraint reduces to $\int_0^1 G_N r\, dr = 0$, which yields

$$\int_0^1 \frac{r^3}{4\pi}\, dr - \frac{1}{2\pi}\int_0^1 r \log r\, dr + A\int_0^1 r\, dr = \frac{1}{16\pi} + \frac{1}{8\pi} + \frac{A}{2} = 0\,,$$

so that $A = -3/(8\pi)$. Thus, $A = R_{N11} = -3/(8\pi)$ is the regular part of the Neumann Green's function at the origin. The two-term expansion (0.23) then becomes

$$\lambda_0 \sim 2\nu + 3\nu^2/2 + \cdots\,. \tag{0.25}$$

The exact eigenvalue relation for the lowest eigenvalue is

$$J_0'(\sqrt{\lambda}) = \frac{J_0(\sqrt{\lambda}\varepsilon)}{Y_0(\sqrt{\lambda}\varepsilon)} Y_0'(\sqrt{\lambda})\,. \tag{0.26}$$

Since $\lambda \to 0$ as $\varepsilon \to 0$, we use the large argument expansion for each term above, and neglect algebraically small terms in $\varepsilon$. We substitute

$$J_0'(z) \sim -z/2 + z^3/16 + \cdots\,; \quad J_0(z) \approx 1\,,$$

$$Y_0(z) \sim \frac{2}{\pi}\left(\log(z/2) + \gamma_e\right) + \cdots\,, \qquad Y_0'(z) \sim \frac{2}{\pi z}\left[1 + \frac{z^2}{4} - \frac{z^2}{2}\left(\log(z/2) + \gamma_e\right)\right]\,,$$

into (0.26), and perform a little algebra to get

$$\frac{\lambda}{2} - \frac{\lambda^2}{16} \sim \frac{\nu}{1 - \nu\left(\log\left(\sqrt{\lambda}/2\right) + \gamma_e\right)}\left[1 + \frac{\lambda}{4} - \frac{\lambda}{2}\left(\log\left(\sqrt{\lambda}/2\right) + \gamma_e\right)\right]\,, \tag{0.27}$$

where $\nu = -1/\log\varepsilon$. We then expand $\lambda = \lambda_{00}\nu$ to get that $\lambda_{00} = 2$. For the next term, we expand $\lambda = 2\nu + \lambda_{01}\nu^2$, so that the equation above, upon using the leading term of the Binomial series, becomes

$$\nu + \lambda_1\nu^2/2 - \nu^2/4 \sim \nu\left(1 + \chi\nu\right)\left(1 + \frac{\nu}{2} - \nu\chi\right)\,, \qquad \chi \equiv \log\left(\sqrt{2\nu}/2\right) + \gamma_e\,. \tag{0.28}$$

Expanding this out, the $\nu\chi$ term cancels and from the $O(\nu^2)$ terms we get $\lambda_1/4 - 1/4 = 1/2$. This gives $\lambda_1 = 3/2$, and so $\lambda \sim 2\nu + 3\nu^2/2$, which agrees with (0.25).

(3) Let $\phi_j$, $\lambda_j$ be the eigenpairs of (0.1) for $j = 0, 1, 2\ldots$ ordered by $\lambda_0 < \lambda_1 < \lambda_2$.... We calculated an asymptotic expansion for the lowest eigenpair $\lambda_0$ and $\phi_0$ above. We will normalize the eigenpairs by $\int_{\Omega\backslash\Omega_p} \phi_j^2\, dx = 1$, and we know that the eigenfunctions are orthogonal in the sense that $\int_\Omega \phi_j\phi_k\, dx = 0$ for $j \neq k$. We then expand the solution $w$ of (0.3) in terms of $\phi_j$ as $w = \sum_{j=0} c_j\phi_j$. By orthogonality, we obtain that

$$c_j = \int_{\Omega\backslash\Omega_p} w\phi_j\, dx \tag{0.29}$$

Next, we multiply the equation in (0.3) by $\phi_j$ and use Green's second identity to obtain

$$\int_{\Omega\backslash\Omega_p} \phi_j \triangle w\, dx - \int_{\Omega\backslash\Omega_p} w \triangle \phi_j\, dx = 0$$

$$-\frac{1}{D}\int_{\Omega\backslash\Omega_p} \phi_j\, dx + \lambda_j \int_{\Omega\backslash\Omega_p} \phi_j w\, dx = 0$$

Thus, $c_j = (D\lambda_j)^{-1}\int_{\Omega\backslash\Omega_p} \phi_j\, dx$, so that from (0.29) we get

$$w = \frac{1}{D}\sum_{j=0}^{\infty} \frac{\phi_j}{\lambda_j}\int_{\Omega\backslash\Omega_p} \phi_j\, dx\,.$$

Now we calculate $\bar{w}$ to get

$$\bar{w} = \frac{1}{|\Omega\backslash\Omega_p|}\int_{|\Omega\backslash\Omega_p|} w\, dx = \frac{1}{D}\sum_{j=0}^{\infty}\frac{1}{\lambda_j}\left(\int_{\Omega\backslash\Omega_p} \phi_j\, dx\right)^2$$

Finally, we notice that $\lambda_0 \to 0$ as $\varepsilon \to 0$ and that $\int_{\Omega \backslash \Omega_p} \phi_j \, dx \to 0$ as $\varepsilon \to 0$ for $j \geq 1$. This follows since for $\varepsilon \to 0$ the first eigenfunction satisfies $\phi_0 \sim |\Omega|^{-1/2}$, and the orthogonality of eigenfunction property holds. Thus only the $j = 0$ term above is retained, and with $\phi_0 \sim |\Omega|^{-1/2}$, we calculate

$$\bar{w} \sim \frac{1}{\lambda_0 D |\Omega|} \left( \int_\Omega |\Omega|^{-1/2} \, dx \right)^2 = \frac{1}{D\lambda_0}$$

Finally, we use our two-term estimate for $\lambda_0$ as given above in (0.22) to get the two-term expansion for the average mean first time

$$\bar{w} \sim \frac{|\Omega|}{2\pi\nu K D} + \frac{|\Omega| p(x_1, \ldots, x_K)}{K^2 D} + \cdots \,, \qquad \nu = -1/\log \varepsilon \,. \tag{0.30}$$

If we want to minimize $\bar{w}$ we must choose the trap locations to minimize $p(x_1, \ldots, x_L)$.

(4) Suppose that the traps have radius $\sigma$ and that the length scale of the domain is $L$. If we assume that $\sigma \ll L$, and define $\varepsilon = \sigma/L$ and scale $U$ by the saturation constant $u = \beta U$, we obtain under steady-state conditions the nonlinear eigenvalue problem

$$\triangle u + \lambda u(1-u) = 0 \,, \quad \mathbf{x} \in \Omega \backslash \Omega_p \,; \qquad \Omega_p \equiv \cup_{j=1}^{K} \Omega_{\varepsilon_j} \,, \tag{0.31 a}$$

$$\partial_n u = 0 \,, \quad \mathbf{x} \in \partial\Omega \,; \qquad u = 0 \,, \quad \mathbf{x} \in \partial\Omega_{\varepsilon_j} \,, \quad j = 1, \ldots, K \,. \tag{0.31 b}$$

Here $\lambda \equiv L^2\mu/D$ is a dimensionless parameter. Notice that $u = 0$ is a solution for all values of $\lambda$. This is the extinct fish solution. We want to know at what minumum value of $\lambda$ will a branch of nontrivial solutions bifurcate from the $u = 0$ solution. Linearizing around $u = 0$, the local bifurcating branch is at the first eigenvalue $\lambda = \lambda_0$ of the Laplacian problem (0.1). Thus

$$\frac{L^2\mu}{D} = \lambda_0(\varepsilon) \sim \frac{2\pi\nu K}{|\Omega|} - \frac{4\pi^2\nu^2}{|\Omega|} p(\mathbf{x}_1, \ldots, \mathbf{x}_K) + \cdots \,, \qquad \nu = -1/\log(\varepsilon d) \,, \tag{0.32}$$

would give a threshold value of $D$ for a bifurcating solution branch.

**Worked Example 2:** Consider the following problem in the **2-D circular disk** $\Omega = \{\mathbf{x} \mid |\mathbf{x}| \leq 2\}$ that contains three small holes

$$\triangle u = 0\,, \qquad \mathbf{x} \in \Omega \backslash \cup_{j=1}^{3} \Omega_{\varepsilon_j}\,, \tag{0.33 a}$$

$$u = 4\cos(2\theta)\,, \qquad |\mathbf{x}| = 2\,. \tag{0.33 b}$$

$$u = \alpha_j\,, \qquad \mathbf{x} \in \partial\Omega_{\varepsilon_j}\,, \quad j = 1,2,3\,. \tag{0.33 c}$$

Suppose that each of the holes has an elliptical shape with semi-axes $\varepsilon$ and $2\varepsilon$. Apply the theory for summing infinite logarithmic expansions to first derive and then numerically solve a linear system for the source strengths. In your implementation assume that the holes are centered at cartesian coordinate locations $\mathbf{x}_1 = (1/2, 1/2)$, $\mathbf{x}_2 = (1/2, 0)$ and $\mathbf{x}_3 = (-1/4, 0)$. Take the boundary values on the holes to be $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = 2$. Plot (on a computer) the source strengths versus $\varepsilon$. (Hint: You will need to recall the method of images for calculating the required Green's function in a circular disk)

**Solution:**

We let the holes be centered at $x_1, \ldots, x_N$. In the outer region, defined away from $\Omega_{\varepsilon_j}$ for $j = 1, \ldots, N$, we expand

$$u(\mathbf{x};\varepsilon) \sim U_{0H}(\mathbf{x}) + U_0(\mathbf{x};\nu) + \sigma(\varepsilon)U_1(\mathbf{x};\nu) + \cdots\,, \tag{0.34}$$

where we assume that $\sigma \ll \nu^m$ for any integer $m > 0$. Since the holes have a common shape, we have that $\nu = -1/\log(\varepsilon d)$ where $d$ is the common logarithmic capacitance of the holes. In (0.34), $U_{0H}(\mathbf{x})$ is the smooth function satisfying the unperturbed problem in the unperturbed domain $\Omega$

$$\triangle U_{0H} = 0\,, \quad \mathbf{x} \in \Omega\,; \qquad U_{0H} = f\,, \quad \mathbf{x} \in \partial\Omega\,. \tag{0.35}$$

Substituting (0.34) into (0.33 $a$) and (0.33 $b$), and letting $\Omega_{\varepsilon_j} \to \mathbf{x}_j$ as $\varepsilon \to 0$, we get that $U_0$ satisfies

$$\triangle U_0 = 0\,, \quad \mathbf{x} \in \Omega \backslash \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}\,, \tag{0.36 a}$$

$$U_0 = 0\,, \qquad \mathbf{x} \in \partial\Omega\,, \tag{0.36 b}$$

$$U_0 \quad \text{is singular as} \quad \mathbf{x} \to \mathbf{x}_j\,, \quad j = 1, \ldots, N\,. \tag{0.36 c}$$

The singularity behavior for $U_0$ as $\mathbf{x} \to \mathbf{x}_j$ will be found below by matching the outer solution to the far-field behavior of the inner solution to be constructed near each $\Omega_{\varepsilon_j}$.

In the $j^{\text{th}}$ inner region near $\Omega_{\varepsilon_j}$ we introduce the inner variables $\mathbf{y}$ and $v(\mathbf{y};\varepsilon)$ by

$$\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j)\,, \qquad v(\mathbf{y};\varepsilon) = u(\mathbf{x}_j + \varepsilon\mathbf{y};\varepsilon)\,. \tag{0.37}$$

We then expand $v(\mathbf{y};\varepsilon)$ as

$$v(\mathbf{y};\varepsilon) = \alpha_j + \nu\gamma_j v_{cj}(\mathbf{y}) + \mu_0(\varepsilon)V_{1j}(\mathbf{y}) + \cdots\,, \tag{0.38}$$

where $\gamma_j = \gamma_j(\nu)$ is a constant to be determined. Here $\mu_0 \ll \nu^k$ as $\varepsilon \to 0$ for any $k > 0$. In (0.38), the logarithmic gauge function $\nu$ is defined by

$$\nu = -1/\log(\varepsilon d)\,, \tag{0.39}$$

where $d$ is specified below. By substituting (0.37) and (0.38) into (0.33 $a$) and (0.33 $c$), we conclude that $v_{cj}(\mathbf{y})$ is the unique solution to

$$\triangle_{\mathbf{y}} v_{cj} = 0\,, \quad \mathbf{y} \notin \Omega_j\,; \quad v_{cj} = 0\,, \quad \mathbf{y} \in \partial\Omega_j\,, \tag{0.40 a}$$

$$v_{cj}(\mathbf{y}) \sim \log|\mathbf{y}| - \log d + o(1)\,, \quad \text{as} \quad |\mathbf{y}| \to \infty\,. \tag{0.40 b}$$

Here $\Omega_j \equiv \varepsilon^{-1}\Omega_{\varepsilon_j}$, and the logarithmic capacitance, $d$, is determined by the shape of $\Omega_j$. Since the holes were assumed to have the same shape then $d$ is independent of $j$.

Writing (0.40 $b$) in outer variables and substituting the result into (0.38), we get that the far-field expansion of $v$ away from each $\Omega_j$ is

$$v \sim \alpha_j + \gamma_j + \nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j|\,, \qquad j = 1, \ldots, N\,. \tag{0.41}$$

Then, by expanding the outer solution (0.34) as $\mathbf{x} \to \mathbf{x}_j$, we obtain the following matching condition between the inner and outer solutions:

$$U_{0H}(\mathbf{x}_j) + U_0 \sim \alpha_j + \gamma_j + \nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j| \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j \,, \quad j = 1, \ldots, N \,. \tag{0.42}$$

In this way, we obtain that $U_0$ satisfies (0.36) subject to the singularity structure

$$U_0 \sim \alpha_j - U_{0H}(\mathbf{x}_j) + \gamma_j + \nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j| + o(1) \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j \,, \quad j = 1, \ldots, N \,. \tag{0.43}$$

Observe that in (0.43) both the singular and regular parts of the singularity structure are specified. Therefore, (0.43) will effectively lead to a linear system of algebraic equations for $\gamma_j$ for $j = 1, \ldots, N$.

The solution to (0.36 *a*) and (0.36 *b*), with $U_0 \sim \nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j|$ as $\mathbf{x} \to \mathbf{x}_j$, can be written as

$$U_0(\mathbf{x}; \nu) = -2\pi\nu \sum_{i=1}^{N} \gamma_i G(\mathbf{x}; \mathbf{x}_i) \,, \tag{0.44}$$

where $G(\mathbf{x}; \mathbf{x}_j)$ is the Green's function satisfying

$$\triangle G = -\delta(\mathbf{x} - \mathbf{x}_j) \,, \quad \mathbf{x} \in \Omega \,; \quad G = 0 \,, \quad \mathbf{x} \in \partial\Omega \,, \tag{0.45 a}$$

$$G(\mathbf{x}; \mathbf{x}_j) \sim -\frac{1}{2\pi} \log|\mathbf{x} - \mathbf{x}_j| + R(\mathbf{x}_j; \mathbf{x}_j) + o(1) \,, \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_j \,. \tag{0.45 b}$$

Here $R_{jj} \equiv R(\mathbf{x}_j; \mathbf{x}_j)$ is the regular part of $G$.

Finally, we expand (0.44) as $\mathbf{x} \to \mathbf{x}_j$ and equate the resulting expression with the required singularity behavior (0.43) to get

$$\nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j| - 2\pi\nu\gamma_j R_{jj} - 2\pi\nu \sum_{\substack{i=1 \\ i \neq j}}^{N} \gamma_i G(\mathbf{x}_j; \mathbf{x}_i) = \alpha_j - U_{0H}(\mathbf{x}_j) + \gamma_j + \nu\gamma_j \log|\mathbf{x} - \mathbf{x}_j| \,, \quad j = 1, \ldots, N \,. \tag{0.46}$$

In this way, we get the following linear algebraic system for $\gamma_j$ for $j = 1, \ldots, N$:

$$-\gamma_j \left(1 + 2\pi\nu R_{jj}\right) - 2\pi\nu \sum_{\substack{i=1 \\ i \neq j}}^{N} \gamma_i G_{ji} = \alpha_j - U_{0H}(\mathbf{x}_j) \,, \quad j = 1, \ldots, N \,. \tag{0.47}$$

Here $G_{ji} \equiv G(\mathbf{x}_j; \mathbf{x}_i)$ and $\nu_j = -1/\log(\varepsilon d_j)$. We summarize the asymptotic construction as follows:

For $\varepsilon \ll 1$, the outer expansion from (0.34) is

$$u \sim U_{0H}(\mathbf{x}) - 2\pi\nu \sum_{i=1}^{N} \gamma_i G(\mathbf{x}; \mathbf{x}_i) \,, \quad \text{for} \quad |\mathbf{x} - \mathbf{x}_j| = \mathcal{O}(1) \,. \tag{0.48 a}$$

The inner expansion near $\Omega_{\varepsilon_j}$ with $\mathbf{y} = \varepsilon^{-1}(\mathbf{x} - \mathbf{x}_j)$, is

$$u \sim \alpha_j + \nu\gamma_j v_{cj}(\mathbf{y}) \,, \quad \text{for} \quad |\mathbf{x} - \mathbf{x}_j| = \mathcal{O}(\varepsilon) \,. \tag{0.48 b}$$

Here $\nu = -1/\log(\varepsilon d)$, $d$ is defined in (0.40 *b*), $v_{cj}(\mathbf{y})$ satisfies (0.40), $U_{0H}$ satisfies the unperturbed problem (0.35), while $G(\mathbf{x}; \mathbf{x}_j)$ and $R(\mathbf{x}_j; \mathbf{x}_j)$ satisfy (0.45). Finally, the constants $\gamma_j$ for $j = 1, \ldots, N$ are obtained from the $N$ dimensional linear algebraic system (0.47).

For the problem under consideration we have $f = 4\cos(2\theta) = 4(\cos^2\theta - \sin^2\theta) = x^2 - y^2$ on $(x^2 + y^2)^{1/2} = 4$. Thus, the solution to the unperturbed problem (0.35) is simply

$$U_{0H}(x, y) = x^2 - y^2 \,. \tag{0.49}$$

Next, the Green's function satisfying (0.45) and its regular part are calculated from the method of images as

$$G(\mathbf{x}; \mathbf{x}_j) = -\frac{1}{2\pi} \log\left(\frac{2|\mathbf{x} - \mathbf{x}_j|}{|\mathbf{x} - \mathbf{x}_j'||\mathbf{x}_j|}\right) \,, \qquad R_{jj} \equiv R(\mathbf{x}_j; \mathbf{x}_j) = -\frac{1}{2\pi} \log\left[\frac{2}{|\mathbf{x}_j - \mathbf{x}_j'||\mathbf{x}_j|}\right] \,. \tag{0.50}$$

Here $\mathbf{x}_j'$ is the image point of $\mathbf{x}_j$ in the unit disk of radius two.

Next, we note that since each of the holes has an elliptic shape with semi-axes $\varepsilon$ and $2\varepsilon$, then from the Table of the class notes their common logarithmic capacitance is $d = 3/2$. The holes are assumed to be centered at $\mathbf{x}_1 = (1/2, 1/2)$, $\mathbf{x}_2 = (1/2, 0)$ and $\mathbf{x}_3 = (-1/4, 0)$, and have the constant boundary values $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = 2$.

Therefore, upon defining $\nu = -1/\log(3\varepsilon/2)$ we obtain from (0.47) that $\gamma_j$ for $j = 1, \ldots, 3$ is the solution of the linear system

$$-\gamma_1 \left[1 + 2\pi\nu R_{11}\right] - 2\pi\nu \left[\gamma_2 G(\mathbf{x}_1; \mathbf{x}_2) + \gamma_3 G(\mathbf{x}_1; \mathbf{x}_3)\right] = 1\,, \qquad (0.51\,a)$$

$$-\gamma_2 \left[1 + 2\pi\nu R_{22}\right] - 2\pi\nu \left[\gamma_1 G(\mathbf{x}_2; \mathbf{x}_1) + \gamma_3 G(\mathbf{x}_2; \mathbf{x}_3)\right] = -1/4\,, \qquad (0.51\,b)$$

$$-\gamma_3 \left[1 + 2\pi\nu R_{33}\right] - 2\pi\nu \left[\gamma_1 G(\mathbf{x}_3; \mathbf{x}_1) + \gamma_2 G(\mathbf{x}_3; \mathbf{x}_2)\right] = 31/16\,. \qquad (0.51\,c)$$

Here $R_{jj}$ and $G(\mathbf{x}_j; \mathbf{x}_i)$ are to be evaluated from (0.50).

We solve this linear system numerically for $\gamma_j$ as a function of $\varepsilon$. The curves $\gamma_j(\varepsilon)$ as a function of $\varepsilon$ are plotted in Fig. 1. We observe that the leading-order approximation to (0.51), valid for $\nu \ll 1$, is simply $\gamma_1 = -1$, $\gamma_2 = 1/4$ and $\gamma_3 = -31/16$. From Fig. 1 we observe that this approximation, which neglects interaction effects between the holes, is rather inaccurate unless $\varepsilon$ is very small.

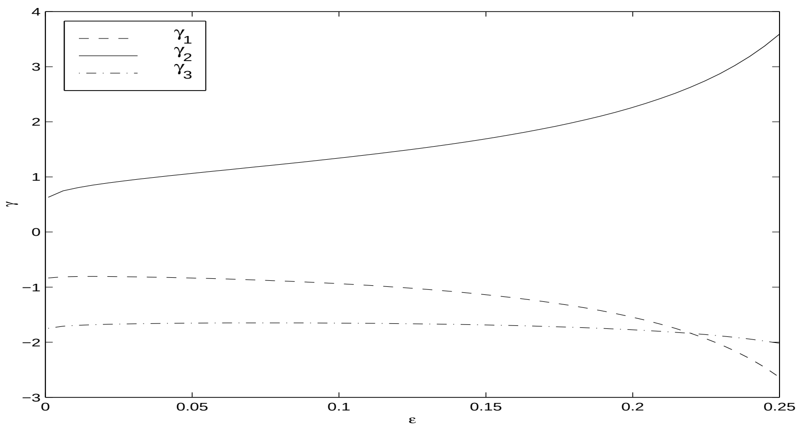


FIGURE 1. Plot of $\gamma_j = \gamma_j(\epsilon)$ for $j = 1, 2, 3$ obtained from the numerical solution to (0.51).